1.

ON THE BODILY TIDES OF VISCOUS AND SEMI-ELASTIC SPHEROIDS, AND ON THE OCEAN TIDES UPON A YIELDING NUCLEUS*.

[*Philosophical Transactions of the Royal Society*, Part I. Vol. 170 (1879), pp. 1—35.]

In a well-known investigation Sir William Thomson has discussed the problem of the bodily tides of a homogeneous elastic sphere, and has drawn therefrom very important conclusions as to the great rigidity of the earth†.

Now it appears improbable that the earth should be perfectly elastic; for the contortions of geological strata show that the matter constituting the earth is somewhat plastic, at least near the surface. We know also that even the most refractory metals can be made to flow under the action of sufficiently great forces.

Although Sir W. Thomson's investigation has gone far to overthrow the old idea of a semi-fluid interior to the earth, yet geologists are so strongly impressed by the fact that enormous masses of rock are being, and have been, poured out of volcanic vents in the earth's surface, that the belief is not yet extinct that we live on a thin shell over a sea of molten lava. Under these circumstances it appears to be of interest to investigate the consequences which would arise from the supposition that the matter constituting the earth is of a viscous or imperfectly elastic nature; for if the interior *is*

* [Since the date of this paper important contributions to the subject have been made by Professor Horace Lamb in his papers on "The Oscillations of a Viscous Spheroid," *Proc. Lond. Math. Soc.*, Vol. XIII. (1881–2), p. 51; "On the Vibrations of an Elastic Sphere," *ibid.*, p. 189, and "On the Vibrations of a Spherical Shell," *ibid.*, Vol. XIV. (1882–3), p. 50. See also a paper by T. J. Bromwich, *Proc. Lond. Math. Soc.*, Vol. XXX. (1898–9), p. 98.]

† Sir William states that M. Lamé had treated the subject at an earlier date, but in an entirely different manner. I am not aware, however, that M. Lamé had fully discussed the subject in its physical aspect.

constituted in this way, then the solid crust, unless very thick, cannot possess rigidity enough to repress the tidal surgings, and these hypotheses must give results fairly conformable to the reality. The hypothesis of imperfect elasticity will be principally interesting as showing how far Sir W. Thomson's results are modified by the supposition that the elasticity breaks down under continued stress.

In this paper, then, I follow out these hypotheses, and it will be seen that the results are fully as hostile to the idea of any great mobility of the interior of the earth as is that of Sir W. Thomson.

The only terrestrial evidence of the existence of a bodily tide in the earth would be that the ocean tides would be less in height than is indicated by theory. The subject of this paper is therefore intimately connected with the theory of the ocean tides.

In the first part the equilibrium tide-theory is applied to estimate the reduction and alteration of phase of ocean tides as due to bodily tides, but that theory is acknowledged on all hands to be quite fallacious in its explanation of tides of short period.

In the second part of this paper, therefore, I have considered the dynamical theory of tides in an equatorial canal running round a tidally-distorted nucleus, and the results are almost the same as those given by the equilibrium theory.

The first two sections of the paper are occupied with the adaptation of Sir W. Thomson's work* to the present hypotheses; as, of course, it was impossible to reproduce the whole of his argument, I fear that the investigation will only be intelligible to those who are either already acquainted with that work, or who are willing to accept my quotations therefrom as established.

As some readers may like to know the results of this inquiry without going into the mathematics by which they are established, I have given in Part III. a summary of the whole, and have as far as possible relegated to that part of the paper the comments and conclusions to be drawn. I have tried, however, to give so much explanation in the body of the paper as will make it clear whither the argument is tending.

The case of pure viscosity is considered first, because the analysis is somewhat simpler, and because the results will afterwards admit of an easy extension to the case of elastico-viscosity.

* His paper will be found in *Phil. Trans.*, 1863, p. 573, and §§ 733—737 and 834—846 of Thomson and Tait's *Natural Philosophy*.

I.

THE BODILY TIDES OF VISCOUS AND ELASTICO-VISCOUS SPHEROIDS.

1. *Analogy between the flow of a viscous body and the strain of an elastic one.*

The general equations of flow of a viscous fluid, *when the effects of inertia are neglected*, are

$$\left.\begin{aligned} -\frac{dp}{dx} + \upsilon\nabla^2\alpha + X = 0 \\ -\frac{dp}{dy} + \upsilon\nabla^2\beta + Y = 0 \\ -\frac{dp}{dz} + \upsilon\nabla^2\gamma + Z = 0 \end{aligned}\right\} \qquad (1)$$

where x, y, z are the rectangular coordinates of a point of the fluid; α, β, γ are the component velocities parallel to the axes; p is the mean of the three pressures across planes perpendicular to the three axes respectively; X, Y, Z are the component forces acting on the fluid, estimated per unit volume; υ is the coefficient of viscosity; and ∇^2 is the Laplacian operation

$$\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}$$

Besides these we have the equation of continuity $\frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz} = 0$.

Also if P, Q, R, S, T, U are the normal and tangential stresses estimated in the usual way across three planes perpendicular to the axes

$$\left.\begin{aligned} P = -p + 2\upsilon\frac{d\alpha}{dx}, \quad Q = -p + 2\upsilon\frac{d\beta}{dy}, \quad R = -p + 2\upsilon\frac{d\gamma}{dz} \\ S = \upsilon\left(\frac{d\beta}{dz} + \frac{d\gamma}{dy}\right), \quad T = \upsilon\left(\frac{d\gamma}{dx} + \frac{d\alpha}{dz}\right), \quad U = \upsilon\left(\frac{d\alpha}{dy} + \frac{d\beta}{dx}\right) \end{aligned}\right\} \qquad (2)$$

Now in an elastic solid, if α, β, γ be the displacements, $m - \frac{1}{3}n$ the coefficient of dilatation, and n that of rigidity, and if $\delta = \frac{d\alpha}{dx} + \frac{d\beta}{dy} + \frac{d\gamma}{dz}$; the equations of equilibrium are

$$\left.\begin{aligned} m\frac{d\delta}{dx} + n\nabla^2\alpha + X = 0 \\ m\frac{d\delta}{dy} + n\nabla^2\beta + Y = 0 \\ m\frac{d\delta}{dz} + n\nabla^2\gamma + Z = 0 \end{aligned}\right\} \qquad (3)^*$$

* Thomson and Tait's *Natural Philosophy*, § 698, eq. (7) and (8).

Also

$$P=(m-n)\,\delta+2n\frac{d\alpha}{dx},\quad Q=(m-n)\,\delta+2n\frac{d\beta}{dy},\quad R=(m-n)\,\delta+2n\frac{d\gamma}{dz} \quad\text{......(4)}$$

and S, T, U have the same forms as in (2), with n written instead of υ.

Therefore if we put $-p=\frac{1}{3}(P+Q+R)$, we have $p=-(m-\frac{1}{3}n)\,\delta$, so that (3) may be written

$$-\frac{m}{m-\frac{1}{3}n}\frac{dp}{dx}+n\nabla^2\alpha+X=0, \text{ \&c., \&c.}$$

Also $$P=-\frac{m-n}{m-\frac{1}{3}n}p+2n\frac{d\alpha}{dx},\quad Q=\text{\&c.},\quad R=\text{\&c.}$$

Now if we suppose the elastic solid to be incompressible, so that m is infinitely large compared to n, then it is clear that the equations of equilibrium of the incompressible elastic solid assume exactly the same form as those of flow of the viscous fluid, n merely taking the place of υ.

Thus every problem in the equilibrium of an incompressible elastic solid has its counterpart in a problem touching the state of flow of an incompressible viscous fluid, when the effects of inertia are neglected; and the solution of the one may be made applicable to the other by merely reading for "displacements" "velocities," and for the coefficient of "rigidity" that of "viscosity."

2. *A sphere under influence of bodily force.*

Sir W. Thomson has solved the following problem:

To find the displacement of every point of the substance of an elastic sphere exposed to no surface traction, but deformed infinitesimally by an equilibrating system of forces acting *bodily* through the interior.

If for "displacement" we read velocity, and for "elastic" viscous, we have the corresponding problem with respect to a viscous fluid, and *mutatis mutandis* the solution is the same.

But we cannot find the tides of a viscous sphere by merely making the equilibrating system of forces equal to the tide-generating influence of the sun or moon, because the substance of the sphere must be supposed to have the power of gravitation.

For suppose that at any time the equation to the free surface of the earth (as the viscous sphere may be called for brevity) is $r=a+\sum_2^\infty\sigma_i$, where σ_i is a surface harmonic. Then the matter, positive or negative, filling the space represented by $\Sigma\sigma_i$ exercises an attraction on every point of the interior; and this attraction, together with that of a homogeneous sphere of radius a, must be added to the tide-generating influence to form the whole force in the interior of the sphere. Also it is a spheroid, and no longer a true sphere with which we have to deal. If, however, we cut a true sphere of radius a out of the spheroid (leaving out $\Sigma\sigma_i$), then by a proper choice of surface actions, the tidal problem may be reduced to finding the state of flow in a true sphere under the action of (i) an external tide-generating influence, (ii) the attraction of the true sphere, and of the positive and negative matter filling the space $\Sigma\sigma_i$, but (iii) subject to certain surface forces.

Since (i) and (ii) together constitute a bodily force, the problem only differs from that of Sir W. Thomson in the fact that there are forces acting on the surface of the sphere.

Now as we are only going to consider small deviations from sphericity, these surface actions will be of small amount, and an approximation will be permissible.

It is clear that rigorously there is tangential action* between the layer of matter $\Sigma\sigma_i$ and the true sphere, but by far the larger part of the action is normal, and is simply the weight (either positive or negative) of the matter which lies above or below any point on the surface of the true sphere.

Thus, in order to reduce the earth to sphericity, the appropriate surface action is a normal traction equal to $-gw\Sigma\sigma_i$, where g is gravity at the surface, and w is the mass per unit volume of the matter constituting the earth.

In order to show what alteration this normal surface traction will make in Sir W. Thomson's solution, I must now give a short account of his method of attacking the problem.

He first shows that, where there is a potential function, the solution of the problem may be subdivided, and that the complete values of α, β, γ consist of the sums of two parts which are to be found in different ways. The first part consists of *any* values of α, β, γ, which satisfy the equations throughout the sphere, without reference to surface conditions. As far as regards the second part, the bodily force is deemed to be non-existent and is replaced by certain surface actions, so calculated as to counteract the surface actions which correspond to the values of α, β, γ found in the first part of the solution. Thus the first part satisfies the condition that there is a

* I shall consider some of the effects of this tangential action in a future paper, viz.: "Problems connected with the Tides of a Viscous Spheroid," read before the Royal Society on December 19th, 1878. [Paper 4.]

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bodily force, and the second adds the condition that the surface forces are zero. The first part of the solution is easily found, and for the second part Sir W. Thomson discusses the case of an elastic sphere under the action of any surface tractions, but without any bodily force acting on it. The component surface tractions parallel to the three axes, in this problem, are supposed to be expanded in a series of surface harmonics; and the harmonic terms of any order are shown to have an effect on the displacements independent of those of every other order. Thus it is only necessary to consider the typical component surface tractions A_i, B_i, C_i of the order i.

He proves that (for an incompressible elastic solid for which m is infinite) this one surface traction A_i, B_i, C_i produces a displacement throughout the sphere given by

$$\alpha = \frac{1}{na^{i-1}}\left\{\frac{a^2-r^2}{2(2i^2+1)}\frac{d\Psi_{i-1}}{dx} + \frac{1}{i-1}\left[\frac{i+2}{(2i^2+1)(2i+1)}r^{2i+1}\frac{d}{dx}(\Psi_{i-1}r^{-2i+1}) + \frac{1}{2i(2i+1)}\frac{d\Phi_{i+1}}{dx} + A_i r^i\right]\right\} \quad \text{..................(5)*}$$

with symmetrical expressions for β and γ; where Ψ and Φ are auxiliary functions defined by

$$\left.\begin{aligned} \Psi_{i-1} &= \frac{d}{dx}(A_i r^i) + \frac{d}{dy}(B_i r^i) + \frac{d}{dz}(C_i r^i) \\ \Phi_{i+1} &= r^{2i+3}\left\{\frac{d}{dx}(A_i r^{-i-1}) + \frac{d}{dy}(B_i r^{-i-1}) + \frac{d}{dz}(C_i r^{-i-1})\right\} \end{aligned}\right\} \quad \text{........(6)}$$

In the case considered by Sir W. Thomson of an elastic sphere deformed by bodily stress and subject to no surface action, we have to substitute in (5) and (6) only those surface actions which are equal and opposite to the surface forces corresponding to the first part of the solution†; but in the case which we now wish to consider, we must add to these latter the components of the normal traction $-gw\Sigma\sigma_i$, and besides must include in the bodily force both the external disturbing force, and the attraction of the matter of the spheroid on itself.

Now from the forms of (5) and (6) it is obvious that the tractions which correspond to the first part of the solution, and the traction $-gw\Sigma\sigma_i$ produce quite independent effects, and therefore we need only add to the complete solution of Sir W. Thomson's problem of the elastic sphere, the terms which arise from the normal traction $-gw\Sigma\sigma_i$. Finally we must pass from the elastic problem to the viscous one, by reading ν for n, and velocities for displacements.

* Thomson and Tait's *Natural Philosophy*, 1867, § 737, equation (52).

† Where the solid is incompressible, this surface traction is normal to the sphere at every point, provided that the potential of the bodily force is expressible in a series of solid harmonics.

I proceed then to find the state of internal flow in the viscous sphere, which results from a normal traction at every point of the surface of the sphere, given by the surface harmonic S_i.

In order to use the formulæ (5) and (6), it is first necessary to express the component tractions $\frac{x}{a} S_i$, $\frac{y}{a} S_i$, $\frac{z}{a} S_i$ as surface harmonics.

Now if V_i be a solid harmonic,

$$\frac{d}{dx}(r^{-2i-1} V_i) = -(2i+1) r^{-(2i+3)} x V_i + r^{-(2i+1)} \frac{dV_i}{dx}$$

So that
$$x V_i = \frac{1}{2i+1} \left\{ r^2 \frac{dV_i}{dx} - r^{2i+3} \frac{d}{dx}(r^{-2i-1} V_i) \right\}$$

Therefore
$$\frac{x}{a} S_i = \frac{1}{2i+1} \left\{ \left[r^{-i+1} \frac{d}{dx}(r^i S_i) \right] - \left[r^{i+2} \frac{d}{dx}(r^{-i-1} S_i) \right] \right\}$$

The quantities within the brackets [ ] being independent of r, and being surface harmonics of orders $i-1$ and $i+1$ respectively, we have $\frac{x}{a} S_i$ expressed as the sum of two surface harmonics A_{i-1}, A_{i+1}, where

$$A_{i-1} = \frac{1}{2i+1} r^{-i+1} \frac{d}{dx}(r^i S_i), \quad A_{i+1} = -\frac{1}{2i+1} r^{i+2} \frac{d}{dx}(r^{-i-1} S_i)$$

Similarly $\frac{y}{a} S_i$, $\frac{z}{a} S_i$ may be expressed as $B_{i-1} + B_{i+1}$ and $C_{i-1} + C_{i+1}$, where the B's and C's only differ from the A's in having y, z written for x.

We have now to form the auxiliary functions Ψ_{i-2}, Φ_i corresponding to A_{i-1}, B_{i-1}, C_{i-1} and Ψ_i, Φ_{i+2} corresponding to A_{i+1}, B_{i+1}, C_{i+1}.

Then by the formulæ (6)

$$(2i+1)\Psi_{i-2} = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) r^i S_i = 0$$

$$\frac{2i+1}{r^{2i+1}} \Phi_i = \frac{d}{dx}\left[r^{-2i+1} \frac{d}{dx}(r^i S_i) \right] + \frac{d}{dy}\left[\quad \right] + \frac{d}{dz}\left[\quad \right] = -\frac{i(2i-1)}{r^{2i+1}} r^i S_i$$

$$-(2i+1)\Psi_i = \frac{d}{dx}\left[r^{2i+3} \frac{d}{dx}(r^{-i-1} S_i) \right] + \frac{d}{dy}\left[\quad \right] + \frac{d}{dz}\left[\quad \right] = -(i+1)(2i+3) r^i S_i$$

$$-\frac{2i+1}{r^{2i+5}} \Phi_{i+2} = \left(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2}\right) r^{-i-1} S_i = 0$$

Thus

$$\Psi_{i-2} = 0, \quad \Phi_i = -\frac{i(2i-1)}{2i+1} r^i S_i, \quad \Psi_i = \frac{(i+1)(2i+3)}{2i+1} r^i S_i, \quad \Phi_{i+2} = 0$$

Then by (5) we form α corresponding to A_{i-1}, B_{i-1}, C_{i-1}, and also to

A_{i+1}, B_{i+1}, C_{i+1}, and add them together. The final result is that a normal traction S_i gives

$$\alpha' = \frac{1}{\upsilon a^i}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2 - \frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{d}{dx}(r^i S_i) - \frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}(r^{-i-1}S_i)\right] \quad\text{.........(7)}$$

and symmetrical expressions for β' and γ'.

α', β', γ' are here written for α, β, γ to show that this is only a partial solution, and υ is written for n to show that it corresponds to the viscous problem. If we now put $S_i = -gw\sigma_i$, we get the state of flow of the fluid due to the transmitted pressure of the deficiencies and excesses of matter below and above the true spherical surface. This constitutes the solution as far as it depends on (iii).

There remain the parts dependent on (i) and (ii), which may for the present be classified together; and for this part Sir W. Thomson's solution is directly applicable. The state of internal strain of an elastic sphere, subject to no surface action, but under the influence of a bodily force of which the potential is W_i, may be at once adapted to give the state of flow of a viscous sphere under like conditions. The solution is

$$\alpha'' = \frac{1}{\upsilon}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2 - \frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{dW_i}{dx} - \frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}(r^{-2i-1}W_i)\right] \quad\text{...........(8)*}$$

with symmetrical expressions for β'' and γ''.

I will first consider (ii); *i.e.*, the matter of the earth is now supposed to possess the power of gravitation.

The gravitation potential of the spheroid $r = a + \sigma_i$ (taking only a typical term of σ) at a point in the interior, estimated per unit volume, is

$$\frac{gw}{2a}(3a^2 - r^2) + \frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i \sigma_i$$

according to the usual formula in the theory of the potential.

The first term, being symmetrical round the centre of the sphere, can clearly cause no flow in the incompressible viscous sphere. We are therefore left with $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i \sigma_i$.

* *Natural Philosophy*, § 834, equation (8) when m is infinite compared with n, and $i-1$ written for i, and υ replaces n.

Now if $\frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i \sigma_i$ be substituted for W_i in (8), and if the resulting expression be compared with (7) when $-gw\sigma_i$ is written for S_i, it will be seen that $-\alpha'' = \frac{3}{2i+1}\alpha'$.

Thus
$$\alpha' + \alpha'' = \alpha''\left(1 - \frac{2i+1}{3}\right)^* = -\tfrac{2}{3}(i-1)\alpha''$$

And if
$$V_i = \frac{3gw}{2i+1}\left(\frac{r}{a}\right)^i \sigma_i$$

$$\alpha' + \alpha'' = -\frac{1}{v}\left[\left\{\frac{i(i+2)}{2(i-1)[2(i+1)^2+1]}a^2 - \frac{(i+1)(2i+3)}{2(2i+1)[2(i+1)^2+1]}r^2\right\}\frac{d}{dx}\{\tfrac{2}{3}(i-1)V_i\}\right.$$
$$\left. - \frac{i}{(2i+1)[2(i+1)^2+1]}r^{2i+3}\frac{d}{dx}\{r^{-2i-1}\tfrac{2}{3}(i-1)V_i\}\right] \quad \text{.........(9)}$$

with symmetrical expressions for $\beta' + \beta''$ and $\gamma' + \gamma''$.

Equation (9) then embodies the solution as far as it depends on (ii) and (iii). And since (9) is the same as (8) when $-\frac{2}{3}(i-1)V_i$ is written for W_i, we may include all the effects of mutual gravitation in producing a state of flow in the viscous sphere, by adopting Thomson's solution (8), and taking instead of the true potential of the layer of matter σ_i, $-\frac{2}{3}(i-1)$ times that potential, and by adding to it the external disturbing potential.

We have now learnt how to include the surface action in the potential; and if W_i be the potential of the external disturbing influence, the *effective* potential per unit volume at a point within the sphere, now free of surface action and of mutual gravitation, is $W_i - \frac{2gw(i-1)}{2i+1}\left(\frac{r}{a}\right)^i \sigma_i = r^i T_i$ suppose.

The complete solution of our problem is then found by writing $r^i T_i$ in place of W_i in Thomson's solution (8)†.

In order however to apply the solution to the case of the earth, it will be convenient to use polar coordinates. For this purpose, write $wr^i S_i$ for W_i, and let r be the radius vector; θ the colatitude; ϕ the longitude. Let ρ, ϖ, ν be the velocities radially, and along and perpendicular to the meridian respectively. Then the expressions for ρ, ϖ, ν will be precisely the same as those for α, β, γ in (8), save that for $\frac{d}{dx}$ we must put $\frac{d}{dr}$; for $\frac{d}{dy}$, $\frac{d}{r\sin\theta d\phi}$; and for $\frac{d}{dz}$, $\frac{d}{rd\theta}$.

* The case of § 815 in Thomson and Tait's *Natural Philosophy* is a special case of this.

† The introduction of the effects of gravitation may be also carried out synthetically, as is done by Sir W. Thomson (§ 840, *Natural Philosophy*); but the effects of the lagging of the tide-wave render this method somewhat artificial, and I prefer to exhibit the proof in the manner here given. Conversely, the elastic problem may be solved as in the text.

Then after some reductions we have

$$\left.\begin{aligned}\rho &= \frac{i^2(i+2)a^2 - i(i^2-1)r^2}{2(i-1)[2(i+1)^2+1]v} r^{i-1}\,\mathrm{T}_i \\ \varpi &= \frac{i(i+2)a^2-(i-1)(i+3)r^2}{2(i-1)[2(i+1)^2+1]v} r^{i-1}\frac{d\mathrm{T}_i}{d\theta} \\ \nu &= \frac{i(i+2)a^2-(i-1)(i+3)r^2}{2(i-1)[2(i+1)^2+1]v}\frac{r^{i-1}}{\sin\theta}\frac{d\mathrm{T}_i}{d\phi}\end{aligned}\right\}\quad\text{..........(10)*}$$

where $\mathrm{T}_i = w\left(\mathrm{S}_i - 2g\dfrac{i-1}{2i+1}\dfrac{\sigma_i}{a^i}\right)$.

These equations for ρ, ϖ, ν give us the state of internal flow corresponding to the external disturbing potential $r^i\mathrm{S}_i$, including the effects of the mutual gravitation of the matter constituting the spheroid.

3. *The form of the free surface at any time.*

If ρ' be the surface value of ρ, then

$$\rho' = \frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{a^{i+1}}{v}\mathrm{T}_i$$

Hence after a short interval of time δt, the equation to the bounding surface of the spheroid becomes $r = a + \sigma_i + \rho'\delta t$; but during this same interval, σ_i has become $\dfrac{d\sigma_i}{dt}\delta t$, whence

$$\frac{d\sigma_i}{dt} = \rho' = \frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{wa^{i+1}}{v}\mathrm{S}_i - \frac{i}{2(i+1)^2+1}\frac{gwa}{v}\sigma_i$$

or
$$\frac{d\sigma_i}{dt} + \frac{i}{2(i+1)^2+1}\frac{gwa}{v}\sigma_i = \frac{i(2i+1)}{2(i-1)[2(i+1)^2+1]}\frac{wa^{i+1}}{v}\mathrm{S}_i \quad\text{....(11)}$$

This differential equation gives the manner in which the surface changes, under the influence of the external potential $r^i\mathrm{S}_i$.

If S_i be not a function of the time, and if s_i be the value of σ_i when $t=0$,

$$\sigma_i = \frac{2i+1}{2(i-1)}\frac{a^i\mathrm{S}_i}{g}\left[1-\exp\left(\frac{-gwait}{[2(i+1)^2+1]v}\right)\right] + s_i\exp\left(\frac{-gwait}{[2(i+1)^2+1]v}\right)\quad\text{(12)†}$$

When t is infinite
$$\sigma_i = \frac{2i+1}{2(i-1)}\frac{a^i\mathrm{S}_i}{g}\quad\text{........................(13)}$$

and there is no further state of flow, for the fluid has assumed the form

* There seems to be a misprint as to the signs of the 𝔊's in the second and third of equations (13) of § 834 of the *Natural Philosophy* (1867). When this is corrected μ and ν admit of reduction to tolerably simple forms. It appears to me also that the differentiation of ρ in (15) is incorrect; and this falsifies the argument in three following lines. The correction is not, however, in any way important.

† I write "exp" for "e to the power of."