1

Coalgebras, bialgebras and Hopf algebras. $U_q(b_+)$

Quantum groups today are like groups were in the nineteenth century, by which I mean

– a young theory, abundant examples, a rich and beautiful mathematical structure. By 'young' I mean that many problems remain wide open, for example the classification of finite-dimensional quantum groups.

– a clear need for something *like* this in the mathematical physics of the day. In our case it means quantum theory, which clearly suggests the need for some kind of 'quantum geometry', of which quantum groups would be the group objects.

These are *algebra* lectures, so we will not be able to say too much about physics. Suffice it to say that the familiar 'geometrical' picture for classical mechanics: symplectic structures, Riemannian geometry, is all thrown away when we look at quantum systems. In quantum systems the classical variables or 'coordinates' are replaced by operators on a Hilbert space and typically generate a noncommutative algebra, instead of a commutative coordinate ring as in the classical case. There is a need for geometrical structures on such quantum systems parallel to those in the classical case. This is needed if geometrical ideas such as gravity are ever to be unified with quantum theory.

From a mathematical point of view, the motivation for quantum groups is:

– the original (dim) origins in cohomology of groups (H. Hopf, 1947); an older name for quantum groups is 'Hopf algebras'

– q-deformed enveloping algebra quantum groups provide an explanation for the theory of q-special functions, which dates back to the 1900s. They are used also in number theory. (For example, there are q-exponentials etc., related to quantum groups as ordinary exponentials are related to the additive group $\mathbb{R}$.)

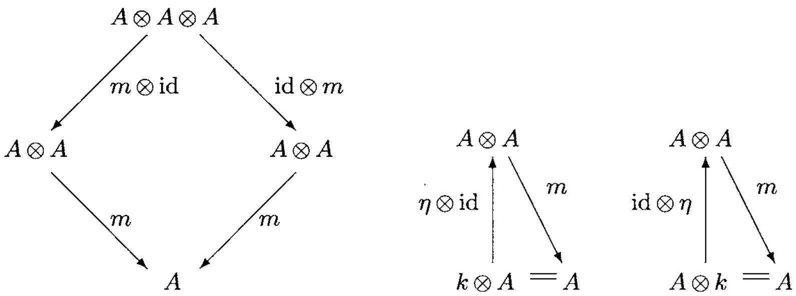

Fig. 1.1. Associativity and unit element expressed as commutative diagrams.

– representations of quantum groups form braided categories, leading to link invariants

– quantum groups are the 'group' objects in some kind of noncommutative algebraic geometry

– quantum groups are the 'transformation' objects in noncommutative algebraic geometry

– quantum groups restore an input–output symmetry to algebraic constructions; for example, they admit Fourier theory.

We fix a field k over which we work. We begin by recalling that an algebra A is

1. A vector space over k.

2. A map $m : A \otimes A \to A$ which is associative in the sense $(ab)c = a(bc)$ for all $a, b, c \in A$. Here $ab = m(a \otimes b)$ is shorthand.

3. A unit element 1_A, which we write equivalently as a map $\eta : k \to A$ by $\eta(1) = 1_A$. We require $a1_A = a = 1_Aa$ for all $a \in A$.

In terms of the maps, these axioms are given by the commutative diagrams in Figure 1.1. Note that most algebraic constructions can, like the axioms themselves, be expressed as commuting diagrams. When all premises, statements and proofs of a theorem are written out like this then reversing all arrows will also yield the premises, statements and proofs of a different theorem, called the 'dual theorem'.

Definition 1.1 *A coalgebra C is*

1. A vector space over k.

2. A map $\Delta : C \to C \otimes C$ (the 'coproduct') which is coassociative in

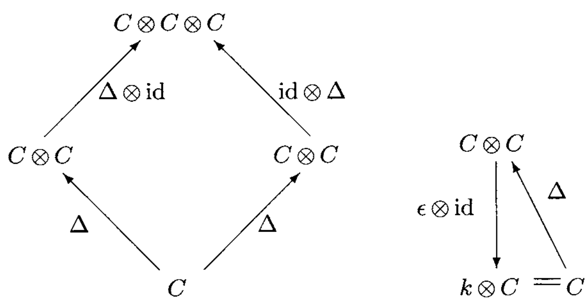

Fig. 1.2. Coassociativity and counit element expressed as commutative diagrams.

the sense

$$\sum c_{(1)(1)} \otimes c_{(1)(2)} \otimes c_{(2)} = \sum c_{(1)} \otimes c_{(2)(1)} \otimes c_{(2)(2)}$$

for all $c \in C$. *Here* $\Delta c \equiv \sum c_{(1)} \otimes c_{(2)}$ *is shorthand.*

3. A map $\epsilon : C \to k$ *(the 'counit') obeying* $\sum \epsilon(c_{(1)})c_{(2)} = c = \sum c_{(1)}\epsilon(c_{(2)})$ *for all* $c \in C$.

In terms of the maps, these axioms are given by the commutative diagrams in Figure 1.2, which is just Figure 1.1 with all arrows reversed.

This notion of reversing arrows has the same status as the idea, familiar in algebra, of having both left and right module versions of a construction. The theory with only left modules is equivalent to the theory with right modules, by a left–right reflection (i.e. reversal of tensor product). But one can also consider theorems with both left and right modules interacting in some way, e.g. bimodules. Similarly, the arrow-reversal operation transforms theorems about algebras to theorems about coalgebras. However, we can also consider theorems involving both concepts. In this way, quantum group theory is a very natural 'completion' of algebra to a setting which is invariant under the arrow-reversal operation.

Definition 1.2 *A bialgebra* H *is*

1. An algebra H, m, η.

2. A coalgebra H, Δ, ϵ.

3. Δ, ϵ *are algebra maps, where* $H \otimes H$ *has the tensor product algebra structure* $(h \otimes g)(h' \otimes g') = hh' \otimes gg'$ *for all* $h, h', g, g' \in H$.

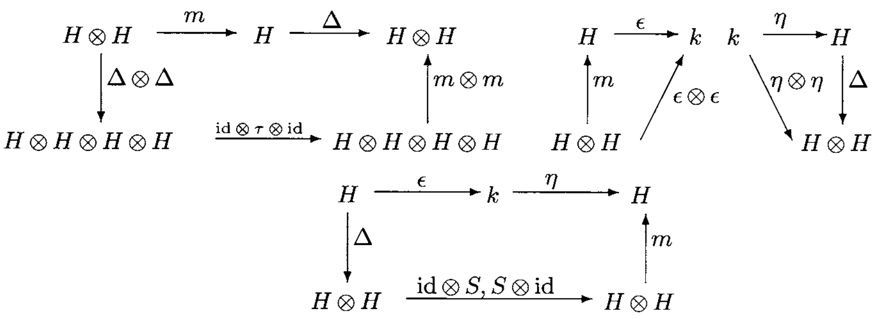

Fig. 1.3. Additional axioms that make the algebra and coalgebra H into a Hopf algebra.

Actually, a bialgebra is more like a quantum 'semigroup'. We need something playing the role of group inversion:

Definition 1.3 *A Hopf algebra H is*

1. A bialgebra $H, \Delta, \epsilon, m, \eta$.

2. A map $S : H \to H$ (the 'antipode') such that $\sum(Sh_{(1)})h_{(2)} = \epsilon(h) = \sum h_{(1)}Sh_{(2)}$ for all $h \in H$.

The axioms that make a simultaneous algebra and coalgebra into a Hopf algebra are shown in Figure 1.3, where $\tau : H \otimes H \to H \otimes H$ is the 'flip' map $\tau(h \otimes g) = g \otimes h$ for all $h, g \in H$.

Proposition 1.4 *(Antihomomorphism property of antipodes). The antipode of a Hopf algebra is unique and obeys $S(hg) = S(g)S(h)$, $S(1) = 1$ (i.e. S is an* antialgebra map*) and $(S \otimes S) \circ \Delta h = \tau \circ \Delta \circ Sh$, $\epsilon Sh = \epsilon h$ (i.e. S is an* anticoalgebra map*), for all $h, g \in H$.*

Proof During proofs, we will usually omit the $\sum$ signs, which should be understood. If S, S_1 are two antipodes on a bialgebra H then they are equal because $S_1 h = (S_1 h_{(1)})\epsilon(h_{(2)}) = (S_1 h_{(1)})h_{(2)(1)}Sh_{(2)(2)} = (S_1 h_{(1)(1)})h_{(1)(2)}Sh_{(2)} = \epsilon(h_{(1)})Sh_{(2)} = Sh$. Here we wrote $h = h_{(1)}\epsilon(h_{(2)})$ by the counit axioms, and then inserted $h_{(2)(1)}Sh_{(2)(2)}$ knowing that it would collapse to $\epsilon(h_{(2)})$. We then used associativity and (the more novel ingredient) coassociativity to be able to collapse $(S_1 h_{(1)(1)})h_{(1)(2)}$ to $\epsilon(h_{(1)})$. Note that the proof is not any harder than the usual one for uniqueness of group inverses, the only complication being that we are working now with parts of linear combinations and have to take care to keep the order of the coproducts. We can similarly collapse such expressions as $(S_1h_{(1)})h_{(2)}$ or $h_{(2)}Sh_{(3)}$ wherever they occur as long as the two collapsing factors are in linear order. This is just the analogue of cancelling $h^{-1}h$ or hh^{-1} in a group. Armed with such techniques, we return now to the proof of the proposition. Consider the identity

$$\begin{aligned}&(S(h_{(1)(1)}g_{(1)(1)}))h_{(1)(2)}g_{(1)(2)}\otimes g_{(2)}\otimes h_{(2)}\\ &\quad=(S((h_{(1)}g_{(1)})_{(1)}))(h_{(1)}g_{(1)})_{(2)}\otimes g_{(2)}\otimes h_{(2)}\\ &\quad=\epsilon(h_{(1)}g_{(1)})1\otimes g_{(2)}\otimes h_{(2)}=1\otimes g\otimes h.\end{aligned}$$

We used that Δ is an algebra homomorphism, then the antipode axiom applied to $h_{(1)}g_{(1)}$. Then we used the counity axiom. Now apply S to the middle factor of both sides and multiply the first two factors. One has the identity

$$\begin{aligned}Sg\otimes h&=(S(h_{(1)(1)}g_{(1)(1)}))h_{(1)(2)}g_{(1)(2)}Sg_{(2)}\otimes h_{(2)}\\ &=(S(h_{(1)(1)}g_{(1)}))h_{(1)(2)}g_{(2)(1)}Sg_{(2)(2)}\otimes h_{(2)}=(S(h_{(1)(1)}g))h_{(1)(2)}\otimes h_{(2)},\end{aligned}$$

where we used coassociativity applied to g. We then use the antipode axiom applied to $g_{(2)}$, and the counity axiom. We now apply S to the second factor and multiply up, to give

$$(Sg)(Sh)=(S(h_{(1)(1)}g))h_{(1)(2)}Sh_{(2)}=(S(h_{(1)}g))h_{(2)(1)}Sh_{(2)(2)}=S(hg).$$

We used coassociativity applied to h, followed by the antipode axioms applied to $h_{(2)}$ and the counity axiom. □

Example 1.5 *The Hopf algebra $H=U_q(b_+)$ is generated by 1 and the elements X,g,g^{-1} with relations*

$$gg^{-1}=1=g^{-1}g,\quad gX=qXg,$$

where q is a fixed invertible element of the field k. Here

$$\Delta X=X\otimes 1+g\otimes X,\quad \Delta g=g\otimes g,\quad \Delta g^{-1}=g^{-1}\otimes g^{-1},$$

$$\epsilon X=0,\quad \epsilon g=1=\epsilon g^{-1},\quad SX=-g^{-1}X,\quad Sg=g^{-1},\quad Sg^{-1}=g.$$

Note that $S^2\neq \mathrm{id}$ in this example (because $S^2X=q^{-1}X$).

Proof We have Δ,ϵ on the generators and extend them multiplicatively to products of the generators (so that they are necessarily algebra maps as required). However, we have to check that this is consistent with the relations in the algebra. For example, $\Delta gX = (\Delta g)(\Delta X) = (g\otimes g)(X\otimes 1 + g\otimes X) = gX\otimes g + g^2\otimes gX$, while equal to this must be $\Delta qXg = q(\Delta X)(\Delta g) = q(X\otimes 1 + g\otimes X)(g\otimes g) = qXg\otimes g + qg^2\otimes Xg$. These expressions are equal, using again the relations in the algebra as stated. Similarly for the other relations. For the antipode, we keep in mind the preceding proposition and extend S as an antialgebra map, and check that this is consistent in the same way. Since S obeys the antipode axioms on the generators (an easy computation), it follows that it obeys them also on the products since Δ, ϵ are already extended multiplicatively. □

It is a nice exercise – we will prove it later in the course, but some readers may want to have fun doing it now – to show that

$$\Delta X^m = \sum_{r=0}^{m} \begin{bmatrix} m \\ r \end{bmatrix}_q X^{m-r} g^r \otimes X^r$$

where

$$\begin{bmatrix} m \\ r \end{bmatrix}_q = \frac{[m]_q!}{[r]_q![m-r]_q!}, \qquad [r]_q! = [r]_q[r-1]_q \cdots [1]_q$$

are the q-binomial coefficients defined in terms of 'q-integers'

$$[r]_q = 1 + q + \cdots + q^{r-1} = \frac{1-q^r}{1-q}.$$

The last expression here should be used only when $q \neq 1$, of course. We should also assume $[r]_q$ are invertible to write the q-binomial coefficients in this way.

Example 1.6 *Let G be a finite group. The* group Hopf algebra *kG is the vector space with basis G, and the algebra structure, unit, coproduct, counit and antipode*

$$\text{product in } G, \quad 1 = e, \quad \Delta g = g\otimes g, \quad \epsilon g = 1, \qquad Sg = g^{-1}$$

on the basis elements $g \in G$ (extended by linearity to all of kG).

Proof The multiplication is clearly associative because the group multiplication is. The coproduct is coassociative because it is so on each of the basis elements $g \in G$. It is an algebra homomorphism because $\Delta(gh) = gh\otimes gh = (g\otimes g)(h\otimes h) = (\Delta g)(\Delta h)$. The other facts are equally easy. G does not actually need to be finite for this construction, but we will be interested in the finite case. □

So all of finite group theory should, in principle, be a special case of Hopf algebra theory. The same is true for Lie theory, if we use the enveloping algebra. We recall that a Lie algebra is:

1. A vector space $\mathfrak{g}$.
2. A map $[\ ,\ ]:\mathfrak{g}\otimes\mathfrak{g}\to\mathfrak{g}$ obeying the Jacobi identity and antisymmetry axioms (when the characteristic of k is not 2).

Example 1.7 *Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over k. The* universal enveloping Hopf algebra *$U(\mathfrak{g})$ is the noncommutative algebra generated by 1 and elements of a basis of $\mathfrak{g}$ modulo the relations $[\xi,\eta]=\xi\eta-\eta\xi$ for all ξ,η in the basis. The coproduct, counit and antipode are*

$$\Delta\xi=\xi\otimes 1+1\otimes\xi,\quad \epsilon\xi=0,\quad S\xi=-\xi$$

extended in the case of Δ,ϵ as algebra maps, and in the case of S as an antialgebra map.

Proof We extend Δ,ϵ as algebra homomorphisms and S as an antialgebra homomorphism, and have to check that this extension is consistent with the relations. For example, $\Delta(\xi\eta)=(\xi\otimes 1+1\otimes\xi)(\eta\otimes 1+1\otimes\eta)=\xi\eta\otimes 1+1\otimes\xi\eta+\xi\otimes\eta+\eta\otimes\xi$. Subtracting from this the corresponding expression for $\Delta\eta\xi$ and using the relations, we obtain $[\xi,\eta]\otimes 1+1\otimes[\xi,\eta]=\Delta[\xi,\eta]$ as required. Similarly for the counit and antipode. □

One can say, informally, that $U(\mathfrak{g})$ is generated by 1 and elements of $\mathfrak{g}$ with the relations stated; it does not depend on a choice of basis. A more formal way to say this is to construct first the tensor Hopf algebra $T(V)=k\oplus V\oplus V\otimes V\oplus V\otimes V\otimes V\oplus\cdots$ on any vector space V. The product here is $(v\otimes\cdots\otimes w)(x\otimes\cdots\otimes y)=(v\otimes\cdots\otimes w\otimes x\otimes\cdots\otimes y)$. This forms a Hopf algebra with

$$\Delta v=v\otimes 1+1\otimes v,\quad \epsilon v=0,\quad Sv=-v$$

for all $v\in V$. The enveloping algebra $U(\mathfrak{g})$ is the quotient of $T(\mathfrak{g})$ modulo the ideal generated by the relations $\xi\otimes\eta-\eta\otimes\xi=[\xi,\eta]$. (Of course, the best definition is as a universal object, but we will not need that.)

So Lie theory is also contained, in principle, as a special case of quantum group theory. In fact, one of the main motivations for Hopf algebras in the 1960s was precisely as a tool that unifies the treatment of results for groups and Lie algebras into one technology, e.g. their cohomology theory. Clearly, our example $U_q(b_+)$ is a mixture of these two kinds of 'classical' Hopf algebras. It has an element g which is *grouplike* in the sense that it obeys $\Delta g = g \otimes g$. And it has an element X which is a bit like the Lie case. But it is neither a group algebra nor an enveloping algebra exactly. What characterises these classical objects, in contrast to $U_q(b_+)$, is:

Definition 1.8 *A Hopf algebra is commutative if it is commutative as an algebra. It is 'cocommutative' if it is cocommutative as a coalgebra, i.e. if $\tau \circ \Delta = \Delta$. This is the arrows-reversed version of commutativity.*

Corollary 1.9 *If H is a commutative or cocommutative Hopf algebra, then* $S^2 = \mathrm{id}$.

Proof We use Proposition 1.4, so that $S^2 h = (S^2 h_{(1)})(Sh_{(2)})h_{(3)} = (S(h_{(1)}Sh_{(2)}))\, h_{(3)} = h$ in the cocommutative case. Here we use a neutral notation $h_{(1)} \otimes h_{(2)} \otimes h_{(3)} \equiv h_{(1)(1)} \otimes h_{(1)(2)} \otimes h_{(2)} = h_{(1)} \otimes h_{(2)(1)} \otimes h_{(2)(2)}$ (just as one writes $abc \equiv (ab)c = a(bc)$). The other case is similar. □

Clearly, kG and $U(\mathfrak{g})$ are cocommutative. The coordinate rings of linear algebraic groups are likewise commutative Hopf algebras, while $U_q(b_+)$ is neither. As a tentative definition, we can say that a truly 'quantum' group (in contrast to a classical group or Lie object viewed as one) is a noncommutative and noncocommutative Hopf algebra. Later on, we will add further properties as well.

2

Dual pairing. $SL_q(2)$. Actions

In the last lecture we showed how to view finite groups and Lie algebras as Hopf algebras, and gave a variant that was truly 'quantum'. We now complete our basic collection of examples with some other classical objects.

Example 2.1 *Let G be a finite group with identity e. The* group function Hopf algebra *$k(G)$ is the algebra of functions on G with values in k and the pointwise product $(fg)(x) = f(x)g(x)$ for all $x \in G$ and $f, g \in k(G)$. The coproduct, counit and antipode are*

$$(\Delta f)(x,y) = f(xy), \quad \epsilon f = f(e), \quad (Sf)(x) = f(x^{-1}),$$

where we identify $k(G) \otimes k(G) = k(G \times G)$ (functions of two group variables).

Proof Coassociativity is evidently $((\Delta \otimes \mathrm{id})\Delta f)(x,y,z) = (\Delta f)(xy,z) = f((xy)z) = f(x(yz)) = (\Delta f)(x,yz) = ((\mathrm{id} \otimes \Delta)\Delta f)(x,y,z)$. Note that it comes directly from associativity in the group. Likewise, the counity and antipode axioms come directly from the group axioms for the unit element and inverse. □

Also, when g is a finite-dimensional complex semisimple Lie algebra (as classified by Dynkin diagrams), it has an associated complex Lie group $G \subset M_n(\mathbb{C})$ (the $n \times n$ matrices with values in $\mathbb{C}$). This subset is of the form $G = \{x \in M_n \mid p(x) = 0\}$, where p is a collection of polynomial equations. Correspondingly, we have an algebraic variety with coordinate algebra $\mathbb{C}[G]$ defined as $\mathbb{C}[x^i{}_j]$ where $i, j = 1, \dots, n$ (polynomials in n^2 variables), modulo the ideal generated by the relations $p(x) = 0$. The group structure inherited from matrix multiplication corresponds to a coproduct and counit

$$\Delta x^i{}_j = \sum_k x^i{}_k \otimes x^k{}_j, \quad \epsilon(x^i{}_j) = \delta^i{}_j$$

where $\delta^i{}_j$ is the Kronecker delta-function. There is also an antipode given algebraically via a matrix of cofactors of the matrix $x^i{}_j$ of generators. In this way, we have a complex linear algebraic group with coordinate algebra $\mathbb{C}[G]$ as a Hopf algebra. In fact, G can be taken so that the coefficients of $p(x)$ are integers (from work of Chevalley) giving a coordinate ring $\mathbb{Z}[G]$. Then, by tensoring with k, the same construction works over any field and provides a Hopf algebra $k[G]$ (and considering all k, one has an affine group scheme).

Example 2.2 *The Hopf algebra $k[SL_2]$ is $k[a, b, c, d]$ modulo the relation*

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1.$$

The coproduct, counit and antipode are

$$\Delta a = a \otimes a + b \otimes c, \quad \Delta b = b \otimes d + a \otimes b, \quad \Delta c = c \otimes a + d \otimes c,$$

$$\Delta d = d \otimes d + c \otimes b, \quad \epsilon(a) = \epsilon(d) = 1, \quad \epsilon(b) = \epsilon(c) = 0,$$

$$Sa = d, \quad Sd = a, \quad Sb = c, \quad Sc = b.$$

The coalgebra and antipode here can be written more concisely as

$$\Delta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \otimes \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \epsilon \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

$$S \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix},$$

where matrix multiplication should be understood in this definition of Δ. This is no more than a shorthand notation. Finally, for a truly 'quantum' variant of this:

Example 2.3 *Let $q \in k^*$. The Hopf algebra $SL_q(2)$ is $k\langle a, b, c, d\rangle$ (the free associative algebra) modulo the ideal generated by the six 'q-*