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Mathematical Preliminaries

This chapter lays the mathematical foundation for the study of optimization that occupies the rest of this book. It focuses on three main topics: the topological structure of Euclidean spaces, continuous and differentiable functions on Euclidean spaces and their properties, and matrices and quadratic forms. Readers familiar with real analysis at the level of Rudin (1976) or Bartle (1964), and with matrix algebra at the level of Munkres (1964) or Johnston (1984, Chapter 4), will find this chapter useful primarily as a refresher; for others, a systematic knowledge of its contents should significantly enhance understanding of the material to follow.

Since this is not a book in introductory analysis or linear algebra, the presentation in this chapter cannot be as comprehensive or as leisurely as one might desire. The results stated here have been chosen with an eye to their usefulness towards the book's main purpose, which is to develop a theory of optimization in Euclidean spaces. The selective presentation of proofs in this chapter reveals a similar bias. Proofs whose formal structure bears some resemblance to those encountered in the main body of the text are spelt out in detail; others are omitted altogether, and the reader is given the choice of either accepting the concerned results on faith or consulting the more primary sources listed alongside the result.

It would be inaccurate to say that this chapter does not presuppose any knowledge on the part of the reader, but it is true that it does not presuppose much. Appendices A and B aim to fill in the gaps and make the book largely self-contained. Appendix A reviews the basic rules of propositional logic; it is taken for granted throughout that the reader is familiar with this material. An intuitive understanding of the concept of an "irrational number," and of the relationship between rational and irrational numbers, suffices for this chapter and for the rest of this book. A formal knowledge of the real line and its properties will, however, be an obvious advantage, and readers who wish to acquaint themselves with this material may consult Appendix B.

The discussion in this chapter takes place solely in the context of Euclidean spaces. This is entirely adequate for our purposes, and avoids generality that we do not need. However, Euclidean spaces are somewhat special in that many of their properties (such as completeness, or the compactness of closed and bounded sets) do not carry over to more general metric or topological spaces. Readers wishing to view the topological structure of Euclidean spaces in a more abstract context can, at a first pass, consult Appendix C, where the concepts of inner product, norm, metric, and topology are defined on general vector spaces, and some of their properties are reviewed.

1.1 Notation and Preliminary Definitions

1.1.1 Integers, Rationals, Reals, $\mathbb{R}^n$

The notation we adopt is largely standard. The set of positive integers is denoted by $\mathbb{N}$, and the set of all integers by $\mathbb{Z}$:

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$
$$\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}.$$

The set of rational numbers is denoted by $\mathbb{Q}$:

$$\mathbb{Q} = \left\{x \mid x = \frac{p}{q},\ p, q \in \mathbb{Z}, q \neq 0\right\}.$$

Finally, the set of all real numbers, both rational and irrational, is denoted by $\mathbb{R}$. As mentioned earlier, it is presumed that the reader has at least an intuitive understanding of the real line and its properties. Readers lacking this knowledge should first review Appendix B.

Given a real number $z \in \mathbb{R}$, its *absolute value* will be denoted $|z|$:

$$|z| = \begin{cases} z & \text{if } z \geq 0 \\ -z & \text{if } z < 0. \end{cases}$$

The *Euclidean distance* between two points x and y in $\mathbb{R}$ is defined as $|x - y|$, i.e., as the absolute value of their difference.

For any positive integer $n \in \mathbb{N}$, the n-fold Cartesian product of $\mathbb{R}$ will be denoted $\mathbb{R}^n$. We will refer to $\mathbb{R}^n$ as *n-dimensional Euclidean space*. When $n = 1$, we shall continue writing $\mathbb{R}$ for $\mathbb{R}^1$.

A point in $\mathbb{R}^n$ is a vector $x = (x_1, \ldots, x_n)$ where for each $i = 1, \ldots, n$, x_i is a real number. The number x_i is called the *i-th coordinate* of the vector x.

We use 0 ιo denote the real number 0 as well as the null vector $(0, \ldots, 0) \in \mathbb{R}^n$. This notation is ambiguous, but the correct meaning will usually be clear from the context.

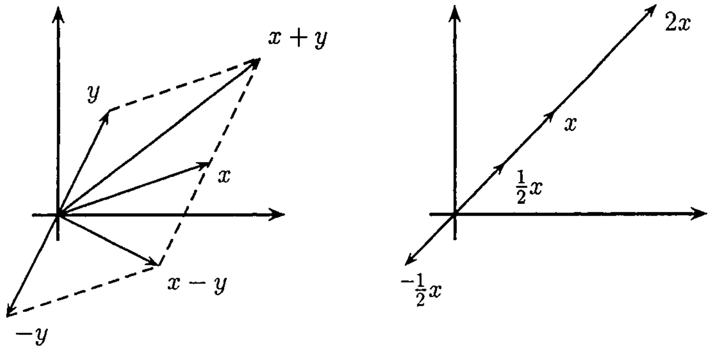

Fig. 1.1. Vector Addition and Scalar Multiplication in $\mathbb{R}^2$

Vector addition and scalar multiplication are defined in $\mathbb{R}^n$ as follows: for $x, y \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$,

$$\begin{aligned} x + y &= (x_1 + y_1, \ldots, x_n + y_n) \\ \alpha x &= (\alpha x_1, \ldots, \alpha x_n). \end{aligned}$$

Figure 1.1 provides a graphical interpretation of vector addition and scalar mutiplication in $\mathbb{R}^2$.

Given any two n-vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$, we write

$$\begin{aligned} x &= y, \quad \text{if } x_i = y_i, \quad i = 1, \ldots, n. \\ x &\geq y, \quad \text{if } x_i \geq y_i, \quad i = 1, \ldots, n. \\ x &> y, \quad \text{if } x \geq y \ \text{ and } \ x \neq y. \\ x &\gg y, \quad \text{if } x_i > y_i, \quad i = 1, \ldots, n. \end{aligned}$$

Note that

- $x \geq y$ does *not* preclude the possibility that $x = y$, and
- for $n > 1$, the vectors x and y need not be comparable under any of the categories above; for instance, the vectors $x = (2, 1)$ and $y = (1, 2)$ in $\mathbb{R}^2$ do not satisfy $x \geq y$, but neither is it true that $y \geq x$.

The *nonnegative* and *strictly positive* orthants of $\mathbb{R}^n$, denoted $\mathbb{R}^n_+$ and $\mathbb{R}^n_{++}$, respectively, are defined as

$$\mathbb{R}^n_+ = \{x \in \mathbb{R}^n \mid x \geq 0\},$$

and

$$\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n \mid x \gg 0\}.$$

1.1.2 Inner Product, Norm, Metric

This subsection describes three structures on the space $\mathbb{R}^n$: the *Euclidean inner product* of two vectors x and y in $\mathbb{R}^n$, the *Euclidean norm* of a vector x in $\mathbb{R}^n$, and the *Euclidean metric* measuring the distance between two points x and y in $\mathbb{R}^n$. Each of these generalizes a familiar concept from $\mathbb{R}$. Namely, when $n = 1$, and x and y are just real numbers, the Euclidean inner product of x and y is just the product xy of the numbers x and y; the Euclidean norm of x is simply the absolute value $|x|$ of x; and the Euclidean distance between x and y is the absolute value $|x - y|$ of their difference.

Given $x, y \in \mathbb{R}^n$, the *Euclidean inner product* of the vectors x and y, denoted $x \cdot y$, is defined as:

$$x \cdot y = \sum_{i=1}^{n} x_i y_i.$$

We shall henceforth refer to the Euclidean inner product simply as the inner product.

Theorem 1.1 *The inner product has the following properties for any vectors $x, y, z \in \mathbb{R}^n$ and scalars $a, b \in \mathbb{R}$:*

1. *Symmetry: $x \cdot y = y \cdot x$.*
2. *Bilinearity: $(ax + by) \cdot z = ax \cdot z + by \cdot z$ and $x \cdot (ay + bz) = x \cdot ay + x \cdot bz$.*
3. *Positivity: $x \cdot x \geq 0$, with equality holding if and only if $x = 0$.*

Proof Symmetry and bilinearity are easy to verify from the definition of the inner product. To check that positivity holds, note that the square of a real number is always nonnegative, and can be zero if and only if the number is itself zero. It follows that as the sum of squared real numbers, $x \cdot x = \sum_{i=1}^{n} x_i^2$ is always nonnegative, and is zero if and only if $x_i = 0$ for each i, i.e., if and only if $x = 0$. □

The inner product also satisfies a very useful condition called the *Cauchy–Schwartz inequality:*

Theorem 1.2 (Cauchy–Schwartz Inequality) *For any $x, y \in \mathbb{R}^n$ we have*

$$|x \cdot y| \leq (x \cdot x)^{1/2} (y \cdot y)^{1/2}.$$

Proof For notational ease, let $X = x \cdot x$, $Y = y \cdot y$, and $Z = x \cdot y$. Then, the result will be proved if we show that $XY \geq Z^2$, since the required inequality will follow simply by taking square roots on both sides.

If $x = 0$, then $Z = X = 0$, and the inequality holds trivially. Suppose, therefore, that $x \neq 0$. Note that by the positivity property of the inner product, we must then have $X > 0$. The positivity property also implies that for any scalar $a \in \mathbb{R}$, we have

$$\begin{aligned} 0 &\leq (ax + y) \cdot (ax + y) \\ &= a^2 x \cdot x + 2ax \cdot y + y \cdot y \\ &= a^2 X + 2aZ + Y. \end{aligned}$$

In particular, this inequality must hold for $a = -Z/X$. When this value of a is used in the equation above, we obtain

$$\left(\frac{Z}{X}\right)^2 X - 2\left(\frac{Z}{X}\right) Z + Y = -\left(\frac{Z^2}{X}\right) + Y \geq 0,$$

or $Y \geq Z^2/X$. Since $X > 0$, this in turn implies $XY \geq Z^2$, as required. □

The *Euclidean norm* (henceforth, simply the *norm*) of a vector $x \in \mathbb{R}^n$, denoted $\|x\|$, is defined as

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2}.$$

The norm is related to the inner product through the identity

$$\|x\| = (x \cdot x)^{1/2}$$

for all $x \in \mathbb{R}^n$; in particular, the Cauchy–Schwartz inequality may be written as

$$|x \cdot y| \leq \|x\| \|y\|.$$

Our next result, which describes some useful properties of the norm, uses this observation.

Theorem 1.3 *The norm satisfies the following properties at all $x, y \in \mathbb{R}^n$, and $a \in \mathbb{R}$:*

1. *Positivity:* $\|x\| \geq 0$, *with equality if and only if* $x = 0$.
2. *Homogeneity:* $\|ax\| = |a| \cdot \|x\|$.
3. *Triangle Inequality:* $\|x + y\| \leq \|x\| + \|y\|$.

Proof The positivity property of the norm follows from the positivity property of the inner product, and the fact that $\|x\| = (x \cdot x)^{1/2}$. Homogeneity obtains since

$$\|ax\| = \left(\sum_{i=1}^{n} a^2 x_i^2\right)^{1/2} = \left(a^2 \sum_{i=1}^{n} x_i^2\right)^{1/2} = |a| \|x\|.$$

The triangle inequality is a little trickier; we will need the Cauchy–Schwartz inequality to establish it. Observe that for any x and y in $\mathbb{R}^n$, we have

$$\|x + y\|^2 = (x + y) \cdot (x + y) = \|x\|^2 + 2x \cdot y + \|y\|^2.$$

By the Cauchy–Schwartz inequality, $x \cdot y \leq \|x\|\|y\|$. Substituting this in the previous equation, we obtain

$$\|x + y\|^2 \leq \|x\|^2 + 2\|x\|\|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

The proof is completed by taking square roots on both sides. □

The *Euclidean distance* $d(x, y)$ between two vectors x and y in $\mathbb{R}^n$ is given by

$$d(x, y) = \left(\sum_{i=1}^{n}(x_i - y_i)^2\right)^{1/2}.$$

The distance function d is called a *metric*, and is related to the norm $\|\cdot\|$ through the identity

$$d(x, y) = \|x - y\|$$

for all $x, y \in \mathbb{R}^n$.

Theorem 1.4 *The metric d satisfies the following properties at all $x, y, z \in \mathbb{R}^n$:*

1. *Positivity: $d(x, y) \geq 0$ with equality if and only if $x = y$.*
2. *Symmetry: $d(x, y) = d(y, x)$.*
3. *Triangle Inequality: $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in \mathbb{R}^n$.*

Proof The positivity property of the metric follows from the positivity property of the norm, and the observation that $d(x, y) = \|x - y\|$. Symmetry is immediate from the definition. The inequality $d(x, z) \leq d(x, y) + d(y, z)$ is the same as

$$\|x - z\| \leq \|x - y\| + \|y - z\|.$$

This is just the triangle inequality for norms, which we have already established. □

The concepts of inner product, norm, and metric can be defined on any abstract vector space, and not just $\mathbb{R}^n$. In fact, the properties we have listed in Theorems 1.1, 1.3, and 1.4 are, in abstract vector spaces, defining characteristics of the respective concepts. Thus, for instance, an inner product on a vector space is *defined* to be any operator on that space that satisfies the three properties of symmetry, bilinearity, and positivity; while a norm on that space is defined to be any operator that meets the conditions of positivity, homogeneity, and the triangle inequality. For more on this, see Appendix C.

1.2 Sets and Sequences in $\mathbb{R}^n$

1.2.1 Sequences and Limits

A *sequence* in $\mathbb{R}^n$ is the specification of a point $x_k \in \mathbb{R}^n$ for each integer $k \in \{1, 2, \ldots\}$. The sequence is usually written as

$$x_1, x_2, x_3, \ldots$$

or, more compactly, simply as $\{x_k\}$. Occasionally, where notational clarity will be enhanced by this change, we will use superscripts instead of subscripts, and denote the sequence by $\{x^k\}$.

A sequence of points $\{x_k\}$ in $\mathbb{R}^n$ is said to *converge* to a limit x (written $x_k \to x$) if the distance $d(x_k, x)$ between x_k and x tends to zero as k goes to infinity, i.e., if for all $\epsilon > 0$, there exists an integer $k(\epsilon)$ such that for all $k \geq k(\epsilon)$, we have $d(x_k, x) < \epsilon$. A sequence $\{x_k\}$ which converges to a limit x is called a *convergent sequence*.

For example, the sequence $\{x_k\}$ in $\mathbb{R}$ defined by $x_k = 1/k$ for all k is a convergent sequence, with limit $x = 0$. To see this, let any $\epsilon > 0$ be given. Let $k(\epsilon)$ be any integer such that $k(\epsilon) > 1/\epsilon$. Then, for all $k > k(\epsilon)$, we have $d(x_k, 0) = d(1/k, 0) = 1/k < 1/k(\epsilon) < \epsilon$, so indeed, $x_k \to 0$.

Theorem 1.5 *A sequence can have at most one limit. That is, if $\{x_k\}$ is a sequence in $\mathbb{R}^n$ converging to a point $x \in \mathbb{R}^n$, it cannot also converge to a point $y \in \mathbb{R}^n$ for $y \neq x$.*

Proof This follows from a simple application of the triangle inequality. If $x_k \to x$ and $y \neq x$, then

$$d(x_k, y) \geq d(x, y) - d(x_k, x).$$

Since $d(x, y) > 0$ and $d(x_k, x) \to 0$, this inequality shows that $d(x_k, y)$ cannot go to zero as k goes to infinity, so $x_k \to y$ is impossible. □

A sequence $\{x_k\}$ in $\mathbb{R}^n$ is called a *bounded* sequence if there exists a real number M such that $\|x_k\| \leq M$ for all k. A sequence $\{x_k\}$ which is not bounded is said to be *unbounded*; that is, $\{x_k\}$ is an unbounded sequence if for any $M \in \mathbb{R}$, there exists $k(M)$ such that $|x_{k(M)}| > M$.

Theorem 1.6 *Every convergent sequence in $\mathbb{R}^n$ is bounded.*

Proof Suppose $x_k \to x$. Let $\epsilon = 1$ in the definition of convergence. Then, there exists $k(1)$ such that for all $k \geq k(1)$, $d(x_k, x) < 1$. Since $d(x_k, x) = \|x_k - x\|$, an application of the triangle inequality yields for any $k \geq k(1)$

$$\begin{aligned} \|x_k\| &= \|(x_k - x) + x\| \\ &\leq \|x_k - x\| + \|x\| \\ &< 1 + \|x\|. \end{aligned}$$

Now define M to be the maximum of the finite set of numbers

$$\{\|x_1\|, \ldots, \|x_{k(1)-1}\|, 1 + \|x\|\}.$$

Then, $M \geq \|x_k\|$ for all k, completing the proof. □

While Theorem 1.5 established that a sequence can have at most one limit, Theorem 1.6 implies that a sequence may have *no* limit at all. Indeed, because every convergent sequence must be bounded, it follows that if $\{x_k\}$ is an unbounded sequence, then $\{x_k\}$ cannot converge. Thus, for instance, the sequence $\{x_k\}$ in $\mathbb{R}$ defined by $x_k = k$ for all k is a non-convergent sequence.[1]

However, unboundedness is not the only reason a sequence may fail to converge. Consider the following example: let $\{x_k\}$ in $\mathbb{R}$ be given by

$$x_k = \begin{cases} \dfrac{1}{k}, & k = 1, 3, 5, \ldots \\ 1 - \dfrac{1}{k}, & k = 2, 4, 6, \ldots \end{cases}$$

This sequence is bounded since we have $|x_k| \leq 1$ for all k. However, it does not possess a limit. The reason here is that the odd terms of the sequence are converging to the point 0, while the even terms are converging to the point 1. Since a sequence can have only one limit, this sequence does not converge.

Our next result shows that convergence of a sequence $\{x^k\}$ in $\mathbb{R}^n$ is equivalent to convergence in each coordinate. This gives us an alternative way to establish convergence in $\mathbb{R}^n$. We use superscripts to denote the sequence in this result to avoid confusion between the k-th element x_k of the sequence, and the i-th coordinate x_i of a vector x.

Theorem 1.7 *A sequence $\{x^k\}$ in $\mathbb{R}^n$ converges to a limit x if and only if $x_i^k \to x_i$ for each $i \in \{1, \ldots, n\}$, where $x^k = (x_1^k, \ldots, x_n^k)$ and $x = (x_1, \ldots, x_n)$.*

[1] This may also be shown directly: for any fixed candidate limit x the distance $d(x_k, x) = |x - x_k| = |x - k|$ becomes unbounded as k goes to infinity. It follows that no $x \in \mathbb{R}$ can be a limit of this sequence, and therefore that it does not possess a limit.

Proof We will use the fact that the Euclidean distance between two points $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ can be written as

$$d(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^2 \right)^{1/2},$$

where $|x_i - y_i|$ is the Euclidean distance between x_i and y_i in $\mathbb{R}$.

First, suppose that $x^k \to x$. We are to show that $x_i^k \to x_i$ for each i, i.e., that, given any i and $\epsilon > 0$, there exists $k_i(\epsilon)$ such that for $k \geq k_i(\epsilon)$, we have $|x_i^k - x_i| < \epsilon$.

So let $\epsilon > 0$ be given. By definition of $x^k \to x$, we know that there exists $k(\epsilon)$ such that $d(x^k, x) < \epsilon$ for all $k \geq k(\epsilon)$. Therefore, for $k \geq k(\epsilon)$ and any i, we obtain:

$$|x_i^k - x_i| = \left(|x_i^k - x_i|^2\right)^{1/2} \leq \left(\sum_{j=1}^{n} |x_j^k - x_j|^2 \right)^{1/2} = d(x^k, x) < \epsilon.$$

Setting $k_i(\epsilon) = k(\epsilon)$ for each i, the proof that $x_i^k \to x_i$ for each i is complete.

Now, suppose that $\{x_i^k\}$ converges to x_i for each i. Let $\epsilon > 0$ be given. We will show that there is $k(\epsilon)$ such that $d(x^k, x) < \epsilon$ for all $k \geq k(\epsilon)$, which will establish that $x^k \to x$.

Define $\eta = \epsilon/\sqrt{n}$. For each i, there exists $k_i(\eta)$ such that for $k \geq k_i(\eta)$, we have $|x_i^k - x_i| < \eta$. Define $k(\epsilon)$ to be the maximum of the finite set of numbers $k_1(\eta), \ldots, k_n(\eta)$. Then, for $k \geq k(\epsilon)$, we have $|x_i^k - x_i| < \eta$ for all i, so

$$d(x^k, x) = \left(\sum_{i=1}^{n} |x_i^k - x_i|^2 \right)^{1/2} < \left(\sum_{i=1}^{n} \left[\frac{\epsilon}{\sqrt{n}} \right]^2 \right)^{1/2} = \epsilon,$$

which completes the proof. □

Theorem 1.7 makes it easy to prove the following useful result:

Theorem 1.8 *Let $\{x^k\}$ be a sequence in $\mathbb{R}^n$ converging to a limit x. Suppose that for every k, we have $a \leq x^k \leq b$, where $a = (a_1, \ldots, a_n)$ and $b = (b_1, \ldots, b_n)$ are some fixed vectors in $\mathbb{R}^n$. Then, it is also the case that $a \leq x \leq b$.*

Proof The theorem will be proved if we show that $a_i \leq x_i \leq b_i$ for each $i \in \{1, \ldots, n\}$. Suppose that the result were false, so for some i, we had $x_i < a_i$ (say). Since $x^k \to x$, it is the case by Theorem 1.7 that $x_j^k \to x_j$ for each $j \in \{1, \ldots, n\}$; in particular, $x_i^k \to x_i$. But $x_i^k \to x_i$ combined with $x_i < a_i$ implies that for all large k, we must have $x_i^k < a_i$. This contradicts the hypothesis that $a_i \leq x_i^k \leq b_i$ for all k. A similar argument establishes that $x_i > b_i$ also leads to a contradiction. Thus, $a_i \leq x_i \leq b_i$, and the proof is complete. □

1.2.2 Subsequences and Limit Points

Let a sequence $\{x_k\}$ in $\mathbb{R}^n$ be given. Let m be any rule that assigns to each $k \in \mathbb{N}$ a value $m(k) \in \mathbb{N}$. Suppose further that m is *increasing*, i.e., for each $k \in \mathbb{N}$, we have $m(k) < m(k+1)$. Given $\{x_k\}$, we can now define a new sequence $\{x_{m(k)}\}$, whose k-th element is the $m(k)$-th element of the sequence $\{x_k\}$. This new sequence is called a *subsequence* of $\{x_k\}$. Put differently, a subsequence of a sequence is any infinite subset of the original sequence that preserves the ordering of terms.

Even if a sequence $\{x_k\}$ is not convergent, it may contain subsequences that converge. For instance, the sequence $0, 1, 0, 1, 0, 1, \ldots$ has no limit, but the subsequences $0, 0, 0, \ldots$ and $1, 1, 1, \ldots$ which are obtained from the original sequence by selecting the odd and even elements, respectively, are both convergent.

If a sequence contains a convergent subsequence, the limit of the convergent subsequence is called a *limit point* of the original sequence. Thus, the sequence $0, 1, 0, 1, 0, 1, \ldots$ has two limit points 0 and 1. The following result is simply a restatement of the definition of a limit point:

Theorem 1.9 *A point x is a limit point of the sequence $\{x_k\}$ if and only if for any $\epsilon > 0$, there are infinitely many indices m for which $d(x, x_m) < \epsilon$.*

Proof If x is a limit point of $\{x_k\}$ then there must be a subsequence $\{x_{m(k)}\}$ that converges to x. By definition of convergence, it is the case that for any $\epsilon > 0$, all but finitely many elements of the sequence $\{x_{m(k)}\}$ must be within ϵ of x. Therefore, infinitely many elements of the sequence $\{x_k\}$ must also be within ϵ of x.

Conversely, suppose that for every $\epsilon > 0$, there are infinitely many m such that $d(x_m, x) < \epsilon$. Define a subsequence $\{x_{m(k)}\}$ as follows: let $m(1)$ be any m for which $d(x_m, x) < 1$. Now for $k = 2, 3, \ldots$ define successively $m(k)$ to be any m that satisfies (a) $d(x, x_m) < 1/k$, and (b) $m > m(k-1)$. This construction is feasible, since for each k, there are infinitely many m satisfying $d(x_m, x) < 1/k$. Moreover, the sequence $\{x_{m(k)}\}$ evidently converges to x, so x is a limit point of $\{x_k\}$. □

If a sequence $\{x_k\}$ is convergent (say, to a limit x), then it is apparent that every subsequence of $\{x_k\}$ must converge to x. It is less obvious, but also true, that if *every* subsequence $\{x_{m(k)}\}$ of a given sequence $\{x_k\}$ converges to the limit x, then $\{x_k\}$ itself converges to x. We do not offer a proof of this fact here, since it may be easily derived as a consequence of other considerations. See Corollary 1.19 below.

In general, a sequence $\{x_k\}$ may have any number of limit points. For instance, *every* positive integer arises as a limit point of the sequence

$$1, 1, 2, 1, 2, 3, 1, 2, 3, 4, \ldots$$