Chapter 2
Muonium–Antimuonium Oscillation in a Modified Standard Model

2.1 Neutrino Masses and Mixings

We modify the Standard Model only by the inclusion of singlet right-handed neutrinos and allowing for general renormalizable interactions. This implies non-zero neutrino masses and mixing matrix elements. The leptonic Yukawa interactions with the Higgs scalar doublet in the modified Standard Model take the form

$$\mathcal{L}_{\text{int}}^{\phi} = -\frac{g}{\sqrt{2}M_W}\phi^{-}\left(\sum_{a,b=1}^{3}\overline{\ell_{Ra}^{(0)}}m_{ab}^{\ell}\nu_{Lb}^{(0)} - \overline{\ell_{La}^{(0)}}m_{ab}^{D\,\dagger}\nu_{Rb}^{(0)}\right) + H.C. \tag{2.1}$$

Here, $\ell_{La}^{(0)}$ and $\nu_{La}^{(0)}$ are, respectively, the charged lepton and its associated neutrino partner of the $SU(2)_L$ doublet, while $\nu_{Ra}^{(0)}$ is the right-handed neutrino singlet. The superscript zero indicates weak interaction eigenstates so that the leptonic charged current interaction is

$$\mathcal{L}_{\text{int}}^{W} = -\frac{g}{\sqrt{2}}W^{-\mu}\sum_{a=1}^{3}\overline{\ell_{La}^{(0)}}\gamma_{\mu}\nu_{La}^{(0)} + H.C. \tag{2.2}$$

After spontaneous symmetry breaking, the mass term for the charged leptons takes the form

$$\mathcal{L}_{\text{mass}}^{\ell} = -\sum_{a,b=1}^{3}\left[\overline{\ell_{Ra}^{(0)}}m_{ab}^{\ell}\ell_{Lb}^{(0)} + \overline{\ell_{La}^{(0)}}m_{ba}^{\ell *}\ell_{Rb}^{(0)}\right] \tag{2.3}$$

where m^{ℓ} is a 3×3 mass matrix. To diagonalize this matrix, one performs the biunitary transformation

$$m^{\ell} = A^{R}m_{\text{diag}}^{\ell}(A^{L})^{\dagger} \tag{2.4}$$

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where A^R and A^L are 3×3 unitary matrices and m^ℓ_{diag} is a diagonal matrix, the entries of which are the charged lepton masses. To implement this basis change, the charged lepton fields participating in the weak interaction are rewritten in terms of the mass diagonal fields as

$$\ell^{(0)}_{La} = \sum_{b=1}^{3} A^L_{ab}\ell_{Lb}, \quad \ell^{(0)}_{Ra} = \sum_{b=1}^{3} A^R_{ab}\ell_{Rb} \tag{2.5}$$

On doing so, the mass term reads

$$\mathcal{L}^\ell_{\mathrm{mass}} = -\sum_{a=1}^{3} m_{\ell a}\left[\overline{\ell_{Ra}}\ell_{Lb} + \overline{\ell_{La}}\ell_{Rb}\right] \tag{2.6}$$

A general neutrino mass term resulting from renormalizable interactions takes the form

$$\mathcal{L}^\nu_{\mathrm{mass}} = -\frac{1}{2}\overline{\left(\left(\nu^{(0)}_L\right)^c \nu^{(0)}_R\right)}\begin{pmatrix} 0 & \left(m^D\right)^T \\ m^D & m^R \end{pmatrix}\begin{pmatrix} \nu^{(0)}_L \\ \left(\nu^{(0)}_R\right)^c \end{pmatrix} + H.C. \tag{2.7}$$

Note that the upper left 3×3 block in the neutrino mass matrix is set to zero. This block matrix involves only left-handed neutrinos and in the (modified) Standard Model its generation requires a nonrenormalizable mass dimension-five operator. Consequently, such a term will be ignored.

For three generations of neutrinos, the six mass eigenvalues, $m_{\nu A}$, are obtained from the diagonalization of the 6×6 matrix

$$M^\nu = \begin{pmatrix} 0 & \left(m^D\right)^T \\ m^D & m^R \end{pmatrix} \tag{2.8}$$

Since M^ν is symmetric, it can be diagonalized by a single unitary 6×6 matrix, U, as

$$M^\nu_{\mathrm{diag}} = U^T M^\nu U. \tag{2.9}$$

This diagonalization is implemented via the basis change on the original neutrino fields organized as the six-dimensional column vector

$$N^{(0)}_L = \begin{pmatrix} \nu^{(0)}_L \\ \left(\nu^{(0)}_R\right)^c \end{pmatrix}, \quad N^{(0)}_R = \begin{pmatrix} \left(\nu^{(0)}_L\right)^c \\ \nu^{(0)}_R \end{pmatrix} \tag{2.10}$$

to the new neutrino fields defined as

$$N^{(0)}_L = UN_L, \quad N^{(0)}_R = U^* N_R \tag{2.11}$$

where

$$N_L = \begin{pmatrix} \nu_L \\ (\nu_R)^c \end{pmatrix}, \quad N_R = \begin{pmatrix} (\nu_L)^c \\ \nu_R \end{pmatrix}. \tag{2.12}$$

The neutrino mass term then takes the form

$$\mathcal{L}^{\nu}_{\text{mass}} = -\frac{1}{2}\sum_{A=1}^{6} m_{\nu A}\left[\nu_A^T C \nu_A + \overline{\nu_A} C \overline{\nu_A^T}\right] = -\sum_{A=1}^{6} m_{\nu A}\overline{\nu_A}\nu_A, \tag{2.13}$$

where $m_{\nu A}$ are the Majorana neutrino masses.

Since a nonzero Majorana mass matrix m^R does not require $SU(2)_L \times U(1)$ symmetry breaking, it is naturally characterized by a much larger scale, M_R, than the elements of the matrix m^D the nontrivial values of which require $SU(2)_L \times U(1)$ symmetry breaking and are thus expected to be somewhere in the order of the charged lepton mass to the W mass. Thus, one can take the elements of m^D, characterized by a scale m_D, to be much less than M_R, the scale of the elements of m^R. One then finds on diagonalization of the 6×6 neutrino mass matrix that three of the eigenvalues are crudely given by

$$m_{\nu a} \sim \frac{m_D^2}{M_R} \ll m_D, \quad a = 1, 2, 3, \tag{2.14}$$

while the other three eigenvalues are roughly

$$m_{\nu i} \sim M_R, \quad i = 4, 5, 6. \tag{2.15}$$

This constitutes the so-called see-saw mechanism (Minkowski 1977; Yanagida 1979; Gell-Mann et al. 1979; Glashow 1979; Mohapatra and Senjanovic 1980) and provides a natural explanation of the smallness of the three light neutrino masses. Moreover, the elements of the mixing matrix are characterized by an M_R mass dependence

$$\begin{aligned} &U_{ab} \sim \mathcal{O}(1), \quad a, b = 1, 2, 3 \\ &U_{ij} \sim \mathcal{O}(1), \quad i, j = 4, 5, 6 \\ &U_{ia} \sim U_{ai} \sim \mathcal{O}\left(\frac{m_D}{M_R}\right), \quad a = 1, 2, 3,\ i = 4, 5, 6. \end{aligned} \tag{2.16}$$

Since the charged lepton mixing matrix is independent of M_R, one finds that elements of the mixing matrix appearing in the charged current have the M_R mass dependence

$$\begin{aligned} &V_{ab} \sim \mathcal{O}(1), \quad a, b = 1, 2, 3 \\ &V_{ai} \sim \mathcal{O}\left(\frac{m_D}{M_R}\right), \quad a = 1, 2, 3,\ i = 4, 5, 6 \end{aligned} \tag{2.17}$$

Inserting the transformations (2.5) and (2.11) in the interaction terms (2.1) and (2.2), and taking into account the mass matrix transformations (2.4) and (2.9), one obtains the explicit interactions of charged bosons with the leptons in their mass diagonal basis as

$$\mathcal{L}_{\text{int}}^{W} = -\frac{g}{\sqrt{2}} W^{-\mu} \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\ell}_{La} \gamma_{\mu} V_{aA} \nu_A - \frac{g}{\sqrt{2}} W^{+\mu} \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\nu}_A V_{aA}^{*} \gamma_{\mu} \ell_{La} \tag{2.18}$$

$$\begin{aligned} \mathcal{L}_{\text{int}}^{\phi} = & -\frac{g}{\sqrt{2} M_W} \phi^{-} \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\ell}_a V_{aA} \left(m_{la} \frac{1-\gamma_5}{2} - m_{\nu A} \frac{1+\gamma_5}{2} \right) \nu_A \\ & -\frac{g}{\sqrt{2} M_W} \phi^{+} \sum_{a=1}^{3} \sum_{A=1}^{6} \bar{\nu}_A V_{aA}^{*} \left(m_{la} \frac{1+\gamma_5}{2} - m_{\nu A} \frac{1-\gamma_5}{2} \right) \ell_a \end{aligned} \tag{2.19}$$

where

$$V_{aA} = \sum_{b=1}^{3} (A_L^{-1})_{ab} U_{bA} \tag{2.20}$$

Note that the mixing matrix V_{aA} satisfies the identities (Ilakovac and Pilaftsis 1995):

$$\sum_{A=1}^{6} V_{aA} V_{bA}^{*} = \delta_{ab} \tag{2.21}$$

$$\sum_{A=1}^{6} V_{aA} V_{bA} m_{\nu A} = 0 \tag{2.22}$$

Identity (2.21) stems from the unitarity of matrices A_L and U, while identity (2.22) is a consequence of the particular form of the neutrino mass matrix. In particular, it requires the Majorana mass term of the left-handed neutrinos be set to zero. A detailed proof of this latter identity is provided in Appendix A.

The propagators in R_{ξ} gauge are (Cheng and Li 1988):

$$\begin{aligned} &\text{Fermions}: \frac{i(\gamma p + m)}{p^2 - m^2 + i\epsilon} \\ &\text{W-bosons } W^{\pm}: \frac{-i}{p^2 - M_W^2 + i\epsilon} \left[g_{\mu\nu} + \frac{(\xi - 1) p_{\mu} p_{\nu}}{p^2 - \xi M_W^2} \right] \\ &\text{Goldstone bosons } \phi^{\pm}: \frac{i}{p^2 - \xi M_W^2} \end{aligned} \tag{2.23}$$

2.2 The Gauge Invariant T-Matrix Elements of Muonium–Antimuonium Oscillation

The lowest order Feynman diagrams accounting for muonium and antimuonium mixing are displayed in Fig. 2.1. We shall consistently employ the R_ξ gauge. The gauge invariance of the T-matrix element will be demonstrated by establishing its ξ independence. In Fig. 2.1, there are two neutrinos in the intermediate state for each graph, while every wavy line represents either a W boson or an R_ξ gauge charged erstwhile Nambu-Goldstone boson.

Note that in the unitary gauge ($\xi \to \infty$), the W boson propagator takes the form $\frac{-i}{p^2-M_W^2+i\epsilon}[g_{\mu\nu} - \frac{p_\mu p_\nu}{M_W^2}]$. A theory with such a propagator has very bad power counting convergence properties. As it turns out, the unitary gauge power counting divergent pieces in the W vector box diagrams vanish, as they must, after application of properties (2.21) and (2.22). Hence, when we calculate the T-matrix elements in R_ξ gauge, we will also apply properties (2.21) and (2.22) to establish the cancelation of the various terms in this case.

As it turns out, graph (a) gives the same contribution as (b), as do graphs (c) and (d). Hence, we need only discuss the gauge invariant T-matrix elements of the

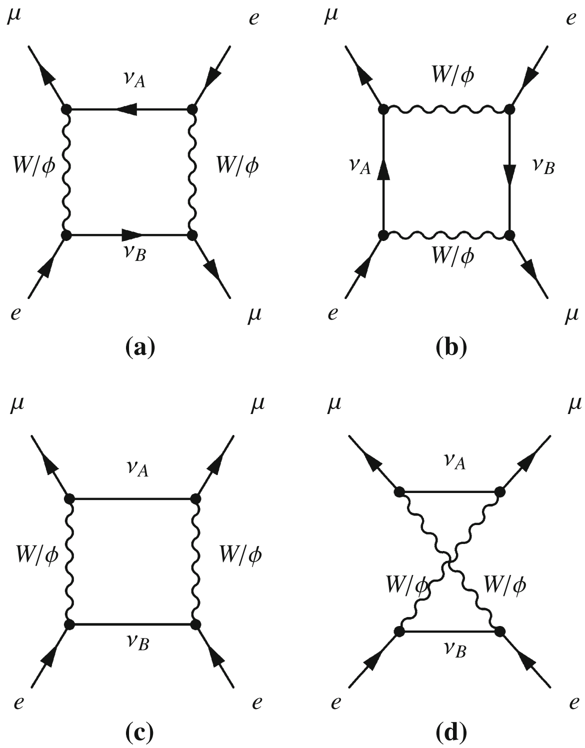

Fig. 2.1 Feynman graphs contributing to the muonium–antimuonium mixing. Each wavy line is either a W boson or an R_ξ gauge charged Nambu–Goldstone boson

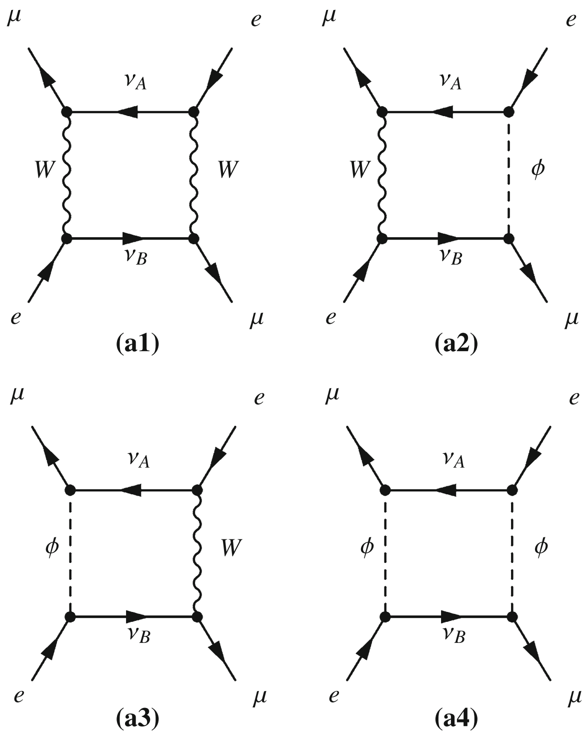

Fig. 2.2 Feynman graphs of type (**a**) in R_ξ gauge, which are presented as one single graph (**a**) in Fig. 2.1

graphs (a) and (c). Figure 2.2 details explicitly the four separate graphs, which are represented by the single graph in Fig. 2.1.

A straightforward application of the R_ξ gauge Feynman rules (Cheng and Li 1988) to the above graphs yields the T-matrix elements

$$T_{a1} = -\frac{g^4}{64\pi^2 M_W^2}\left[\bar{\mu}(3)\gamma_\mu \frac{1-\gamma_5}{2} e(2)\right]\left[\bar{\mu}(4)\gamma^\mu \frac{1-\gamma_5}{2} e(1)\right]\sum_{A=1}^{6}\sum_{B=1}^{6}(V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)$$
$$\cdot\int_0^\infty dt\left[\frac{x_A x_B}{(t+x_A)(t+x_B)(t+1)^2}\cdot\left\{1+\frac{2(\xi-1)}{t+\xi}\cdot t+\frac{(\xi-1)^2}{4(t+\xi)^2}\cdot t^2\right\}\right] \tag{2.24}$$

$$T_{a2} = T_{a3} = -\frac{g^4}{64\pi^2 M_W^2}\left[\bar{\mu}(3)\gamma_\mu \frac{1-\gamma_5}{2} e(2)\right]\left[\bar{\mu}(4)\gamma^\mu \frac{1-\gamma_5}{2} e(1)\right]$$
$$\cdot\sum_{A=1}^{6}\sum_{B=1}^{6}(V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)\int_0^\infty dt\left[\frac{x_A x_B}{(t+x_A)(t+x_B)(t+1)(t+\xi)}\right.$$
$$\left.\cdot\left\{t+\frac{\xi-1}{4(t+\xi)}\cdot t^2\right\}\right] \tag{2.25}$$

$$T_{a4} = -\frac{g^4}{64\pi^2 M_W^2}\left[\bar{\mu}(3)\gamma_\mu \frac{1-\gamma_5}{2} e(2)\right]\left[\bar{\mu}(4)\gamma^\mu \frac{1-\gamma_5}{2} e(1)\right]\sum_{A=1}^{6}\sum_{B=1}^{6}(V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)$$
$$\cdot\int_0^\infty dt\left[\frac{x_A x_B}{(t+x_A)(t+x_B)(t+\xi)^2}\cdot\frac{t^2}{4}\right] \tag{2.26}$$

where $\bar{\mu}(3) = \bar{\mu}(p_3, s_3)$, $\bar{\mu}(4) = \bar{\mu}(p_4, s_4)$, $e(1) = e(p_1, s_1)$ and $e(2) = e(p_2, s_2)$ are the spinors of the muons and electrons and $x_A = \frac{m_{\nu A}^2}{M_W^2}$, $A = 1, \ldots, 6$. Note that in obtaining these results, we already applied properties (2.21) and (2.22) to eliminate various self-canceling terms. As such, the integrals in (2.24)–(2.26) are finite even in the $\xi \to \infty$ limit. See Appendix B for detailed calculations of T-matrix elements.

In order to discuss the ξ dependence in a manifest way, we rewrite these T-matrix elements as

$$\begin{aligned}
T_{a1} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{A}(x_A, x_B, t)\cdot\left(h(t)\cdot\frac{1}{(t+\xi)^2} - 2g(t)\cdot\frac{1}{t+\xi} + f(t)\right)\\
T_{a2} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{A}(x_A, x_B, t)\cdot\left(-h(t)\cdot\frac{1}{(t+\xi)^2} + g(t)\cdot\frac{1}{t+\xi}\right)\\
T_{a3} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{A}(x_A, x_B, t)\cdot\left(-h(t)\cdot\frac{1}{(t+\xi)^2} + g(t)\cdot\frac{1}{t+\xi}\right)\\
T_{a4} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{A}(x_A, x_B, t)\cdot\left(h(t)\cdot\frac{1}{(t+\xi)^2}\right)
\end{aligned} \tag{2.27}$$

where

$$\mathcal{A}(x_A, x_B, t) = -\frac{g^4}{64\pi^2 M_W^2}\left[\bar{\mu}(3)\gamma_\mu\frac{1-\gamma_5}{2}e(2)\right]\left[\bar{\mu}(4)\gamma^\mu\frac{1-\gamma_5}{2}e(1)\right](V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)$$
$$\cdot\frac{x_A x_B}{(t+x_A)(t+x_B)} \tag{2.28}$$

with

$$h(t) = \frac{t^2}{4}, \quad g(t) = \frac{t+\frac{t^2}{4}}{t+1} \text{ and } f(t) = \frac{1+2t+\frac{t^2}{4}}{(t+1)^2} \tag{2.29}$$

Note that the $\frac{1}{(t+\xi)^2}$ terms from the second and third graphs totally cancel against the ones from the first and fourth graphs, while the $\frac{1}{t+\xi}$ terms from the second and third graphs exactly cancel the one from the first graph. All ξ dependent contributions thus vanish and the only remaining piece is the term containing $f(t)$

from the first graph, which is ξ independent. Hence, we have the gauge invariant T-matrix element for graphs of type (a)

$$\begin{aligned} T_a = & -\frac{g^4}{64\pi^2 M_W^2}\left[\bar{\mu}(3)\gamma_\mu \frac{1-\gamma_5}{2} e(2)\right]\left[\bar{\mu}(4)\gamma^\mu \frac{1-\gamma_5}{2} e(1)\right] \\ & \cdot \sum_{A=1}^{6}\sum_{B=1}^{6}(V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)x_A x_B \cdot \int_0^\infty dt \frac{1+2t+\frac{t^2}{4}}{(t+x_A)(t+x_B)(t+1)^2} \\ = & -\frac{G_F^2 M_W^2}{8\pi^2}[\bar{\mu}(3)\gamma_\mu(1-\gamma_5)e(2)][\bar{\mu}(4)\gamma^\mu(1-\gamma_5)e(1)] \\ & \cdot \left[\sum_{A=1}^{6}(V_{\mu A}V_{eA}^*)^2 S(x_A) + \sum_{A,B=1;A\neq B}^{6}(V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)T(x_A,x_B)\right] \end{aligned} \tag{2.30}$$

Here, we have introduced the Fermi scale

$$\frac{G_F}{\sqrt{2}} = \frac{g^2}{8M_W^2} \tag{2.31}$$

along with the Inami–Lim (Inami and Lim 1981) function

$$S(x_A) = \frac{x^3 - 11x^2 + 4x}{4(1-x)^2} - \frac{3x^3}{2(1-x)^3}\ln(x) \tag{2.32}$$

We have also defined

$$T(x_A, x_B) = x_A x_B \left(\frac{J(x_A) - J(x_B)}{x_A - x_B}\right) = T(x_B, x_A) \tag{2.33}$$

with

$$J(x) = \frac{x^2 - 8x + 4}{4(1-x)^2}\ln(x) - \frac{3}{4}\frac{1}{(1-x)} \tag{2.34}$$

In a similar manner, the graph (c) in Fig. 2.1 represents the four graphs in Fig. 2.3.

The T-matrix elements of these four graphs are

$$\begin{aligned} T_{c1} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{B}(x_A, x_B, t)\cdot\left(t\cdot\frac{1}{(t+\xi)^2} - 2\cdot\frac{1}{t+\xi} + \tilde{f}(x_A, x_B, t)\right) \\ T_{c2} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{B}(x_A, x_B, t)\cdot\left(-t\cdot\frac{1}{(t+\xi)^2} + \frac{1}{t+\xi}\right) \\ T_{c3} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{B}(x_A, x_B, t)\cdot\left(-t\cdot\frac{1}{(t+\xi)^2} + \frac{1}{t+\xi}\right) \\ T_{c4} &= \sum_{A=1}^{6}\sum_{B=1}^{6}\int_0^\infty dt\mathcal{B}(x_A, x_B, t)\cdot\left(t\cdot\frac{1}{(t+\xi)^2}\right) \end{aligned} \tag{2.35}$$

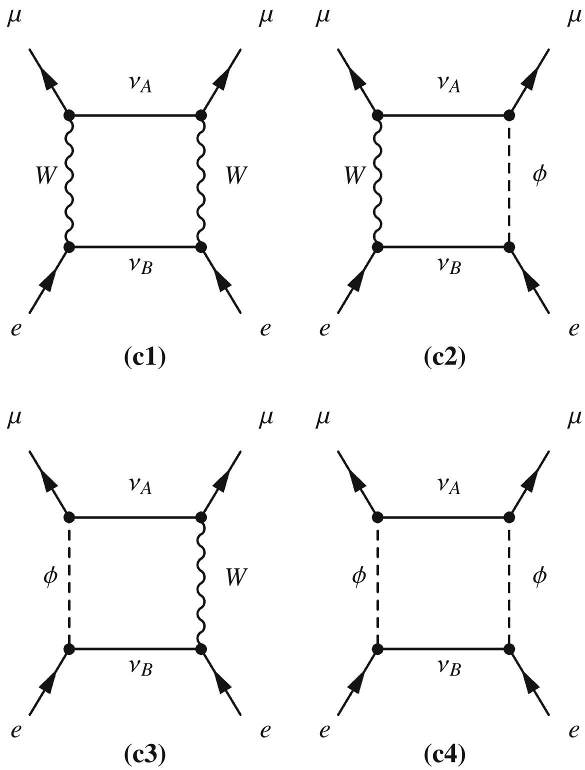

Fig. 2.3 Feynman graphs of type (**c**) in R_ξ gauge, which are presented as one single graph (**a**) in Fig. 2.1

where

$$\mathcal{B}(x_A, x_B, t) = \frac{g^4}{64\pi^2 M_W^4}\left[\bar{\mu}(3)\gamma^\mu \frac{1-\gamma_5}{2} e(2)\right]\left[\bar{\mu}(4)\gamma_\mu \frac{1-\gamma_5}{2} e(1)\right](V_{\mu A})^2 (V_{eB}^*)^2 \cdot \frac{m_A \cdot m_B}{2} \cdot \frac{x_A x_B}{(t+x_A)(t+x_B)} \tag{2.36}$$

and

$$\tilde{f}(x_A, x_B, t) = \frac{\frac{4t}{x_A \cdot x_B} + t + 2}{(t+1)^2} \tag{2.37}$$

In a similar fashion to the case for the graphs of type (a), all the ξ dependent terms again cancel against each other leaving only the ξ independent $\tilde{f}(x_A, x_B, t)$ term. Thus the type (c) T-matrix element is gauge invariant and is given by

$$T_c = \frac{G_F^2 M_W^2}{8\pi^2}[\bar{\mu}(3)\gamma^\mu(1-\gamma_5)e(2)][\bar{\mu}(4)\gamma_\mu(1-\gamma_5)e(1)]\sum_{A=1}^{6}(V_{\mu A})^2 \sum_{B=1}^{6}(V_{eB}^*)^2 \frac{\sqrt{x_A x_B}}{2} \cdot \int_0^\infty dt \left\{\frac{4t + x_A x_B(t+2)}{(t+x_A)(t+x_B)(t+1)^2}\right\} \tag{2.38}$$

If $x_A = x_B$, the relevant integral is

$$I(x_A) = \frac{\sqrt{x_A x_B}}{2} \cdot \int_0^\infty dt \frac{4t + x_A x_B (t+2)}{(t+x_A)(t+x_B)(t+1)^2} = \frac{(x_A - 4)x_A}{(x_A - 1)^2} + \frac{(x_A^3 - 3x_A^2 + 4x_A + 4)x_A}{2(x_A - 1)^3} \ln x_A \tag{2.39}$$

while for $x_A \neq x_B$, it takes the form

$$K(x_A, x_B) = \frac{\sqrt{x_A x_B}}{2} \cdot \int_0^\infty dt \frac{4t + x_A x_B (t+2)}{(t+x_A)(t+x_B)(t+1)^2} = \sqrt{x_A x_B} \frac{L(x_A, x_B) - L(x_B, x_A)}{x_A - x_B} \tag{2.40}$$

with

$$L(x_A, x_B) = \frac{4 - x_A x_B}{2(x_A - 1)} + \frac{x_A(2x_B - x_A x_B - 4)}{2(x_A - 1)^2} \ln x_A \tag{2.41}$$

The T-matrix element of graph (c) is thus secured as

$$T_c = \frac{G_F^2 M_W^2}{8\pi^2} [\bar{\mu}(3)\gamma^\mu (1 - \gamma_5) e(2)] [\bar{\mu}(4)\gamma_\mu (1 - \gamma_5) e(1)] \cdot \left[\sum_{A=1}^{6} (V_{\mu A} V_{eA}^*)^2 I(x_A) + \sum_{A,B=1;A \neq B}^{6} (V_{\mu A})^2 (V_{eB}^*)^2 K(x_A, x_B) \right] \tag{2.42}$$

2.3 The Effective Lagrangian

Combining the various contributions, the T-matrix element can be reproduced using the gauge invariant effective Lagrangian given by:

$$\mathcal{L}_{\text{eff}} = \frac{G_{\bar{M}M}}{\sqrt{2}} [\bar{\mu}\gamma^\mu (1 - \gamma_5) e] [\bar{\mu}\gamma_\mu (1 - \gamma_5) e] \tag{2.43}$$

where

$$\frac{G_{\bar{M}M}}{\sqrt{2}} = -\frac{G_F^2 M_W^2}{16\pi^2} \left[\sum_{A=1}^{6} (V_{\mu A} V_{eA}^*)^2 S(x_A) + \sum_{A,B=1;A \neq B}^{6} (V_{\mu A} V_{eA}^*)(V_{\mu B} V_{eB}^*) T(x_A, x_B) - \sum_{A=1}^{6} (V_{\mu A} V_{eA}^*)^2 I(x_A) - \sum_{A,B=1;A \neq B}^{6} (V_{\mu A})^2 (V_{eB}^*)^2 K(x_A, x_B) \right]$$

$$= -\frac{G_F^2 M_W^2}{16\pi^2}\left[\sum_{A=1}^{6}(V_{\mu A}V_{eA}^*)^2(S(x_A)-I(x_A)) + \sum_{A,B=1;A\neq B}^{6}\Big((V_{\mu A}V_{eA}^*)(V_{\mu B}V_{eB}^*)T(x_A,x_B)-(V_{\mu A})^2(V_{eB}^*)^2K(x_A,x_B)\Big)\right] \tag{2.44}$$

2.4 Limit on M_R

Muonium (antimuonium) is a nonrelativistic Coulombic bound state of an electron and an anti-muon (positron and muon). The nontrivial mixing between the muonium ($|M\rangle$) and antimuonium ($|\bar{M}\rangle$) states is encapsulated in the effective Lagrangian of Eq. (2.43) and leads to the mass diagonal states given by the linear combinations (see Appendix C for detailed derivation)

$$|M_\pm\rangle = \frac{1}{\sqrt{2(1+|\varepsilon|^2)}}[(1+\varepsilon)|M\rangle \pm (1-\varepsilon)|\bar{M}\rangle] \tag{2.45}$$

where

$$\varepsilon = \frac{\sqrt{\mathcal{M}_{M\bar{M}}}-\sqrt{\mathcal{M}_{\bar{M}M}}}{\sqrt{\mathcal{M}_{M\bar{M}}}+\sqrt{\mathcal{M}_{\bar{M}M}}} \tag{2.46}$$

$$\mathcal{M}_{M\bar{M}} = \frac{\langle M| - \int d^3r\mathcal{L}_{\text{eff}}|\bar{M}\rangle}{\sqrt{\langle M|M\rangle\langle\bar{M}|\bar{M}\rangle}}, \quad \mathcal{M}_{\bar{M}M} = \frac{\langle \bar{M}| - \int d^3r\mathcal{L}_{\text{eff}}|M\rangle}{\sqrt{\langle M|M\rangle\langle\bar{M}|\bar{M}\rangle}} \tag{2.47}$$

Since the neutrino sector is expected to be CP violating, these will be independent, complex matrix elements. If the neutrino sector conserves CP, with $|M\rangle$ and $|\bar{M}\rangle$ CP conjugate states, then $\mathcal{M}_{M\bar{M}} = \mathcal{M}_{\bar{M}M}$ and $\epsilon = 0$. In general, the magnitude of the mass splitting between the two mass eigenstates is

$$|\Delta M| = 2\left|\text{Re}\sqrt{\mathcal{M}_{M\bar{M}}\mathcal{M}_{\bar{M}M}}\right| \tag{2.48}$$

Since muonium and antimuonium are linear combinations of the mass diagonal states, an initially prepared muonium or antimuonium state will undergo oscillations into one another as a function of time. The muonium–antimuonium oscillation timescale, $\tau_{\bar{M}M}$, is given by

$$\frac{1}{\tau_{\bar{M}M}} = |\Delta M|. \tag{2.49}$$

We would like to evaluate $|\Delta M|$ in the nonrelativistic limit. A nonrelativistic reduction of the effective Lagrangian of Eq. (2.43) produces a local, complex effective potential

$$V_{\text{eff}}(\mathbf{r}) = 8\frac{G_{\bar{M}M}}{\sqrt{2}}\delta^3(\mathbf{r}) \tag{2.50}$$

Taking the muonium (antimuonium) to be in their respective Coulombic ground states, $\phi_{100}(\mathbf{r}) = \frac{1}{\sqrt{\pi a_{\bar{M}M}^3}}e^{-r/a_{\bar{M}M}}$, where $a_{\bar{M}M} = \frac{1}{m_{\text{red}}\alpha}$ is the muonium Bohr radius with $m_{\text{red}} = \frac{m_e m_\mu}{m_e + m_\mu} \simeq m_e$ the reduced mass of muonium, it follows that

$$\begin{aligned}\frac{1}{\tau_{\bar{M}M}} &\simeq 2^3 r\phi^*_{100}(\mathbf{r})|\text{Re}V_{\text{eff}}(\mathbf{r})|\phi(\mathbf{r})_{100} \\ &= 16\frac{|\text{Re}G_{\bar{M}M}|}{\sqrt{2}}|\phi_{100}(0)|^2 = \frac{16}{\pi}\frac{|\text{Re}G_{\bar{M}M}|}{\sqrt{2}}\frac{1}{a^3_{\bar{M}M}}\end{aligned} \tag{2.51}$$

Thus, we secure an oscillation timescale

$$\frac{1}{\tau_{\bar{M}M}} \simeq \frac{16}{\pi}\frac{|\text{Re}G_{\bar{M}M}|}{\sqrt{2}}m_e^3\alpha^3 \tag{2.52}$$

The present experimental limit (Willmann et al. 1999) on the non-observation of muonium–antimuonium oscillation translates into the bound $|\text{Re}G_{\bar{M}M}| \leq 3.0 \times 10^{-3}G_F$ where $G_F \simeq 1.16 \times 10^{-5}\,\text{GeV}^{-2}$ is the Fermi scale. This limit can then be used to construct a crude lower bound on M_R. For the case when the neutrino masses arise from a see-saw mechanism and taking m_D to be of order M_W, the M_R dependence of $G_{\bar{M}M}$ is obtained from Eq. (2.44) as:

$$\begin{aligned}&\textit{Case}\ 1: \quad |\text{Re}G_{\bar{M}M}| \sim \frac{G_F^2 M_W^4}{M_R^2}\ln\frac{M_R}{M_W}, \quad A = 1,2,3, \quad B = 1,2,3 \\ &\textit{Case}\ 2: \quad |\text{Re}G_{\bar{M}M}| \sim \frac{G_F^2 M_W^4}{M_R^2}\ln\frac{M_R}{M_W}, \quad A = 4,5,6, \quad B = 4,5,6 \\ &\textit{Case}\ 3: \quad |\text{Re}G_{\bar{M}M}| \sim \frac{G_F^2 M_W^6}{M_R^4}\ln\frac{M_R}{M_W}, \quad A = 1,2,3, \quad B = 4,5,6\end{aligned} \tag{2.53}$$

Cases 1 and 2 give the same order M_R dependence, while case 3 is suppressed by an additional factor of $\frac{M_W^2}{M_R^2}$. Hence, the term $\frac{G_F^2 M_W^4}{M_R^2}\ln\frac{M_R}{M_W}$ gives the dominant contribution. We then roughly calculate a bound of M_R as

$$\frac{G_F^2 M_W^4}{M_R^2}\ln\frac{M_R}{M_W} \leq 3.0 \times 10^{-3}G_F \tag{2.54}$$

which has also been obtained in reference (Cvetic et al. 2005). Using $M_W \simeq 80.4$ GeV and $G_F = 1.166 \times 10^{-5}\,\text{GeV}^{-2}$, we finally secure

$$M_R \geq 6 \times 10^2\,\text{GeV} \tag{2.55}$$

Note that this is just a rough estimate, since we are retaining only the dependence on M_R while neglecting all numerical dependence on the mixing angles and CP violating phases in V_{aA}.

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