One

Japan and Temple Geometry

Temple bells die out.
The fragrant blossoms remain.
A perfect evening!
—Bashō

Temples

No visitor to a foreign country has failed to experience the fascination and unease that accompanies an encounter with unknown traditions and customs. Some visitors attempt to overcome their fears, while the majority quickly retreats to familiar shores, and in this lies a distinction: Those who embrace culture shock are travelers; those who do not are tourists.

The most profound culture shock comes about when one is confronted by a different way of thinking. Most of us can hardly imagine walking into a Western church or cathedral to encounter stained glass windows covered by equations and geometrical figures. Even if we can conceive of it, the thought strikes us as alien, out of place, perhaps sacrilegious. Yet for well over two centuries, Japanese mathematicians—professionals, amateurs, women, children—created what was essentially mathematical stained glass, wooden tablets adorned with beautiful geometric problems that were simultaneously works of art, religious offerings, and a record of what we might call "folk mathematics." The creators of these *sangaku*—a word that literally means "mathematical tablet"—hung them by the thousands in Buddhist temples and Shinto shrines throughout Japan, and for that reason the entire collection of *sangaku* problems has come to be known as "temple geometry," sacred mathematics.

In this book you will be invited not only to encounter temple geometry but to appreciate it. There is a bit of culture shock to be overcome. A single

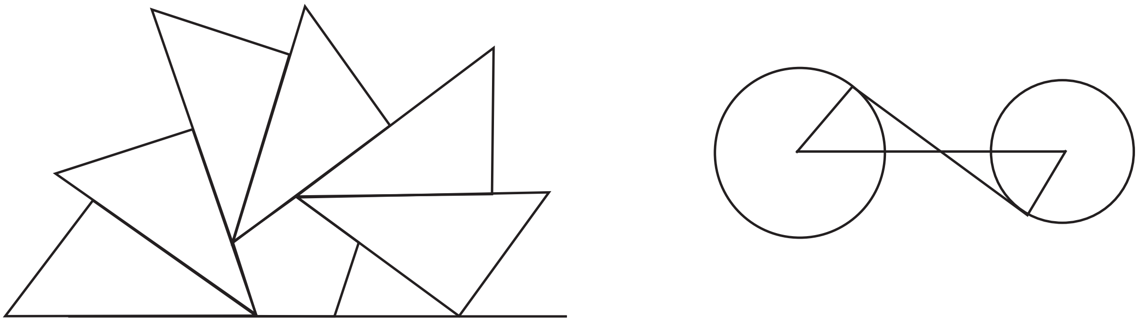

Figure 1.1. Which diagram would you guess came from an American geometry text?

glance at a *sangaku* is enough for one to realize that they were created by a profoundly different esthetic than the Greek-inspired designs found in Western geometry books. On a deeper level, one learns that the methods Japanese geometers employed to solve such problems differed, sometimes significantly, from those of their Western counterparts. Ask any professional mathematician whether the laws of mathematics would be the same in another universe and he or she will reply, of course. Real mathematicians are Pythagoreans—they cannot doubt that mathematics exists independently of the human mind. At the same time, during their off hours, mathematicians frequently speculate about how different mathematics could look from the way it is taught in Western schools.

Temple geometry provides a partial answer to both questions. Yes, the rules of mathematics are the same in East and West, but yes again, the traditional Japanese geometers who created *sangaku* saw their mathematical world through different eyes and sometimes solved problems in distinctly non-Western ways. To learn traditional Japanese mathematics is to learn another way of thinking.

Traditional Japanese mathematics, and with it temple geometry, arose in the seventeenth century under a nearly unique set of circumstances. In 1603, three years after defeating his rival *daimyo*— warlords—at the battle of Sekigahara, Tokugawa Ieyasu became shogun of Japan, establishing the Tokugawa shogunate. (A contemporary depiction of the battle of Sekigahara can be seen in the color plate 1.) The Tokugawa family ruled Japan for the better part of three hundred years, until 1868, when a decade after

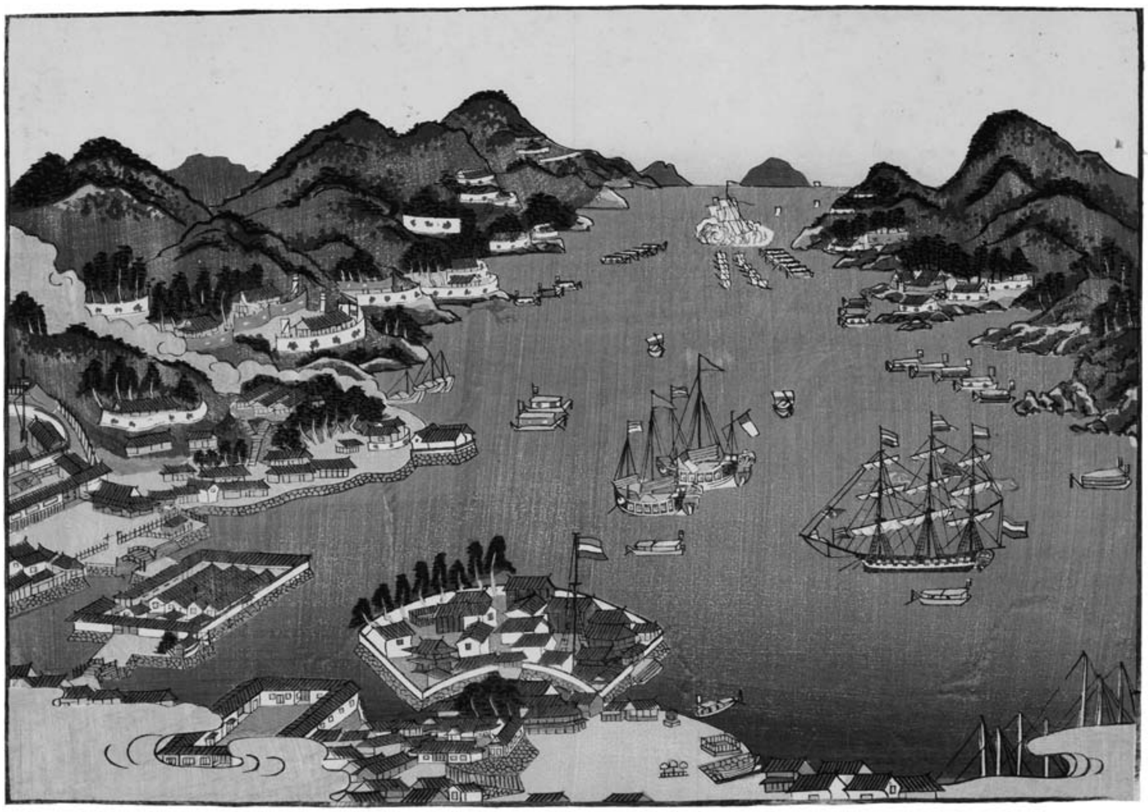

Plate 1.2. An anonymous and undated woodblock print (probably mid-nineteenth century) shows Nagasaki harbor with the small fan-shaped island of Deshima in the foreground. Another view of Deshima can be seen in color plate 2. (© Nagasaki Museum of History and Culture.)

Commodore Matthew C. Perry forcefully opened Japan to the West, the shogunate collapsed.

One of Ieyasu's first moves after Sekigahara was to establish his headquarters at a small fortress town in central Japan, a town that became known as Edo (pronounced "Yedo")—today's Tokyo. For that reason the rule of the Tokugawa is also known as the Edo period. During the first years of the Tokugawa shogunate, Ieyasu (who, although living until 1616, officially remained shogun only until 1605) consolidated power by confiscating the lands of other warlords, but nevertheless continued many of the foreign policies of his predecessor, the great *daimyo* Toyotomi Hideyoshi (1537–1598). At the turn of the seventeenth century, Japan carried on substantial trade with foreign countries, both Eastern and Western. Nagasaki on the island of Kyūshū had become the base for the "southern barbarians" to import their goods, as well as to print translations of Western literature, much of it religious.

Foreign missionaries had by then been in Japan for over fifty years. In autumn of 1543, three Portuguese were shipwrecked off Kyūshū. The misfortune proved decisive in terms of Japan's relations with outsiders, for the men were carrying arquebuses, which were rapidly adopted by the Japanese warlords. Of equal or greater importance was that, within a few years of the fateful shipwreck, Portuguese merchants and Jesuit missionaries began to arrive, seeking both trade and converts. The Jesuits were particularly successful, converting as many as two hundred thousand Japanese over the next forty years and becoming de facto rulers of the Nagasaki region.

All of this alarmed the proponents of Buddhism and raised the distrust of Hideyoshi himself; he in 1587 took direct control of Nagasaki and issued two edicts designed to curb the spread of Christianity. But the Spanish soon arrived, with Spanish merchants vying with Portuguese for trade and Franciscans vying with Jesuits for converts. In 1596, after a Spaniard supposedly boasted that the missionaries were merely the vanguard of an Iberian conquest, Hideyoshi ordered the execution of twenty-six priests and converts. The warlord, though, had other affairs on his mind, in particular the conquest of China, and he failed to pursue a resolution of the growing tensions between the Japanese and Westerners.

The tensions were resolved, in a particularly decisive and brutal fashion, at the very end of Tokugawa Ieyasu's life and in the two decades that followed. In 1614 Ieyasu reissued an earlier edict with which he summarily ordered that all Christian missionaries leave the country, that places of worship be torn down, and that the practice of Christianity be outlawed. But other internal affairs intervened and Ieyasu died in 1616 without having taken much action. After his death, though, persecution of Christian converts began in earnest and by 1637, according to some estimates, three hundred thousand converts had apostasized or been killed. Throughout the 1630s Ieyasu's grandson, Togukawa Iemitsu, issued a series of decrees that offered rewards for the identification of *kirishitan*, forbade the sending of Japanese ships overseas, and forbade any Japanese from traveling abroad, on pain of death.

By 1641 the last of the Portuguese merchants had been expelled, leaving only the Dutch. The Dutch had arrived comparatively late to Japan, in 1609, and had shown more interest in trade than mission. For that reason they were allowed to remain after the expulsion of the Iberians. The Japanese, however, by now utterly suspicious of Westerners, put severe strictures on the Dutch presence: The representatives of the Dutch East India Company were forced to move onto a small, man-made island called Deshima in Nagasaki harbor (see color plate 2 and plate 1.2). The fan-shaped island, originally created for the Portuguese, measured only 200 by 70 meters. A wall surrounded Deshima, posted with signs warning the Japanese to keep away, and it was entirely cut off from the mainland except for a bamboo water pipe and a single, guarded bridge. On this oasis, twenty or so members of the East India Company lived among the few warehouses, sheep, pigs, and chickens, and awaited the summer arrival of the Dutch ships. Upon making port, captains locked all Bibles and Christian literature into barrels, while Japanese laborers unloaded cargo.

That, for the next two hundred years, constituted Japan's trade with the West, and so began the policy of what would eventually become known as *sakoku*, "closed country." It is impossible to claim that *sakoku* was one hundred percent effective; certainly trade with Korea and China continued. Two Japanese did escape to Holland around 1650 in order to study mathematics. We know the scholars only by their adopted names, Petrius Hartsingius and Franciscus Carron, the former at least having achieved some distinction. Whether they ever returned to Japan we do not know. One doctor, Nakashima Chōzaburō, traveled abroad with a Dutch trader and risked beheading to come home. According to tradition, the local *daimyo* spared Nakashima's life because he healed one of the warlord's injured pigeons.

Such scraps of information do lead one to believe that by any ordinary standards the isolation from the West was nearly complete. In terms of mathematics, it is extremely unlikely that anyone in Japan learned about the creation of modern calculus by Newton and Leibnitz later in the seventeenth century, and there is certainly no evidence from *sangaku* problems and traditional Japanese mathematics texts that its practitioners understood the fundamental theorem of calculus.

One should not conclude from this state of affairs that *sakoku* had entirely negative consequences. To the contrary, the policy was so successful

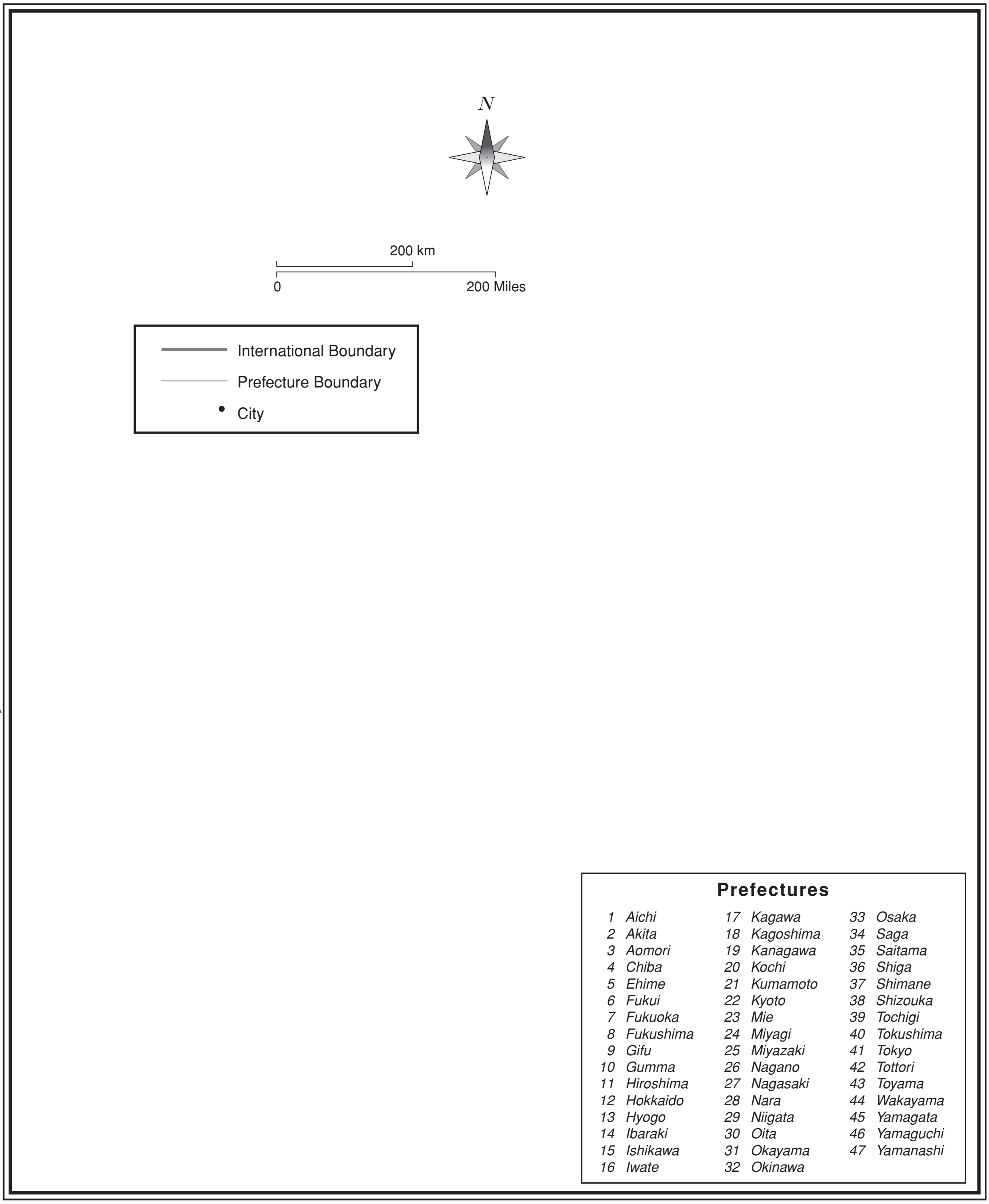

Plate 1.3. The map shows Japan's 47 administrative divisions, known as prefectures, which are roughly akin to states or provinces, at least insofar as that each has a popularly elected government and single-chamber parliament. Prefectures are further divided into cities (*shi*) and towns (*machi*). Because prefectures are usually named after the largest city within their borders, one often sees "Nagano city" to distinguish it from "Nagano prefecture." The map also indicates some of the more important cities mentioned in the text.

at eliminating foreign conflicts that the 250 years of the Edo period became known as the "Great Peace." Moreover, with the stability provided by the Tokugawa shogunate, Japanese culture experienced a brilliant flowering, so much so that the years of the late seventeenth century are referred to as *Genroku*, Renaissance. At the time a gentleman was expected to cultivate skills in "medicine, poetry, the tea ceremony, music, the hand drum, the noh dance, etiquette, the appreciation of craft work, arithmetic and calculation . . . not to mention literary composition, reading and writing. There are other things as well . . ."[1]

We do not have space here to delve into the riches of Genroku culture, but one should recognize that during this era many of the arts for which Japan is renowned attained their highest achievements: Noh dance flourished; the great dramatist Chikamatsu Monzaemon (1653–1725) produced plays for both the Kabuki and puppet theatres; tea ceremonies, flower arranging, and garden architecture were on the ascendant, as well as painting in several schools, including the ubiquitous *ukiyo-e*, or "floating world" prints that illuminated the demimonde of courtesans and erotic love and fairly defined the entire epoch. *Ukiyo-e* prints were made using wood blocks, not because the Japanese lacked movable type, which had been imported from Korea during Hideyoshi's day, but because printers preferred the calligraphic and artistic possibilities afforded by block printing. Poetry was not to be eclipsed, especially haiku, which achieved some of its greatest expression in the works of Matsuo Bashō (1644–1694), who long ago achieved worldwide renown.

What is strikingly absent in the standard reviews of Japanese cultural achievements of the period is any mention of science or mathematics. And yet the isolation that produced such a distinctive esthetic in the arts certainly had no less an impact on these fields. The stylistic form of the impact on geometry will gradually become apparent to readers who delve into the mathematical aspects of this book, but it isn't coincidental that many *sangaku* problems resemble origami designs, nor that the practice of hanging the tablets began precisely during the Genroku, for, as we will see shortly, it was in the mid-to-late seventeenth century that traditional Japanese mathematics began to flourish.

[1] See Conrad Totman, *Early Modern Japan* p. 186 ("For Further Reading, Chapter 1," p. 338).

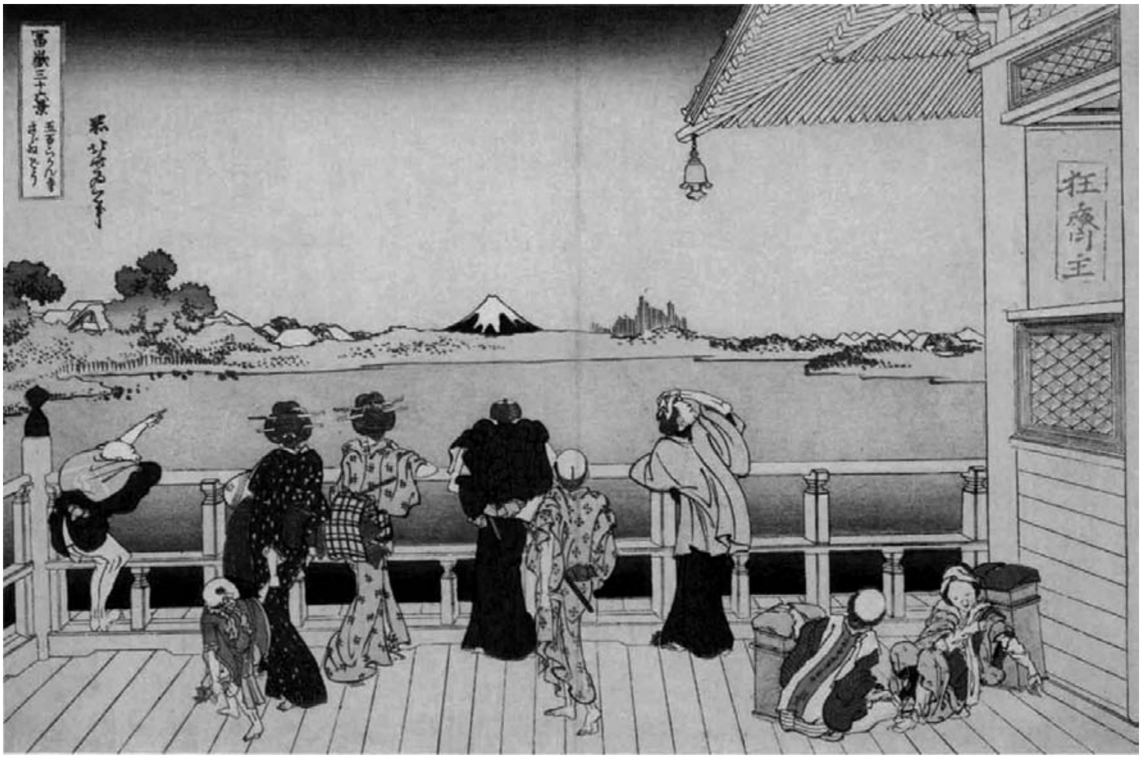

Plate 1.4. This ukiyo-e, or "floating world," print is from the series "Thirty-Six views of Mt. Fuji" by Katsushika Hokusai (1760–1849), one of the most famous artists of the Edo period. The print shows a distant view of Mt. Fuji from the Rakan-ji temple in Honjo. The original is in color. (© Nagoya TV-Japan.)

Regardless of the formal developments in mathematics at the time, Western readers invariably want to know how the strange custom of hanging tablets in shrines and temples arose. In the context of Japan, it was fairly natural. Shintoism, Japan's native religion, is populated by "eight hundred myriads of gods," the *kami*, whose spirits infuse everything from the sun and moon to rivers, mountains, and trees. For centuries before *sangaku* came into existence, worshippers would bring gifts to local shrines. The *kami*, it was said, love horses, but horses were expensive, and a worshipper who couldn't afford to offer a living one might present a likeness drawn on a piece wood instead. In fact, many tablets from the fifteenth century and earlier depict horses.

And so it would not have seemed extremely strange to the Japanese to hang a mathematical tablet in a temple. We cannot say exactly in what year, or even decade, the tradition began, but the oldest surviving *sangaku* has been found in Tochigi prefecture and dates from 1683, while the nineteenth-century mathematician Yamaguchi Kanzan, whose travel diary we excerpt in chapter 7, mentions an even older tablet dating from 1668; that one is now lost. Over the next two centuries the tablets appeared all over Japan, about two-thirds in Shinto shrines, one-third in Buddhist temples. We do not know how many were originally produced. From *sangaku* mentioned in contemporary mathematics texts, we are certain that at least 1,738 have been lost; moreover, only two percent of the tablets recorded in Yamaguchi's diary survive. So it is reasonable to guess that there were originally thousands more than the 900 tablets extant today. The practice of hanging *sangaku* gradually died out after the fall of the Tokugawa shogunate, but some devotees continued to post them as late as 1980, and *sangaku* continue to be discovered even now. In 2005, five tablets were found in the Toyama prefecture alone. The "newest" one was discovered by Mr. Hori Yoji at the Ubara shrine and dates from 1870. Two problems in chapter 4 are taken from the tablet and we present a photo of it in the color section, color plate 13.

Most *sangaku* contain only the final answer to a problem, rarely a detailed solution. (In *Sacred Mathematics* we usually give both answers and solutions, many drawn from traditional Japanese texts.) Apart from considerations of space, there seems to have been a certain bravado involved: Try this one if you dare! Nevertheless, as you will discover yourself from reading the inscriptions, the presenters of *sangaku* also took the spiritual, and even religious, aspect of the practice seriously, seeing nothing odd in offering a tablet to God in return for progress in mathematics. But just who were the creators of sacred mathematics? *Sangaku* are inscribed in a language called Kanbun, which used Chinese characters and essentially Chinese grammar, but included diacritical marks to indicate Japanese meaning. Kanbun played a role similar to Latin in the West and its use on *sangaku* would indicate that whoever set down the problems was highly educated. The majority of the presenters, in fact, seem to have been members of the samurai class. During the Edo period most samurai were not charging around the countryside, sword in hand, but worked as government functionaries; many became mathematicians, some famous ones. Nevertheless, the inscriptions on the tablets make clear that whole classes of students, children, and occasionally women dedicated *sangaku*. So the best answer to the question “Who created them?” seems to be “everybody.”

While contemplating this lesson, let us paint a fuller picture of the context in which *sangaku* were created by backing up as far as possible and briefly exploring the development of Japanese mathematics.

The Age of Arithmetic

The early history of Japan is inextricably bound up with that of China, from which it imported much of its culture, the Buddhist religion, as well as its system of government. This is true of Japanese mathematics as well; however, our knowledge about the state of mathematics in Japan prior to the eighth century is almost nonexistent. Perhaps the only definite piece of information from the earliest times is that the Japanese had some system of exponential notation that could be used for writing high powers of ten, similar to what Archimedes employed in the *Sand Reckoner.* Traditionally, the system was in place before the legendary Jimmu founded Japan in the seventh century B.C., but the date and the exact nature of the system are open to dispute.

More concrete information dates only from the onset of the Nara period (710–794), when a government was established at the city of Heijo, today’s Nara, near Osaka. By then the unification of Japan had been in progress for four hundred years. Buddhism arrived from China in the mid-sixth century and by the eighth century had becomes extremely powerful, as evidenced by the “Great Eastern Temple” Todaiji that was built at Nara in 752. At the opening of the eighth century, the Nara rulers established a university and prescribed nine Chinese mathematical texts, six of them from what became known as the *Ten Classics*. The most important of these would have been the *Jiu zhang Suanshu*, or *Nine Chapters on the Mathematical Art*. The “mathematical art” of the *Nine Chapters* and the other books is for the most part arithmetic and elementary algebra; in Japan they were introduced principally to aid in land surveying and tax collection. Although their full impact would not be evident for nearly one thousand years, the Chinese texts provided the foundation for all Japanese mathematics and their importance cannot be overstated. They also offer an illuminating window onto Chinese society of the time, and the reader

Plate 1.5. The Great Eastern temple of Todaiji was built in A.D. 752 in Nara, near Osaka. Today it is one of the most popular tourist destinations in Japan. (© Todaiji.)

can get a taste of them by sampling the problems from the most influential classics in Chapter 2.

One impact of the Chinese texts was felt as early as 718. In that year the government passed the law *yoryō ritsuryō*, literally "law of the yoryō age,"[2] by which it created the office of *San Hakase*, meaning approximately "Arithmetic Intelligence." The Department of Arithmetic Intelligence consisted of about 70 midlevel functionaries whose job was apparently to measure the size of fields and levy taxes. According to the law, the members of Arithmetic Intelligence were to learn only enough math from the Chinese books to calculate taxes, and so, although the Japanese became proficient in arithmetic operations, higher mathematics did not develop at that time.

Calculations of the period were performed by a precursor to the abacus that consisted of a set of small bamboo sticks known as *saunzi* in Chinese, *sangi* in Japanese. Certain configurations of the sticks represented numbers, not dissimilar to the simple strokes that represent roman numerals in

[2] *Yoryō* is a proper name that literally means "cherish aged people."

the West. A member of the Department of Arithmetic Intelligence, intending to calculate some taxes, would place the *sangi* on a ruled piece of paper that resembed a chessboard, and with a series of prescribed operations he could carry out addition, subtraction, multiplication, division and extraction of roots, very much in the spirit that Western students performed long division before the advent of the calculator. (See color plate 3 for a photo of a *sangi* set.)

At the time, Japan's two religions, the native Shintoism and the recently imported Buddhism, coexisted in relative peace. Buddhist temples in particular—as monasteries did during those centuries in the West—became repositories of learning. In chapter 7 you will have the opportunity to visit the major Shinto shrines Ise Jingū and Izumo Taisya with mathematician Yamaguchi Kanzan as he tours Japan collecting *sangaku* problems. Although they are not mentioned in the part of the diary we excerpt, he also visited the two great Buddhist complexes of Hōryūji and Todaiji; at the latter Japan's largest statue of the Buddha was constructed with the temple in 752. Next to the temple is a wooden storehouse where a number of historical documents pertaining to tax collection in the Nara period reside.

These documents, which include maps drawn and signed by the members of the Department of Arithmetic Intelligence, reveal some sophisticated bookkeeping, for instance a rather involved expense account for an inspection of Suruga province[3] in 738. The *San Hakase* staff consisted of two directors, nine subdirectors, six officers, ten clerks, and forty assistants. A first group, made up of one director, one officer, one clerk, and six helpers, inspects one village in twelve days. A second group, made up of a director, three subdirectors, three clerks, and twenty helpers, inspects seven villages and stays four days in each village. There are seven such groups, all of differing composition, and the total number of people involved is 1,330. Directors, subdirectors, and officers all get the same daily allocation of rice, salt, and sake, but clerks and helpers get less. It is a substantial arithmetic calculation to determine the total expenditure of rice, salt, and sake, but the Department of Arithmetic Intelligence got it exactly right.

In order to quiet the various power struggles that plagued Japan during the Nara period, in 794 the seat of government was moved to Heian-kyō, "the city of peace and tranquility"— present-day Kyoto. Heian-kyō remained

[3]Now Shizuoka prefecture.

the capital until 1192, and for that reason the period is known as the Heian era. During this relatively stable epoch, Japan began to develop a culture independent of China, and a writing system independent of Chinese. The most significant developments of the time were in literature: *The Tales of the Genji* by Lady Murasaki Shikibu is considered the world's first novel, and Sei Shonagon's diary of court life *The Pillow Book* has also achieved renown. Only a handful of names even tangentially connected with mathematics have come down to us, from the Nara period through to the seventeenth century, and there are almost no advances to report for nine hundred years. The Chinese texts written at the time may well have been imported into Japan but, as in the West, clergy were little interested in science and mathematics, and as far as mathematics goes it was very much a dark age.

From the Kamakura period (1192–1333), when the Minamoto shogunate established its government in Kamakura, far from Kyoto, there do survive a few literary references to *sangi*, which indicate that they were still used for arithmetic calculation. For instance, in Kamo no Chōmei book *Hosshinsyu* (*Stories about Buddhism*), which was written around 1241 and consisted of one hundred stories, there are two mentions of *sangi*. One is to count the number of repetitions of a Buddhist chant; in another story the author describes houses destroyed by a flood as "like *sangi*," because *sangi* are scattered on paper. In the anonymous *Uji Syui* (*Stories Edited by the Uji Minister*) from the beginning of the thirteenth century, one of the 197 humorous stories concerns a man who wants to learn how to use the *sangi*.

Such meager scraps lead us to conclude that *sangi* continued to be used for arithmetic calculation, but there as yet appear to have been no developments in higher mathematics. This state of affairs continued through the Muromachi period, from 1338 to 1573, which takes its name from the Muromachi area of Kyoto, where the Ashikaga family reestablished the government. During this era, the story goes, one could hardly find in all Japan a person versed in the art of division. Nevertheless, not only was this an age when Japan carried out extensive trade with Southeast Asia and rich merchants appeared, but it was also an age of burgeoning culture. At this time, contemporaneous to the Italian Renaissance, Kanami Kiyotsugu (1333–1384) invented Noh drama, while his son Zeami Motokiyo (1363–1443) brought it to the peak of its development. A few hundred years later, Sen no Rikyū perfected the tea ceremony, which concerns far more than the drinking of tea; even today, millions of Japanese study the ritual as a path toward perfecting Zen principles.

As in the West, the final decades of the sixteenth century in Japan were far from peaceful. The Ashikaga shogunate came to an end in 1573 when warlord Oda Nobunaga (1534–1582) drove the last Ashikaga shogun from Kyoto. The following decades saw Nobunaga's successor, Toyotomi Hideyoshi, with the help of Tokugawa Ieyasu, conquer one province after another, until by 1590 Japan was finally unified. As it happened, both Nobunaga and Hideyoshi were great patrons of the arts and helped set the stage for the cultural blossoming that was shortly to come. Toward the end of his life, however, Hideyoshi appears to have begun behaving in an erratic and dangerous fashion, in 1591 forcing his friend and tea master Sen no Rikyū to commit ritual suicide. Not satisfied with the unification of Japan, the following year he launched a massive invasion of Korea, which ultimately failed. It did, however, have profound consequences for Japanese mathematics.

One of Hideyoshi's soldiers at the port of Hakata, which the warlord had established as his staging ground for the invasion, had in his possession an abacus, *soroban* in Japanese, which evidently came from China. The soldier's *soroban* is in fact the oldest surviving abacus in Japan (see plate 1.6 on page 15).

Whether or not the soldier's was actually the first abacus to reach Japanese shores, Japan's thriving trade at that time with other Asian countries made the importation of the Chinese *suan phan*, literally "calculating plate," inevitable. We discuss the development of the *suan phan* in slightly more detail in the next chapter, but its appearance as the *soroban* around 1592 revolutionized Japanese mathematics; traditional Japanese mathematics can be said to have begun with introduction of the calculating plate, aided by the peace of the Tokugawa shogunate.

The Ascendence of Traditional Japanese Mathematics

The *soroban*'s advent in Japan also heralded the first record of an identifiable Japanese mathematician, Mōri Shigeyoshi,[4] who flourished around 1600. Little more is known about him, except that he lived in Osaka until the city was taken in 1615 by Tokugawa Ieyasu and thereafter lived in

[4] In the West usually referred to as Kambei Mōri.

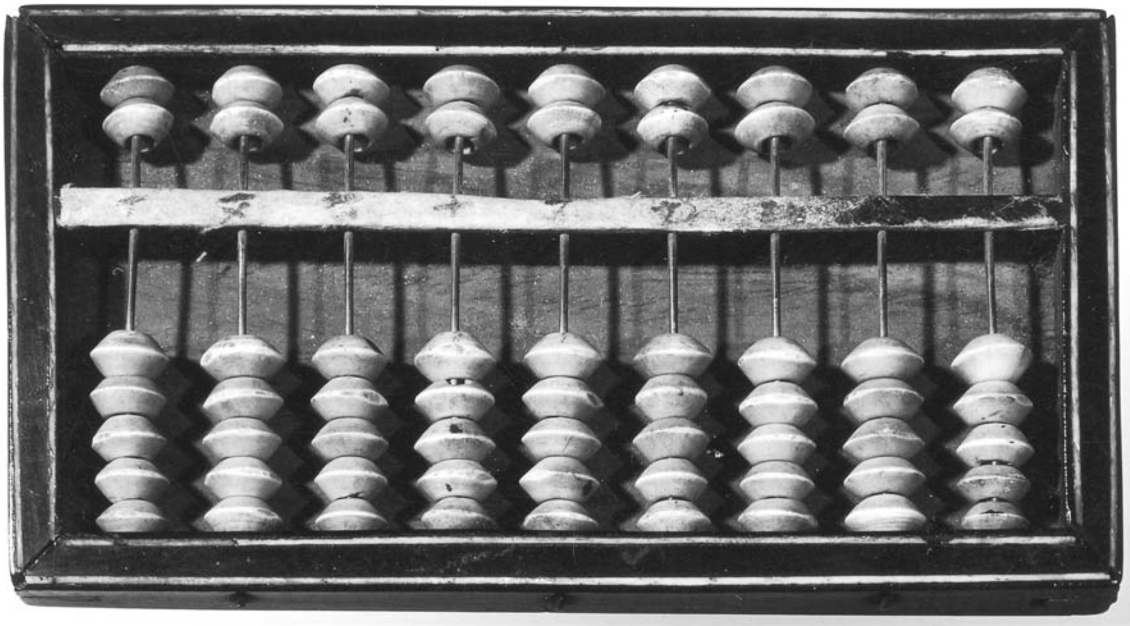

Plate 1.6. Here is the oldest known abacus, or *soroban*, in Japan. It dates from about A.D. 1592 and was in the possession of one of Hideyoshi's soldiers at the port of Hakata. (© Maeda Ikutokukai.)

Kyoto. There are stories, almost certainly untrue, that Mōri himself brought the *soroban* from China, but in any case he was an expert at its use and did more than anyone at the time to popularize numerical calculations. In 1622 Mori published a small primer *Warizansyo*, or *Division Using the Soroban*.[5]

Mōri himself was in possession of a Chinese book on the *soroban*, Cheng Da-Wei's famous *Suanfa Tong Zong*, or *Systematic Treatise on Arithmetic*, which was published in China in 1593 and made its way to Japan shortly afterward, in other words, at the same time as the *soroban* itself. Cheng's book (see Chapter 2) had a great influence on the course of Japanese mathematics independently of Mōri's work. Not only was a Japanese edition published in 1676 by Yuasa Ichirōzaemon (?–?), but already, in the 1620s, Yoshida Mitsuyoshi (1598–1672) studied the *Suanfa Tong Zong* closely, changing problems to suit Japan and adding many illustrations. Thus was born his

[5]The actual title of the booklet is uncertain because the title page is lost.

Jinkō-ki, or *Large and Small Numbers*, which appeared in 1627, becoming the first complete mathematics book published in Japan.

The title *Jinkō-ki* originated from an old religious book of the twelfth century, *Ryōjin Hishō*, or *Poems of Those Days*. Yoshida's *Jinkō-ki* mostly concerned computational algorithms for which we use a calculator today, such as the extraction of square and cube roots. The book achieved immediate and enduring popularity, running through about three hundred different versions over the next three centuries. There was the *New Jinkō-ki*, the *Treasure of Jinkō-ki*, the *The Concise Jinkō-ki*, the *The Riches of Jinkō-ki*. . . . Of course, most were ghost written, literally, but Yoshida did publish at least seven editions himself. In the 1641 version, he presented some open problems for readers to solve. When readers provided the answers, he published the next edition adding more open problems, and so on. In this way many Japanese mathematics books were published, the readers contributing their solutions.

One of the problems treated in the *Jinkō-ki* concerned the calculation of π. In response, mathematician Muramatsu Shigekiyo (1608–1695) published *Sanso*, or *Stack of Mathematics*, in which he uses a 2^{15}- or 32,768-sided polygon to obtain $\pi = 3.14195264877$. Nineteen hundred years earlier, Archimedes had arrived at his value of π by inscribing an n-sided polygon within a circle and approximating the circumference of the circle by the perimeter of the polygon. The more sides, the more accurate the approximation, and the better the value of π.[6] Muramatsu used essentially the same technique, as did his contemporary Isomura Yoshinori (1640?–1710), who employed a $2^{17} = 131,072$–sided polygon inscribed in a circle of radius 1 and obtained 3.141592664 for the perimeter. For some reason he wrote only $\pi = 3.1416$.

The most famous mathematician of the age, Seki Takakazu[7] (1640?–1708) also took up the challenge to calculate π. Using his own method, which was published posthumously by his disciples in the 1712 *Katsuyō Sanpō*, or *Collection of Important Mathematical Results*, he obtained $\pi = 3.14159265359$, which is correct to the eleven digits calculated. Seki's disciple Takebe Katahiro (1664–1739) succeeded in obtaining a value of π correct to 41 digits.

[6]More precisely, Archimedes bounded π by placing the circle between two polygons, knowing that the circumference of the circle was greater than the perimeter of the inscribed polygon and less than the perimeter of the circumscribed polygon. Employing 96-sided polygons, he achieved a numerical value of $3\ 10/71 < \pi < 3\ 1/7$.

[7]Usually known in the West as Kowa Seki.

Plate 1.7. From a 1715 edition of the *Jinkō-ki*, this woodblock print illustrates the advantages of using an abacus in business transactions. (Collection of Fukagawa Hidetoshi.)

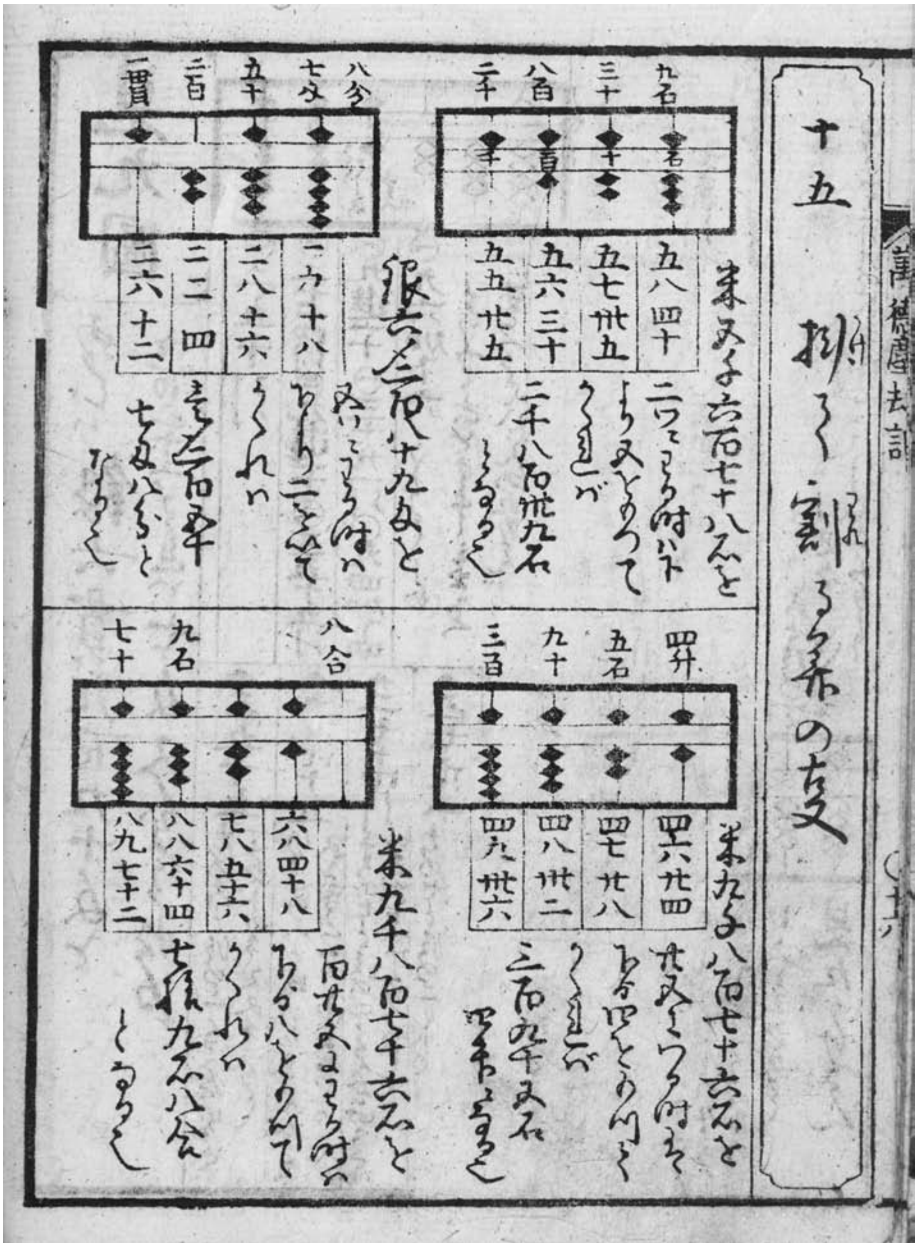

Plate 1.8. A *soroban* exercise from a later edition of the *Jinkō-ki*, c. 1818–1829. (Collection of Fukagawa Hidetoshi.)

Because of their importance, we devote much of chapter 3 to the *Jinkō-ki*, Seki, and the various calculations of π.

The start of traditional Japanese Mathematics is usually dated from the publication of the *Jinkō-ki* in 1627. The *Nine Chapters*, Cheng's *Treatise*, and the other Chinese classics continued to exert their influence on Japanese mathematics, either directly or through translations, but with the onset of *sakoku* the development of Japanese mathematics rapidly became independent of China. Strangely, though, the *Jinkō-ki* did not herald the immediate death of the *sangi*. The *soroban* did quickly replace *sangi* in everyday business calculations, but it was not so well suited for complex algebraic operations, in particular the solution of high-degree equations, of which Japanese mathematicians became very fond. As a result the *sangi* persisted side by side with the *soroban* well into the nineteenth century.

The samurai posed a major problem for the peace of the Edo era. Centuries of warfare had turned them into barely literate brutes who needed to be pacified and civilized. The Tokugawa expended much effort in this direction, with the result that within several generations the samurai were transformed into a highly educated class, literate and versed in the finer things of life, as a European noble of the time would be. Most of the warriors, having lost their jobs so to speak, became ordinary civil servants. For three or four days a week a samurai might go to the provincial castle to work, but the salary was so terrible that he often had to take on a side job.[8] During the Edo period there were no universities in Japan. Consequently, many samurai moonlighted as teachers in small private schools called *juku*, which were devoted to the three R's: reading, 'riting, and 'rithmetic, the last "r" standing for *soroban*. Whereas in previous ages samurai visited villages to levy soldiers, now their visits were less frequent, and farmers found they had to calculate the area of their fields by themselves. As a result, they also began to attend the *juku*, which was made possible by a low attendance fee. With people from every caste—from the rich to the poor, from samurai to farmers—going to school, *juku* began to flourish. The roster at one school, the Yōken *juku*, shows that 2,144 students, including many adults, attended it over the course of fifty years.

[8]The Tokugawa shogunate limited local warlords to one castle per domain and, with little fighting during the entire Edo period, previous military strongholds, "castle towns," became administrative centers. Mathematician Yamaguchi Kanzan speaks of them frequently in his diary, chapter 7.

Plate 1.9. The Yōken *juku*, where mathematician Sakuma Yōken (1819–1896) taught 2,144 students over a half-century. (Tamura city.)

Their teacher was mathematician Sakuma Yōken (1819–1896), and the small wooden school room still stands. A recent study[9] indicates that, by the nineteenth century, late in the Edo period, about 80,000 *juku* existed throughout Japan. Although, as in the West, children were considered laborers, not students, the home-grown schooling provided by the *juku* resulted in a literacy level that was high compared to other countries at the time.

Many mathematicians, usually samurai who had received "Master of Mathematics" licenses, visited these rural schools to teach mathematics—

[9]Ohishi Manabu, http:/library.u-gakugei.ac.jp (in Japanese).

and evidently more than simple ’rithmetic. For it was from this milieu, isolated from the Western world and increasingly divergent from China, that *wasan*, literally “Japanese mathematics,” arose. Ordinary people at the *juku*, who could not afford to publish their own books, took up the ancient custom of bringing votive tablets to temples and began to hang *sangaku*, in this way both making a religious offering and advertising their results. Incidently, they created wonderful art.

Sangaku were not the only medium for disseminating geometry. The Koreans had been printing books with movable type 100 years prior to Gutenberg,[10] and both Korean and European presses had found their way to Japan before the opening of the seventeenth century. As mentioned, however, printing by wooden blocks on rice paper became the favored method of producing books in Japan, and, by the end of the seventeenth century, a plethora of mathematics texts had begun to appear, many of which contained problems from or identical to those found on *sangaku*. Sometimes these texts provide solutions not written on the tablets, and we often quote from them in the solution sections of our own book. Many of the illustrations found throughout *Sacred Mathematics* also come directly from rice-paper books, originally printed in the seventeenth through nineteenth centuries.

Thus, by the end of the seventeenth century, *wasan*, traditional Japanese mathematics, was firmly established. It would be over the next two hundred years, however, that traditional methods produced their most striking and original results.

The Flowering and Decline of Traditional Japanese Mathematics

It was the eighteenth and nineteenth centuries, as Japan’s isolation deepened, that saw the greatest number of traditional Japanese mathematics texts published, the most interesting theorems proved, and most of the *sangaku* problems created.

The majority of results obtained by traditional Japanese mathematicians were not path breaking by Western standards, partly because Japan

[10]If, indeed, Gutenberg used movable type. See Tony Rothman, *Everything’s Relative and Other Fables from Science and Technology* (Wiley, Hoboken, N.J, 2003).

never developed a fully fledged theory of calculus. Traditional Japanese mathematicians found the areas and volumes of geometric figures in much the same way Eudoxus and Archimedes did in ancient Greece (and much as we do numerically today with computers). For example, one can divide up a circle into rectangular strips, as in plate 1.10. The narrower the strips, the more closely the area of the strips will approximate that of the circle. By letting the width of the strips go to zero, one can get an exact formula for the circle's area. This idea served as the basis of the *Enri* ("Yenri"), a vague term meaning "circle principle," but which amounts to what calculus students know as definite integration. And, as students know, there are any number of methods for computing the area of geometric figures by slicing up those figures in a convenient way and letting the width of the slices go to zero. One can do this informally for each situation, without proving the theorems about limits that students detest, but unless you have those theorems, in particular the first and second fundamental theorems of calculus, you are restricted to performing so-called "definite" integrals, and do not have a theory for doing integration in general, that is, for performing "indefinite" integration. This was more or less the situation in traditional Japanese mathematics. We discuss it more fully in chapter 9.

Despite such drawbacks, one cannot accuse the Japanese mathematicians of lacking ingenuity. Seki developed a theory of determinants before Leibnitz, and other Japanese geometers proved a handful of famous theorems prior to their Western counterparts, or at least independently of them. We'll encounter some of these in chapter 8. They include the Descartes circle theorem, the Malfatti problem, the Casey theorem, the Soddy hexlet, and a few others.

Additionally, the Japanese were extremely skilful at handling equations of high degree. We do mean high degree. In Yamaguchi's diary, we will run across a famous problem, the Gion shrine problem, which involves an equation of the 1,024th degree. (And students fear quadratics!) The mathematician Ajima Naonobu[11] (1732–1798) became famous for reducing the problem to one of the tenth degree, which was then solved numerically. Ajima proved a number of difficult theorems in geometry, which we discuss in chapter 3, and also came the closest of the Japanese mathematicians to producing a full theory of definite integration.

Ajima's work built on that of his predecessor Matsunaga Yoshisuke (1692?–1744), who studied infinite series and their applicability to the calculation of

[11] Sometimes known as Chokuyen Ajima.

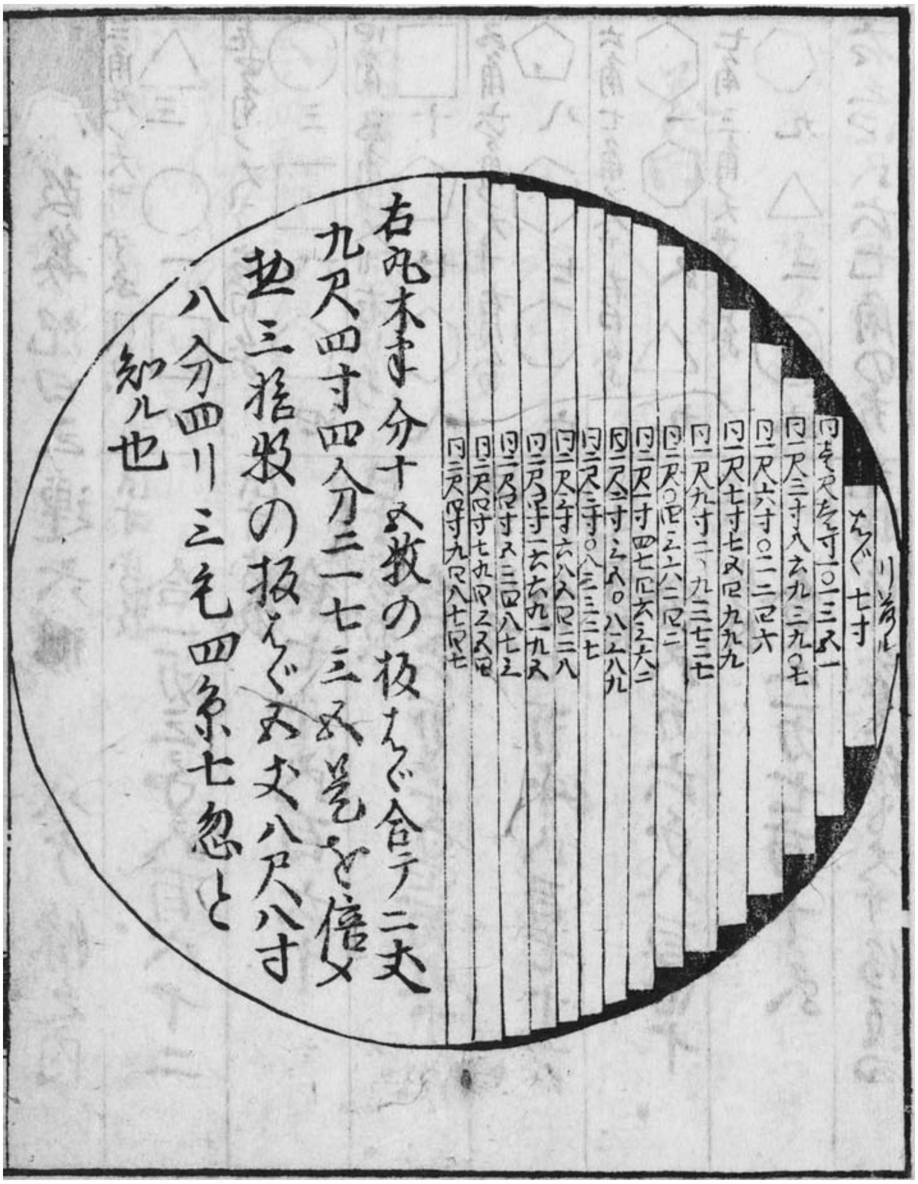

Plate 1.10. From Sawaguchi Kazuyuki's 1671 book *Kokon Sanpōki*, or *Old and New Mathematics*, this figure illustrates how to approximate the area of a circle by slicing it into rectangular strips. In a first-year calculus course one calculates the area of a circle in the same way—by slicing up the circle, and adding up the area of the strips in the limit that their width goes to zero. (Aichi University of Education.)

areas through integration. They were followed by Wada Yasushi (1787–1840), who lived in poverty and produced a number of tables of definite integrals. Uchida Kyō (1805–1882) studied integration in Wada's *juku*. He then produced a series of books that treat integration of solids, including areas formed by the intersection of cylinders, spheres, and so on. You will be challenged to try this sort of problem in chapters 5 and 6.

As we have said, many *sangaku* problems also appeared in traditional Japanese texts. Fujita Sadasuke (1734–1807) published a book *Seiyō Sanpō* (*Detailed Mathematics*), and his son, Fujita Kagen (1772–1828), carried on the tradition by publishing *Shinpeki Sanpō*, rendered sometimes as *Mathematics of Shrines and Temples* and sometimes as *Sacred Mathematics*, the first collection of *sangaku* problems. (See color plate 14 for a portrait of Fujita Kagen.) A few famous problems appearing in the present *Sacred Mathematics* have been taken from those works.

As the centuries progressed, a few hints of Occidental mathematics seeped in to Japan via China and the Dutch at Nagasaki. For example, the Japanese evidently learned of logarithms from a 1713 Chinese publication, the *Su-li Ching-Yin*. Nevertheless, even in 1824, a Japanese mathematician seemed surprised to learn of a drawing in a Dutch work that showed an ellipsograph—a mechanism for drawing an ellipse known in the West at least since Leonardo da Vinci. By the mid-nineteenth century one does find manuscripts containing both Eastern and Western notation.

But *wasan* held its ground until, as a direct consequence of the opening of Japan to the West by Perry, the Tokugawa family lost power in 1868. The new Meiji government decided that, in order for Japan to be an equal partner to foreign nations, it must rapidly modernize. Their program included mathematics. Governmental schools were established all over Japan and in the 1872 *Gakurei*, or "Fundamental Code of Education," the Meiji leaders decreed that "*wasan* was not to be taught at school, but Western mathematics only."

Due to the *juku*, mathematics had been flourishing in Japan and Western mathematics—*yosan*—proved easy to introduce and was quickly adopted. Of course, diehards fought back. One of the last samurai, Takaku Kenjirō (1821–1883), wrote, "Astronomy and the physical sciences as found in the West are truth eternal and unchangeable, and this we must learn; but as to mathematics, there Japan is leader of the world."[12] In the end, resistance was futile. With the Meiji Fundamental Code, teachers of traditional mathematics lost their jobs and *wasan* was destined to perish. From a strictly mathematical point of view, its death is perhaps not to be mourned overmuch, but from an esthetic point of view we surely lost something when lovers of traditional mathematics finally ceased creating their beautiful problems and the tablets that they offered to the world. We can only be grateful for what remains.

[12]Smith and Mikami, p. 273 ("For Further Reading, Chapter 1," p. 338).