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Global Analysis of Minimal Surfaces

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Chapter 1

Minimal Surfaces with Supporting Half-Planes

In Chapter 2 of Vol. 2 we have investigated the regularity of stationary minimal surfaces in the class $\mathcal{C}(\Gamma, S)$. Such stationary surfaces had been introduced in Section 4.6 of Vol. 1 (cf. also Chapter 1 of Vol. 2). We have shown that, for a uniformly smooth surface S with a smooth boundary ∂S, the stationary surfaces X belong to the class $C^{1,1/2}(B \cup I, \mathbb{R}^3)$. One of the consequences of results proved in the present chapter will be that this regularity result is optimal.

Recall that, according to the results of Chapter 2 of Vol. 2, the nonoriented tangent of the free trace $\Sigma = \{X(w) \colon w \in I\}$ of a stationary minimal surface X in $\mathcal{C}(\Gamma, S)$ changes continuously. This, in particular, means that the free trace cannot have corners at points where it attaches to the border of the supporting surface S. On the other hand, since isolated branch points of odd order cannot be excluded, there might exist cusps on the free trace. In fact, experimental evidence suggests that cusps do appear for certain shapes of the boundary configuration $\langle \Gamma, S \rangle$.

In Section 1.1 we shall describe soap film experiments, demonstrating the generation of cusps by a suitable bending process of the arc Γ. Such a physical proof for the existence of cusps is, of course, not conclusive in the mathematical sense although it bears strong evidence for the existence of this phenomenon. In Section 1.2 we therefore present several examples of stationary minimal surfaces with cusps on their traces. In fact, such examples are already well known to us (see, for example, Henneberg's surface and Catalan's surface) and have been discussed in Section 3.5 of Vol. 1.

The main part of this chapter is devoted to the study of the free trace of a stationary surface X within a boundary configuration $\langle \Gamma, S \rangle$ consisting of a half-plane S and a symmetric curve Γ which has a convex projection onto a plane E orthogonal to ∂S and which connects the two sides of S. After classifying the possible sets of contact of the free trace Σ with the boundary ∂S of the supporting half-plane, we prove a representation theorem for stationary surfaces in $\mathcal{C}(\Gamma, S)$ which is the key to all further results of this

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chapter. It essentially states that X can be viewed as a nonparametric surface with respect to the plane E. One of the main consequences drawn from this representation theorem is a uniqueness theorem stating that there can be at most one stationary minimal surface whose trace is touching ∂S, and this surface is area minimizing among all surfaces of $\mathcal{C}(\Gamma, S)$.

Furthermore we shall derive asymptotic expansions of a stationary surface along its free boundary I which will imply that $C^{1,1/2}$-regularity is in general the optimal regularity result. Finally, we describe the geometrical shape of the free trace, and we exhibit conditions on Γ which prevent the occurrence of branch points.

1.1 An Experiment

Let S be a half-plane and consider some arc Γ that starts in some point P_1 on the upper side of S, leads about the edge ∂S, and ends in some point P_2 on the lower side of S, as depicted in Fig. 1. It is assumed that Γ has no points

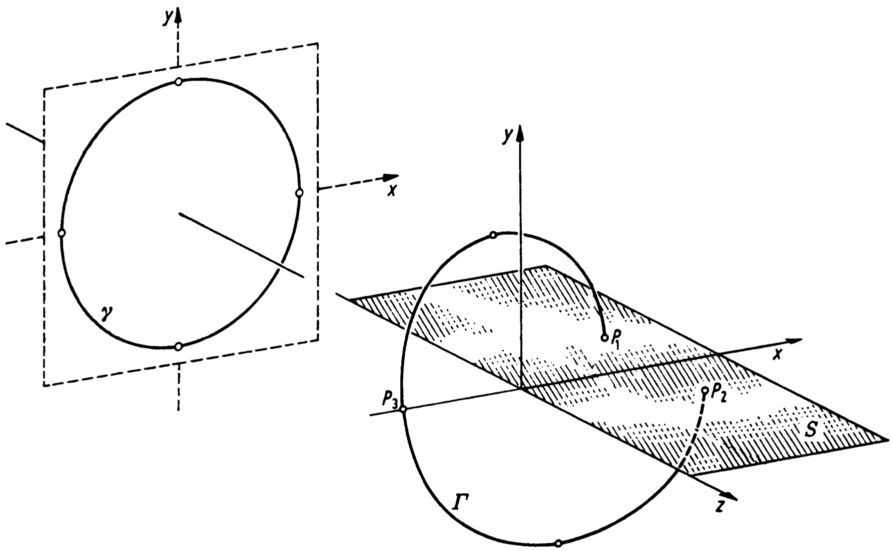

Fig. 1.

in common with S, except for P_1 and P_2. We can imagine that Γ is obtained from a circle by cutting it and pulling its ends slightly apart. Suppose that S is the part $\{x \geq 0, y = 0\}$ of the x, z-plane and that ∂S coincides with the z-axis. Then we may assume that the projection of Γ onto some plane E orthogonal to the z-axis is nearly circular and certainly convex, and that the z-component of a suitable Jordan representation of Γ is monotonically increasing. In this

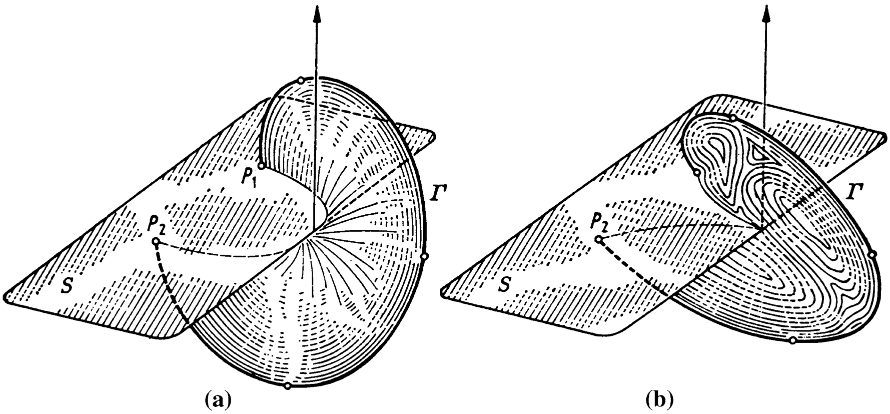

Fig. 2. (a) Tongue. **(b)** Cusp

case, the free trace of a soap film spanned in $\langle \Gamma, S\rangle$ is depicted in Fig. 2. Let us now define the arc Γ in such a way that its endpoints on S are kept fixed and the projection of Γ onto the plane is only slightly altered, whereas the z-component of the representation of Γ changes its signs repeatedly (an odd number of times). During this deformation process the free trace may develop a cusp (see Fig. 2). This can be seen by looking at the free trace in various stages of the bending procedure; cf. Fig. 3. Let us deform the arc Γ by twisting it about some axis in the supporting plane orthogonal to the edge. If the twisting is carried sufficiently far, the originally tongue-shaped free trace narrows more and more, forms for a moment a cusp, which then opens and changes into a loop. This loop as well as the original tongue are attached to the border of S along an interval.

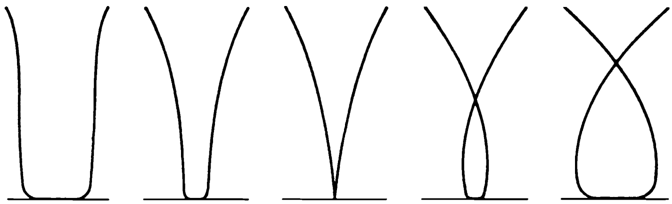

Fig. 3. The free trace during various stages of the bending process

Three different forms of the free trace that were actually observed and photographed during an experiment are reproduced in Plate II.

It is interesting to contrast the situation depicted in Fig. 4 with another, related experiment where Γ is a circle, cut at some point, which again has its endpoints on opposite sides of the supporting half-plane S, but this time not spread apart. If the circle is turned about its horizontal diameter, the free

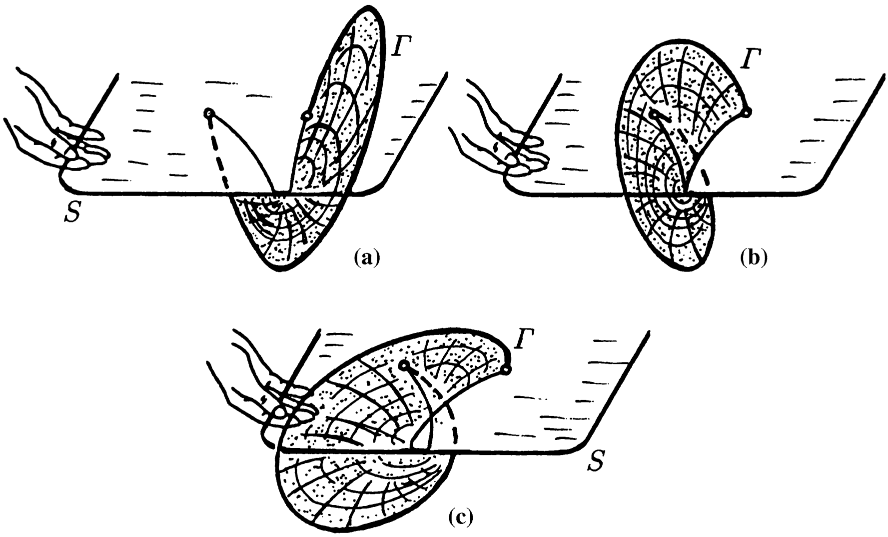

Fig. 4. **(a)** Tongue. **(b)** Cusp. **(c)** Loop

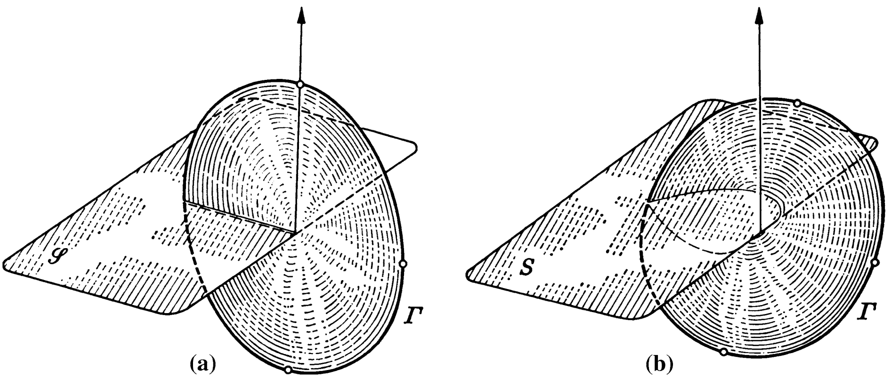

Fig. 5. **(a, b)** Another bending process where no cusps are formed

boundary, originally consisting of two matching segments on either side of S (cf. Fig. 5a), opens and develops a shape, depicted in Fig. 5b, which does not contain a cusp at any stage of the turning process.

The symmetry assumptions on S and Γ stated above are essential for the following mathematical discussion, but they are by no means essential for the experiment. The supporting surface S can be an arbitrary smooth surface, planar or not, and Γ can be an arbitrary arc which has no points in common with S except for its endpoints. Of course, the free trace of a soapfilm in the frame $\langle \Gamma, S \rangle$ will then be more complicated and can develop several cusps and selfintersections. A mathematical discussion of this general case has not yet been carried out.

1.2 Examples of Minimal Surfaces with Cusps on the Supporting Surface

In the sequel, B will not denote the unit disk $\{|w| < 1\}$ but the semidisk

$$B := \{w \in \mathbb{C}\colon |w| < 1, \operatorname{Im} w > 0\},$$

and I denotes the interval

$$I := \{u \in \mathbb{R}\colon |u| < 1\}$$

on the real axis. Finally we introduce the circular arc

$$C := \partial B \setminus I.$$

Definitions, theorems, etc. concerning surfaces previously defined on the whole disk $\{|w| < 1\}$ are then carried over to surfaces defined on the semidisk B by means of a conformal map $\tau\colon \{|w| < 1\} \to B$ keeping the three points $1, -1, i$ fixed.

As in (1), we consider the half-plane

$$S = \{(x, y, z) \in \mathbb{R}^3\colon x \ge 0, y = 0\}$$

as supporting surface.

In Sections 3.4 and 3.5 of Vol. 1 we have seen how Schwarz's formula solving Björling's problem can be used to construct stationary surfaces $X\colon B \to \mathbb{R}^3$ which intersect S perpendicularly in a given curve Σ having a cusp at the origin of the system of coordinates. The surfaces of Henneberg and Catalan are prominent examples of such minimal surfaces.

Let us consider the following rescaled version of Henneberg's surface, a portion of which is pictured in Figs. 1 and 2:

$$\begin{aligned} x &= \cosh(2\lambda u)\cos(2\lambda v) - 1, \\ y &= -\sinh(\lambda u)\sin(\lambda v) - \frac{1}{3}\sinh(3\lambda u)\sin(3\lambda v), \\ z &= -\sinh(\lambda u)\cos(\lambda v) + \frac{1}{3}\sinh(3\lambda u)\cos(3\lambda v). \end{aligned} \tag{1}$$

It follows from

$$\begin{aligned} X(u, 0) &= \left(\cosh(2\lambda u) - 1, 0, -\sinh(\lambda u) + \frac{1}{3}\sinh(3\lambda u)\right) \\ &= \left(2\sinh^2(\lambda u), 0, \frac{4}{3}\sinh^3(\lambda u)\right) \end{aligned}$$

that (1) intersects the plane $y = 0$ in Neil's parabola

$$2x^3 = 9z^2, \quad y = 0. \tag{2}$$

For small values of w we have the expansion

$$x(w) = \operatorname{Re}\{2\lambda^2 w^2 + \cdots\}, \quad y(w) = \operatorname{Re}\{2i\lambda^2 w^2 + \cdots\},$$
$$z(w) = \operatorname{Re}\left\{\frac{4}{3}\lambda^3 w^3 + \cdots\right\}.$$

Let us denote by $\mathcal{M}$ the portion of (1) which corresponds to the closed semidisk $\overline{B} = \{w\colon |w| \le 1, v \ge 0\}$ in the parameter plane. The surface $\mathcal{M}$ is bounded by a configuration $\langle \Gamma, S\rangle$ where S is the half-plane $\{x \ge 0, y = 0\}$, and Γ is the image of the circular arc C under the mapping (1), that is, the arc $\{X(e^{i\theta})\colon 0 \le \theta \le \pi\}$. The free boundary of $\mathcal{M}$ on S is Neil's parabola (2); $\mathcal{M}$ and S meet at a right angle along this curve.

The orthogonal projection of Γ onto the x, y-plane is a smooth closed curve. For a later reference we observe that this curve is convex as long as the parameter λ remains in the interval $0 < \lambda \le \lambda_0 \mathrel{\hat{=}} 1.014379\ldots$. (It turns out that λ_0 is the first positive root of the equation $\tan(2\lambda) = -2\lambda$.)

Certain other algebraic singularities of the free boundary are also possible. For the minimal surface represented by the equations

$$\begin{aligned} x &= \operatorname{Re}\{w^2\}, \\ y &= \operatorname{Re}\left\{2i\int_0^w \omega\sqrt{1+\omega^{4n-2}}\,d\omega\right\}, \\ z &= \operatorname{Re}\left\{\frac{2}{2n+1}w^{2n+1}\right\}, \end{aligned} \tag{3}$$

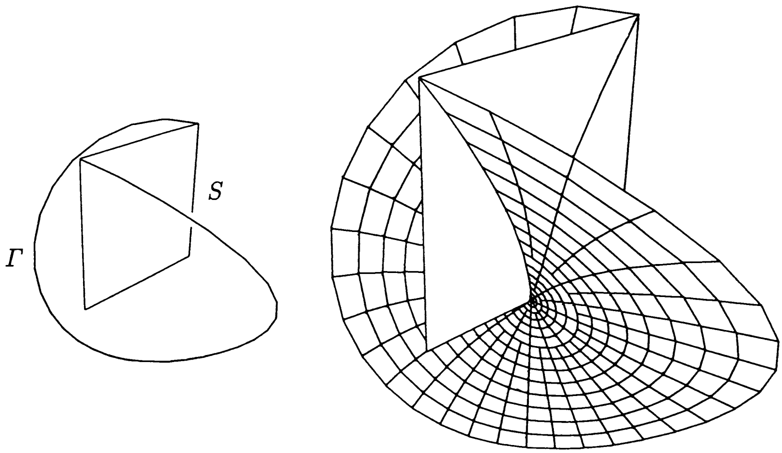

Fig. 1. A part of Henneberg's surface as solution in a configuration $\langle \Gamma, S\rangle$ whose free trace on S has a cusp

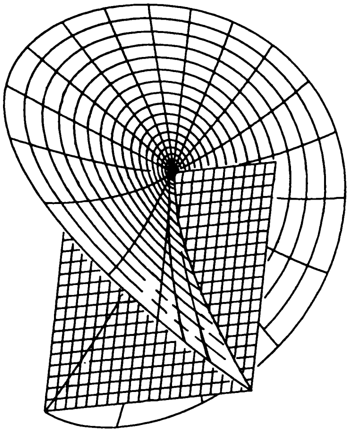

Fig. 2. Another view of Henneberg's surface in a configuration $\langle \Gamma, S \rangle$. Courtesy of I. Haubitz

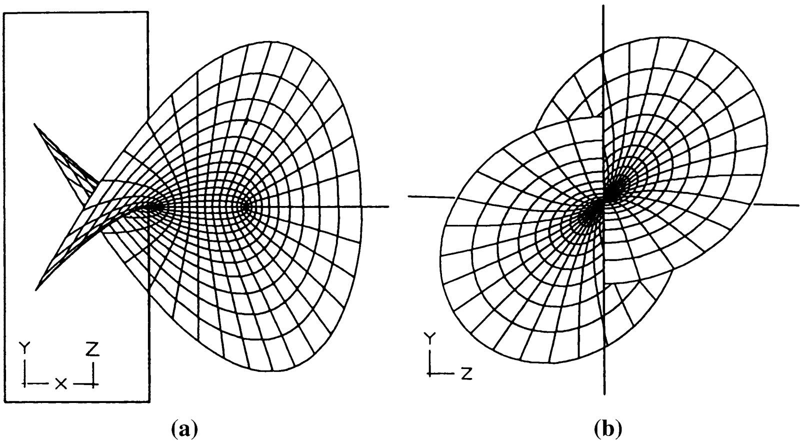

Fig. 3. Two views of two cusps in Henneberg's surface

the free boundary, i.e., the image of I on the half-plane $S = \{x \geq 0, y = 0\}$, is the curve

$$4x^{2n+1} = (2n+1)^2 z^2, \quad y = 0.$$

We can state even simpler examples if we do not insist on classical curves as free boundaries. One very simple example is furnished by the minimal surface

$$\begin{aligned} x &= \operatorname{Re}\{w^2 - 18\lambda^2 w^4\}, \\ y &= \operatorname{Re}\{iw^2 + 18i\lambda^2 w^4\}, \quad \lambda > 0, \\ z &= \operatorname{Re}\{8\lambda w^3\} \end{aligned} \tag{4}$$

which meets the half-plane S orthogonally along the curve

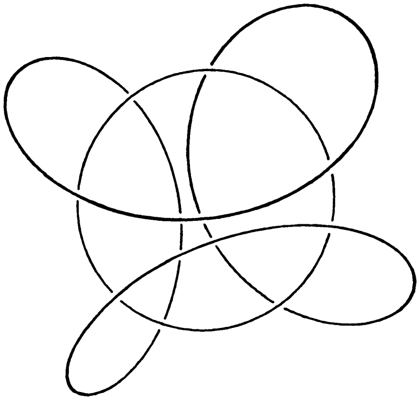

Fig. 4. A boundary configuration $\langle \Gamma, S \rangle$ consisting of a disk S and a disjoint Jordan curve Γ. It bounds a stationary minimal surface of annulus type which meets S perpendicularly at an asteroid

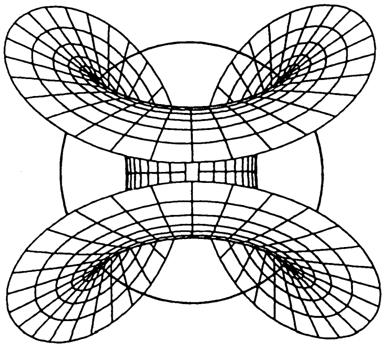

Fig. 5. The annulus-type stationary minimal surface within the configuration $\langle \Gamma, S \rangle$ depicted in Fig. 4 is part of the adjoint of Henneberg's surface. The four cusps correspond to four branch points

$$x(u) = u^2 - 18\lambda^2 u^4, \quad y(u) = 0, \quad z(u) = 8\lambda u^3$$

which has the expansion

$$z = 8\lambda x^{3/2} + \cdots , \quad y = 0$$

about the origin. As arc Γ we shall again use the image of the circular arc C, this time under the mapping (4). The orthogonal projection of Γ onto the x, y-plane is the closed curve

$$x = \cos\theta - 18\lambda^2 \cos 2\theta,$$
$$y = \sin\theta + 18\lambda^2 \sin 2\theta, \quad 0 \le \theta \le 2\pi.$$

This curve is convex if $0 < 18\lambda^2 \le \frac{1}{4}$, that is, if $0 < \lambda \le \lambda_0 := \sqrt{2}/12 = 0.117851\ldots$.

It is not at all a priori clear that the above surfaces are solutions of the minimum problem

(5) $$D(X) \to \min \quad \text{in } \mathcal{C}(\Gamma, S).$$

This will, in fact, follow from the uniqueness theorem proved in Section 1.5. In particular Henneberg's surface (1) provides us with a simple example of a solution of the minimum problem (5) which possesses a cusp on its trace.

1.3 Setup of the Problem. Properties of Stationary Solutions

We will now prepare the mathematical discussion to be carried out in the following sections. We begin by fixing the assumptions on the boundary configuration $\langle \Gamma, S \rangle$ which are supposed to hold throughout Sections 1.3–1.9.

Assumption A. *Let S be the half-plane $\{(x,y,z) \colon x \geq 0, y = 0\}$ in $\mathbb{R}^3$. Moreover, the curve Γ is assumed to be a regular arc of class $C^{1,\alpha}, 0 < \alpha < 1$, with the endpoints P_1 and $P_2, P_1 \neq P_2$, which issues from S at right angles and meets S only in its endpoints. Close to P_1, the arc Γ is supposed to lie in the half-space $\{y \geq 0\}$. Assume also that Γ is symmetric with respect to the x-axis, and that the orthogonal projection of Γ onto the x,y-plane is a closed, strictly convex and regular curve γ of class $C^{1,\alpha}$. Finally, suppose that the projection of Γ onto γ is one-to-one, except for the endpoints P_1 and P_2 of Γ which are projected onto the same point of γ.*

This assumption is satisfied by the examples discussed in Section 1.2.

Assume that $P(s) = (p^1(s), p^2(s), p^3(s))$, $0 \leq s \leq L$, is a parametrization of Γ by the arc length s such that

(1) $$P(0) = P_1 = (a, 0, -c), \quad P(L) = P_2 = (a, 0, c)$$

where $a > 0$ and $c > 0$. Then $P_3 := P(L/2)$ is the uniquely determined intersection point of Γ with the x-axis which must be of the form

(2) $$P(L/2) = P_3 = (-b, 0, 0), \quad b > 0.$$

This is illustrated in Fig. 1.

Let us now recall that the definition of *stationary minimal surfaces* was phrased in such a way that these surfaces are precisely the stationary points of Dirichlet's integral within the class $\mathcal{C}(\Gamma, S)$. In Chapter 2 of Vol. 2 we have formulated the following result:

Lemma 1. *Every stationary minimal surface in $\mathcal{C}(\Gamma, S)$ is continuous in the closure $\overline{B}$ of the parameter domain B.*

From the regularity theory of Chapter 2 in Vol. 2 we can also derive the following result:

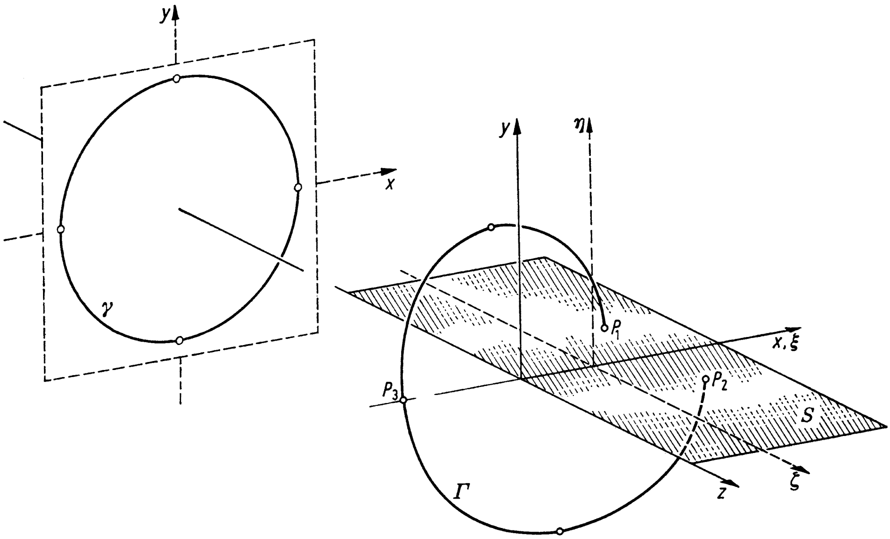

Fig. 1. Assumption A

Lemma 2. *If X is a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ and if $\langle \Gamma, S\rangle$ satisfies Assumption* A*, then X belongs to $C^1(\overline{B}, \mathbb{R}^3)$ and satisfies*

$$y(u) = 0, \quad z_v(u) = 0 \quad \textit{for } u \in I. \tag{3}$$

Thus $y(w)$ and $z(w)$ can be continued analytically across the interval $I = \{|u| < 1\}$ of the u-axis, and the extended functions are harmonic in the whole disk $\{w : |w| < 1\}$. The set

$$I_1 := \{u \in I : x(u) > 0\}$$

is an open subset of $\mathbb{R}$ containing the intervals $(-1, -1+2\delta_0)$ and $(1-2\delta_0, 1)$ for some sufficiently small $\delta_0 > 0$. Hence the set of contact

$$I_2 = \{u \in I : x(u) = 0\}$$

is closed in $\mathbb{R}$. In addition, we have

$$x_v(u) = 0 \quad \textit{for } u \in I_1. \tag{4}$$

Proof. The regularity theory of Chapter 2 of Vol. 2 yields that X is of class C^1 on $\overline{B} \setminus \{\pm 1\}$. Since X is stationary in $\mathcal{C}(\Gamma, S)$, it follows that

$$y(u) = 0 \quad \text{for } u \in I$$

holds as well as

$$x_v(u) = 0, \quad z_v(u) = 0 \quad \text{for } u \in I_1.$$

The first equation implies

$$y_u(u) = 0 \quad \text{for } u \in I.$$

Furthermore, the relations

$$x(u) \geq 0 \quad \text{for } u \in I, \quad x(u) = 0 \quad \text{for } u \in I_2$$

imply that

$$x_u(u) = 0 \quad \text{for } u \in I_2,$$

whence

$$X_u(u) = (0, 0, z_u(u)) \quad \text{for } u \in I_2$$

and $z_u(u) \neq 0$ on I_2 except for isolated points. By virtue of $\langle X_u, X_v \rangle = 0$, we infer that

$$z_v(u) = 0 \quad \text{for } u \in I_2,$$

and therefore

$$z_v(u) = 0 \quad \text{for all } u \in I.$$

Thus we have verified (3) and (4), and, in view of the reflection principle, the functions $y(w)$ and $z(w)$ can be continued analytically across the interval I on the u-axis by setting

$$y(u - iv) := -y(u + iv), \quad z(u - iv) := z(u + iv)$$

for $v \geq 0$. The extended functions $y(w)$ and $z(w)$ are harmonic in the disk

$$\{w \colon |w| < 1\}.$$

Since $X \in C^0(\overline{B}, \mathbb{R}^3)$ and $a > 0$, the points $X(u)$ lie in the interior of the half-plane S if u is close to ± 1. Hence there is a number $\delta_0 > 0$ such that the intervals $(-1, -1 + 2\delta_0)$ and $(1 - 2\delta_0, 1)$ on the u-axis are contained in I_1. On the part $X(I_1)$ of the free trace, the surface X meets S perpendicularly. Hence we can continue $X(w)$ analytically across I_1 by a reflection with respect to S, and the extended surface $\hat{X}(w)$ is a minimal surface on $\{w \colon |w| < 1, w \notin I_2\}$. Moreover, $\hat{X}(w)$ is continuous on $\{w \colon |w| < 1\}$. Since Γ issues from S perpendicularly, the surface $\hat{X}$ maps the unit circle $\{w \colon |w| = 1\}$ bijectively onto a closed regular curve of class $C^{1,\alpha}$. Then the regularity results stated in Section 2.12 of Vol. 2 imply that $\hat{X}(w)$ is of class $C^{1,\alpha}$ in the strip $\{1 - \delta_0 \leq |w| \leq 1\}$. Thus X is of class C^1 on $\overline{B}$.

1.4 Classification of the Contact Sets

The principal result of this section is the following observation: *The free trace of a stationary minimal surface in $\mathcal{C}(\Gamma, S)$ either meets the boundary ∂S of the half-plane S in a single point, or in a single subinterval, or in no point at all.*

More precisely, we shall prove:

Theorem. *Let $X(w) = (x(w), y(w), z(w))$ be a stationary minimal surface in $\mathcal{C}(\Gamma, S)$, and set $I_1 := \{u \in I \colon x(u) > 0\}, I_2 := \{u \in I \colon x(u) = 0\}$, and $x_0 := \min\{x(u) \colon u \in I\}$. Then only the following three cases can occur:*

(I) *$x_0 = 0$, and I_2 consists of a single point u_0;*
(II) *$x_0 = 0$, and I_2 is a closed interval of positive length;*
(III) *$x_0 > 0$, that is, I_2 is empty, and there is exactly one point u_0 in I such that $x_0 = x(u_0)$. Consequently, we have $x(u) > x_0$ for $u \in I$ with $u \neq u_0$.*

Remark. Case I may indeed occur as we see from the examples given in Section 1.2. If we introduce the new supporting surface

$$S_\varepsilon = \{(x, y, z) \in \mathbb{R}^3 \colon y = 0, x \geq -\varepsilon\}, \quad \varepsilon > 0,$$

for some sufficiently small $\varepsilon > 0$ as well as the new coordinates

$$\xi = x + \varepsilon, \quad \eta = y, \quad \zeta = z,$$

a surface $X(w) = (x(w), y(w), z(w))$ of type I is transformed into a surface $\Xi(w) = (\xi(w), \eta(w), \zeta(w))$ of type III. Hence also the case III may appear. On the other hand, we shall see in Section 1.6 that minima of Dirichlet's integral are never of type III.

As a first step towards the proof of the Theorem we draw some preliminary information from the maximum principle which is formulated as

Lemma 1. *The trace $X(I)$ is contained in the strip $\{(x, y, z) \colon 0 \leq x < a, y = 0\}$ of the half-plane S whence*

$$0 \leq x_0 < a. \tag{1}$$

Moreover, we have

$$-b < x(w) < a \quad \text{for all } w \in B. \tag{2}$$

Proof. In fact, if there were some $u \in I$ with $x(u) \geq a$, then there would exist some $u^* \in I$ such that

$$x(u^*) = \max_I x(u) \geq a > 0,$$

since $x(\pm 1) = a$. Since $x(w)$ is harmonic and nonconstant in B, the lemma of E. Hopf[1] implies that[1]

$$x_v(u^*) < 0.$$

Since u^* belongs to I_1, this contradicts Lemma 2 of the preceding section.

Thus we have proved that $x_0 := \min\{x(u) \colon u \in I\}$ satisfies (1). Moreover, the x-component $p^1(s)$ of the representation $P(s)$ of Γ satisfies

[1] Cf. Gilbarg and Trudinger [1], p. 33.

$$-b \le p^1(s) \le a \quad \text{for } 0 \le s \le L$$

whence

$$-b \le x(w) \le a \quad \text{for } w \in C.$$

On account of

$$0 \le x(u) < a \quad \text{for } u \in I$$

we infer relation (2) from the maximum principle. □

The next lemma is the crucial step for the proof of the Theorem. We need the following notations:

For each value $\mu \in \mathbb{R}$ we define the open (and possibly empty) subsets $B(\mu), B^+(\mu)$, and $B^-(\mu)$ of B by

$$\begin{aligned} B(\mu) &:= \{w \in B \colon x(w) \ne \mu\}, \\ B^+(\mu) &:= \{w \in B \colon x(w) > \mu\}, \\ B^-(\mu) &:= \{w \in B \colon x(w) < \mu\}. \end{aligned}$$

By virtue of Lemma 1, we obtain

$$\begin{aligned} &B^+(\mu) = \emptyset, \quad B^-(\mu) = B \quad \text{if } \mu \ge a, \\ &B^+(\mu) = B, \quad B^-(\mu) = \emptyset \quad \text{if } \mu \le -b. \end{aligned}$$

Recall that $X(w)$ provides a topological mapping of the circular arc C onto Γ. By Assumption A there are, for each value $\mu \in (-b, a)$, exactly two points $w_1(\mu) = e^{i\theta_1(\mu)}$ and $w_2(\mu) = e^{i\theta_2(\mu)}$ on $C, 0 < \theta_1(\mu) < \theta_2(\mu) < \pi$, with the property that

$$x(w_1(\mu)) = x(w_2(\mu)) = \mu.$$

In addition we set

$$w_1(-b) = w_2(-b) := i, \quad \theta_1(-b) = \theta_2(-b) := \frac{\pi}{2},$$

$$w_1(a) := 1, \quad w_2(a) := -1, \quad \theta_1(a) := 0, \quad \theta_2(a) := \pi.$$

For $\mu \in (-b, a)$, we define the following open subarcs of C:

$$\begin{aligned} C_1^+(\mu) &:= \{w = e^{i\theta} \colon 0 < \theta < \theta_1(\mu)\}, \\ C^-(\mu) &:= \{w = e^{i\theta} \colon \theta_1(\mu) < \theta < \theta_2(\mu)\}, \\ C_2^+(\mu) &:= \{w = e^{i\theta} \colon \theta_2(\mu) < \theta < \pi\}. \end{aligned}$$

Lemma 2. *For each $\mu \in (-b, a)$, the set $B^-(\mu)$ is connected, and the set $B^+(\mu)$ can have at most two components.*

Proof. We proceed as follows:

(i) First we fix some $\mu \in (-b, a)$, and denote by Q the component of $B^-(\mu)$, the boundary of which contains the arc $C^-(\mu)$. If $B^-(\mu)$ were not connected, there would exist another nonempty component R of $B^-(\mu)$. Clearly, $\partial R \subset B \cup I \cup \{w_1(\mu), w_2(\mu)\}, x(w) = \mu$ for $w \in \partial R \cap (B \cup C)$, and $x(w) \leq \mu$ for $w \in \partial R \cap I$. (Note that $\partial R \cap I$ is void for $\mu < 0$.) If $\partial R \cap I$ were empty, the maximum principle would imply that $x(w) \equiv \mu$ on R, so that, contrary to the facts, $x(w) \equiv \mu$ on $\overline{B}$. If $\partial R \cap I$ is nonempty (this is only possible for $\mu \geq 0$), let $m := \inf\{x(u)\colon u \in \partial R \cap I\}$. We claim that $m < \mu$. Otherwise, if $m = \mu$, we could obtain a contradiction as before. From $x(u) \geq 0$ we conclude that $0 \leq m < \mu$. Thus, R has to be void if $\mu \leq 0$. If $0 < \mu < a$, the value

$$\overline{u} := \sup\{u \in I \cap \partial R\colon x(u) = m\}$$

satisfies $-1 < \overline{u} < 1$. Since $m < \mu$, there is a number $\varepsilon > 0$ such that

$$(\overline{u} - \varepsilon, \overline{u} + \varepsilon) \subset \partial R.$$

By the maximum principle and E. Hopf's lemma it follows that $x_v(\overline{u}) > 0$. On the other hand, by the definitions of m and $\overline{u}$, a right neighbourhood $\mathcal{U}$ of $\overline{u}$ on I must belong to I_1. By Lemma 2 of Section 1.3, it follows that $x_v(u) = 0$ for $u \in \mathcal{U}$, and x_v is continuous on I. Thus we arrive at the contradictory conclusion $x_v(\overline{u}) = 0$. We have proved that $B^-(\mu)$ is connected for all $\mu \in (-b, a)$.

(ii) Again, we select a value $\mu \in (-b, a)$. Denote by Q_1 and Q_2 the two components of $B^+(\mu)$, the boundary of which contains $C_1^+(\mu)$ and $C_2^+(\mu)$ respectively. It is of course possible that Q_1 and Q_2 are identical. We assert that $B^+(\mu)$ cannot have further components. Otherwise, if R were such a nonempty component different from Q_1 and Q_2, we would have $\partial R \subset B \cup I \cup \{w_1(\mu), w_2(\mu)\}, x(w) = \mu$ for $w \in \partial R \cap (B \cup C)$, and $x(w) \geq \mu$ for $w \in I \cap \partial R$. If $\partial R \cap I$ were empty, the maximum principle would lead to a contradiction, as in (i). We may therefore assume that $\partial R \cap I$ is nonvoid. If $-b < \mu < 0$, the level set $l(\mu) = \{w\colon x(w) = \mu\}$ cannot touch I. In fact, there is a strip $s_\varepsilon = \{w = u + iv; 0 \leq v < \varepsilon\}$ abutting on I which is not penetrated by $l(\mu)$, so that $s_\varepsilon \subset Q_1 \cup Q_2$. But this is incompatible with the assumption $\partial R \cap I \neq \emptyset$. We turn to the case $0 \leq \mu < a$. Neighbourhoods in B of the corner points $w = \pm 1$ belong to the components Q_1 and Q_2. Hence there is a $\delta > 0$, such that $\partial R \cap I \subset \{u\colon |u| < 1 - \delta\}$. Then there exists a point $\overline{u} \in \partial R \cap I$ in which $x(\overline{u})$ attains the maximum value $m = \max\{x(u)\colon u \in I \cap \partial R\}$. As in (i) we conclude from the maximum principle that $m > \mu$. Then there is a $\sigma > 0$ such that the interval $(\overline{u} - \sigma, \overline{u} + \sigma)$ on I belongs to the boundary ∂R, whence $x_v(\overline{u}) < 0$, again on account of E. Hopf's lemma. On the other hand, $0 \leq \mu < m = x(\overline{u})$ implies that the point $\overline{u}$ belongs to I_1 which leads to the contradicting statement $x_v(\overline{u}) = 0$. Therefore, $B^+(\mu)$ has no components other than Q_1 and Q_2. □

Remark. The proof of Lemma 2 yields further information regarding the set $B^+(\mu)$. We see for instance that $B^+(\mu)$ is connected for $-b < \mu < 0$. If $B^+(\mu)$ consists of two different components, then the boundary of one of these components, Q_1, contains all points of the arc $C_1^+(\mu)$, while $C_2^+(\mu)$ is part of the boundary of the other component Q_2.

Now we turn to the *proof of the Theorem.* Set

$$u_0 := \min\{u \in I\colon x(u) = x_0\}, \quad u_0' := \max\{u \in I\colon x(u) = x_0\}.$$

Clearly we have $-1 < u_0 \le u_0' < 1$. Then one of the following three, mutually exclusive cases must hold:

$$\begin{aligned}
&(\alpha) \quad u_0 = u_0'; \\
&(\beta) \quad u_0 < u_0', \quad x(u) = x_0 \quad \text{for all } u \in [u_0, u_0']; \\
&(\gamma) \quad u_0 < u_0', \quad x(\overline{u}) > x_0 \quad \text{for some } \overline{u} \in (u_0, u_0').
\end{aligned}$$

We shall show first that case (γ) cannot occur. In case (γ) we would be able to find two points $u_1, u_2 \in [u_0, u_0']$, $u_1 < u_2$, such that $x(u_1) = x(u_2) = x_0$ and $x(u) > x_0$ for $u_1 < u < u_2$. Set

$$m := \max\{x(u)\colon u_1 < u < u_2\}, \quad 0 \le x_0 < m < a,$$

and assume that $x(u') = m, u_1 < u' < u_2$. Then $x_u(u') = 0$; moreover we have $x_v(u) = 0$ for $u_1 < u < u_2$, since $(u_1, u_2) \subset I_1$. Therefore, $x(w)$ can be continued analytically as a harmonic function across the segment $u_1 < u < u_2$ of the u-axis into the lower half of the w-plane. In a (full) neighbourhood of the point $w = u'$ this function has an expansion

$$x(w) = m + \operatorname{Re}\{\kappa(w - u')^\nu + \cdots\}, \quad \kappa \ne 0, \quad \nu \ge 2,$$

since $\nabla x(u') = 0$ and $x_v(u) = 0$ for $u_1 < u < u_2$. From the fact that $u = u'$ is a local maximum of $x(u)$ on I we conclude that $\kappa < 0$ and $\nu = 2n$, $n \ge 1$. A neighbourhood of $w = u'$ in B is divided into $2n+1$—at least three—open sectors $\sigma_1, \sigma_2, \ldots, \sigma_{2n+1}$ such that $x(w) < m$ in $\sigma_1, \sigma_3, \ldots, \sigma_{2n+1}$, and that $x(w) > m$ in $\sigma_2, \sigma_4, \ldots, \sigma_{2n}$. Now consider two points w_1 and w_{2n+1} in σ_1 and σ_{2n+1} respectively. As we know from Lemma 2, the set $B^-(m)$ is connected and contains the points w_1 and w_{2n+1}. Thus we can connect w_1 and w_{2n+1} by a path $\tilde{\gamma}$ contained in $B^-(m)$. Connecting w_1 and w_{2n+1} with u' in σ_1 and σ_{2n+1} respectively we obtain a closed curve which separates the component Ω_2 of $B^+(m)$ containing the sector σ_2 from the components Q_1 and Q_2 that were introduced in the proof of the preceding lemma. In other words, the case (γ) would imply that $B^+(m)$ has at least three components, which is not true.

Having ruled out case (γ), we shall now prove that (β) cannot hold unless $x_0 = 0$. In fact, the inequality $x_0 > 0$ would imply $x_v(u) \equiv 0$ on I, and then the unique continuation principle would yield $x(w) \equiv x_0$ in $\overline{B}$ if (β) were true. This is again not possible.

Therefore the relation $x_0 > 0$ implies that we are in case (α), and the proof is completed. $\square$

1.5 Nonparametric Representation, Uniqueness, and Symmetry of Solutions

the following representation theorem which will be proved in Section 1.9 is the key to all the other results of this chapter. It states that *all stationary minimal surfaces* X *in* $\mathcal{C}(\Gamma, S)$ *are graphs.*

Theorem 1 (Representation theorem). *Let* X *be a stationary minimal surface in* $\mathcal{C}(\Gamma, S)$*, and let* x_0 *be the lowest* x*-level of the free trace of* X*, that is,* $x_0 := \min\{x(u) : u \in I\}$*. Moreover, denote by* $D = D(x_0)$ *the two-dimensional domain in the* x, y*-plane which is obtained from the interior of the orthogonal projection* γ *of* Γ *by slicing this interior along the* x*-axis from* $x = x_0$ *to* $x = a$*. In defining the boundary* $\hat{\partial} D$ *of the slit domain* D*, both borders of the slit* $x_0 < x \leq a$ *will appear, with opposite orientation.*

Then the functions $x(w), y(w)$ *provide a* C^1*-mapping of* $\overline{B}$ *onto* $D \cup \hat{\partial} D$ *which is topological, except in case* II*, where the interval of coincidence*

$$I_2 = \{u \in I : x(u) = 0\}$$

corresponds wholly to the point $(0, 0)$ *on* $\hat{\partial} D$*. Moreover, the minimal surface* $\mathcal{M}$ with the position vector $X(w)$ *admits a nonparametric representation* $z = Z(x, y)$ *over the domain* D*. The function* $Z(x, y)$ *is real analytic in* D*, and on both shores of the open segment* $x_0 < x < a$*, and*

$$\lim_{y \to +0} \frac{\partial}{\partial y} Z(x, y) = \lim_{y \to -0} \frac{\partial}{\partial y} Z(x, y) = 0 \quad \text{for } x_0 < x < a. \tag{1}$$

$Z(x, y)$ *is continuous on* $D \cup \hat{\partial} D$ *in cases* I *and* III*. In case* II*,* $Z(x, y)$ *is continuous on* $D \cup \hat{\partial} D \setminus \{(0, 0)\}$ *and remains bounded upon approach of the point* $(0, 0)$*.*

As we shall immediately see, this result implies the following

Theorem 2 (Uniqueness theorem). *If* X_1 *and* X_2 *are two stationary minimal surfaces in* $\mathcal{C}(\Gamma, S)$ *which are normed in the same way, say,* $X_1, X_2 \in \mathcal{C}^*(\Gamma, S)$*, and whose free traces* $X_1(\mathrm{I})$ *and* $X_2(\mathrm{I})$ *have the same lowest* x*-levels, then*

$$X_1(w) \equiv X_2(w) \quad \text{on } \overline{B}.$$

In particular, two stationary minimal surfaces in $\mathcal{C}^*(\Gamma, S)$ *coincide on* $\overline{B}$ *if both are not of type* III*.*

Let $X(w) = (x(w), y(w), z(w))$, $w = u + iv$, be a stationary minimal surface in $\mathcal{C}^*(\Gamma, S)$. Then also

$$\hat{X}(u + iv) := (x(-u + iv), -y(-u + iv), -z(-u + iv))$$

is a stationary minimal surface in $\mathcal{C}^*(\Gamma, S)$, and the surfaces X and $\hat{X}$ have the same lowest x-levels. Then the uniqueness theorem implies that $X(w) \equiv \hat{X}(w)$ on $\overline{B}$, and we obtain

Theorem 3 (Symmetry theorem). *Every stationary minimal surface* $X \in \mathcal{C}^*(\Gamma, S)$ *is symmetric with respect to the* x*-axis. More precisely, we have*

$$x(u+iv) = x(-u+iv), \tag{2}$$

$$y(u+iv) = -y(-u+iv), \tag{3}$$

$$z(u+iv) = -z(-u+iv). \tag{4}$$

In cases I *or* III *we have*

$$x_0 = x(0) \quad \textit{and} \quad x_0 < x(u) < a \quad \textit{for } u \in I, u \neq 0.$$

In case II, I_2 *is of the form* $[u_1, u_2]$, *where* $0 < u_2 < 1$ *and* $u_1 = -u_2$. *Clearly, relations* (3) *and* (4) *imply*

$$y(iv) = z(iv) = 0 \quad \textit{for all } v \in [0,1]. \tag{5}$$

Finally, the nonparametric representation $z = Z(x,y)$ *of the minimal surface* $\mathcal{M}$, *given by* $X \colon \overline{B} \to \mathbb{R}^3$, *satisfies*

$$Z(x,y) = -Z(x,-y) \quad \textit{for } (x,y) \in D(x_0),$$

and therefore also

$$\lim_{y \to +0} Z(x,y) = -\lim_{y \to -0} Z(x,y), \quad x \neq 0,$$

in case II.

Now we come to the *proof of Theorem 2*. The domain D introduced in the representation theorem is the same for X_1 and X_2, even if the diffeomorphisms $B \to D$ given by the first two components differ. Therefore we have the nonparametric representations $z = Z_1(x,y)$ and $z = Z_2(x,y)$ respectively with $(x,y) \in D$, for the two surfaces X_1 and X_2. The functions $Z_1(x,y)$ and $Z_2(x,y)$ have the properties stated in Theorem 1 and satisfy

$$Z_1(x,y) = Z_2(x,y) \quad \text{for all } (x,y) \in \gamma, \tag{6}$$

where γ is the projection of Γ onto the x,y-plane. We will show that Z_1 and Z_2 coincide in D.

For $j = 1, 2$, we set

$$p_j := \frac{\partial}{\partial x} Z_j, \quad q_j := \frac{\partial}{\partial y} Z_j, \quad W_j := \sqrt{1 + p_j^2 + q_j^2}.$$

For fixed $(x,y) \in D$ and for $t \in [0,1]$ we introduce the notations

$$\begin{aligned} p(t) &:= p_1 + t(p_2 - p_1), \\ q(t) &:= q_1 + t(q_2 - q_1), \\ W(t) &:= \{1 + p^2(t) + q^2(t)\}^{1/2}, \end{aligned}$$

as well as

$$f(t) := (p_2 - p_1)\left\{\frac{p(t)}{W(t)} - \frac{p_1}{W_1}\right\} + (q_2 - q_1)\left\{\frac{q(t)}{W(t)} - \frac{q_1}{W_1}\right\}.$$

Note that $f(0) = 0$. Then, in view of the mean value theorem, there is some $t = t(x, y) \in (0, 1)$ such that

$$f(1) = f'(t). \tag{7}$$

Furthermore, a brief calculation yields

$$f'(t) \geq W^{-3}(t)[(p_2 - p_1)^2 + (q_2 - q_1)^2].$$

Since $(W^2(t))'' \geq 0$, we obtain

$$f'(t) \geq (\max\{W_1, W_2\})^{-3}[(p_2 - p_1)^2 + (q_2 - q_1)^2]. \tag{8}$$

For $\delta > 0$ and $\varepsilon > 0$ we now introduce the set $D_{\delta,\varepsilon}$ consisting of all points in D the distance of which from $(x_0, 0)$ and $(a, 0)$ exceeds ε, and whose distance from $\hat{\partial} D$ is greater than δ. Let Q be an arbitrary compact subset of $D_{\delta,\varepsilon}$, and set

$$m(Q) := \max\{W_1(x, y), W_2(x, y) \colon (x, y) \in Q\},$$

and

$$I(Q) := \int_Q [(p_2 - p_1)^2 + (q_2 - q_1)^2]\, dx\, dy.$$

Invoking (7) and (8), we arrive at

$$I(Q) \leq m^3(Q) \int_Q f(1)\, dx\, dy \leq m^3(Q) \int_{D_{\delta,\varepsilon}} f(1)\, dx\, dy.$$

Inserting

$$f(1) = (p_2 - p_1)\left[\frac{p_2}{W_2} - \frac{p_1}{W_1}\right] + (q_2 - q_1)\left[\frac{q_2}{W_2} - \frac{q_1}{W_1}\right],$$

and applying an integration by parts, we obtain that

$$I(Q) \leq m^3(Q) \int_{\partial D_{\delta,\varepsilon}} (Z_1 - Z_2)\left[-\left(\frac{q_1}{W_1} - \frac{q_2}{W_2}\right) dx + \left(\frac{p_1}{W_1} - \frac{p_2}{W_2}\right) dy\right].$$

Letting δ decrease to zero, keeping ε fixed, we infer from the boundary conditions (1) and (6) that

$$I(Q) \leq m^3(Q) \int_{C_\varepsilon} (Z_1 - Z_2)\left[-\left(\frac{q_1}{W_1} - \frac{q_2}{W_2}\right) dx + \left(\frac{p_1}{W_1} - \frac{p_2}{W_2}\right) dy\right] \tag{9}$$

where C_ε denotes the parts of the circles $\{x^2+y^2=\varepsilon^2\}$ and $\{(x-a)^2+y^2=\varepsilon^2\}$ which are contained in $D \cup \hat{\partial} D$. Since the integrand of the right-hand side of (9) is bounded, the line integral tends to zero as $\varepsilon \to 0$ whence $I(Q)=0$ for every compact subset Q of D. It follows that

$$\nabla Z_1(x,y) \equiv \nabla Z_2(x,y) \quad \text{in } D,$$

and therefore also

$$Z_1(x,y) \equiv Z_2(x,y) \quad \text{in } D,$$

on account of (6). Consequently X_1 and X_2 are conformal representations of the same nonparametric minimal surface $\mathcal{M}$, with the same parameter domain B and satisfying the same three-point condition. From this we conclude that $X_1(w) \equiv X_2(w)$ because a conformal map of B onto itself has to be the identical map if it leaves three points on ∂B fixed. □

1.6 Asymptotic Expansions for Surfaces of Cusp-Types I and III. Minima of Dirichlet's Integral

The central result of this section is the following

Theorem 1. *Minima of Dirichlet's integral in* $\mathcal{C}^*(\Gamma, S)$ *are not of type* III.

In Chapter 4 of Vol. 1 we have proved that there is always a solution of the minimum problem in $\mathcal{C}^*(\Gamma, S)$. By Theorem 1, this minimum has to be of type I or II. On the other hand, the uniqueness theorem of Section 1.5 states that there is at most one stationary minimal surface in $\mathcal{C}^*(\Gamma, S)$ if surfaces of type III are excluded. Hence Theorem 1 implies the following result:

Theorem 2. (i) *Stationary minimal surfaces in* $\mathcal{C}(\Gamma, S)$ *furnish the absolute minimum of Dirichlet's integral in* $\mathcal{C}(\Gamma, S)$ *if and only if they are of type* I *or* II.

(ii) *There exists one and only one minimum of Dirichlet's integral in* $\mathcal{C}^*(\Gamma, S)$.

Hence the stationary surfaces of type III constructed in Section 1.4 do not minimize Dirichlet's integral within $\mathcal{C}(\Gamma, S)$.

A proof of Theorem 1 can be based on the following asymptotic expansions for surfaces of type I or III:

Theorem 3. *Let* $X(w) = (x(w), y(w), z(w))$ *be of class* I *or* III. *Then* $w=0$ *is a first order branch point of* $X(w)$, *and we have the expansion*

$$\begin{aligned} x(w) &= x_0 + \operatorname{Re}\{\kappa w^2 + \cdots\}, \\ y(w) &= \operatorname{Re}\{i\kappa w^2 + \cdots\}, \\ z(w) &= \operatorname{Re}\{\mu w^{2n+1} + \cdots\}, \end{aligned} \tag{1}$$

where $\kappa > 0$, μ *is real and* $\neq 0$, *and* n *is an integer* ≥ 1.

Proof. We note that $\nabla y(0) = 0$, since $y_u(u) = 0$ for all $u \in I$, and $y_v(0) = 0$ by (5) of Section 1.5. In cases I and III we have

$$x_u^2(u) + z_u^2(u) = y_v^2(u) \quad \text{for all } u \in I. \tag{2}$$

Combining this identity with $y_v(0) = 0$, we conclude that $x_u(0) = 0$ and $z_u(0) = 0$. Since $x_v(u) = z_v(u) = 0$ for all $u \in I$ in cases I and III, we see that $\nabla x(0) = \nabla z(0) = 0$.

Since $x(u) > x_0$ for $0 < |u| \leq 1$, the arguments employed in the proofs of Lemma 2 and the Theorem of Section 1.4 lead, for small w, to an expansion

$$x(w) = x_0 + \text{Re}\{\kappa w^2 + \cdots\}, \quad \kappa > 0. \tag{3}$$

Hence, $w = 0$ is a branch point of order one for X.

From the relation $z_v(u) = 0, u \in I$, it follows that $z(w)$ can be extended harmonically across the u-axis and that, in view of Section 1.5, (5), an expansion

$$z(w) = \text{Re}\{\mu w^m + \cdots\}$$

is obtained in which μ is real and $\neq 0$ and m is an integer ≥ 2. Formula (4) of Section 1.5 shows that this integer must be odd so that, near $w = 0$,

$$z(w) = \text{Re}\{\mu w^{2n+1} + \cdots\}, \quad n \geq 1. \tag{4}$$

Recall now that the vector $A \neq 0$ appearing in the general expansion formula

$$X(w) = X_0 + \text{Re}\{A w^n + \cdots\}$$

satisfies $\langle A, A \rangle = 0$. Therefore we obtain, in conjunction with the formulas (2), (3) and the relations $y_v(0) = 0$, $y(u) = 0$ on I, the following local expansion for $y(w)$:

$$y(w) = \text{Re}\{\pm i\kappa w^2 + \cdots\}.$$

Here the plus sign must be chosen because $y_v(u) < 0$ for $0 < u < 1$. This follows from E. Hopf's lemma if one notes that $y(w) \leq 0$ on the boundary of the set $Q = \{w \colon |w| < 1, u > 0, v > 0\}$, so that by virtue of the maximum principle $y(w) < 0$ for $w \in Q$. □

Proof of Theorem 1. Because of Section 1.5, (5), $y(iv)$ vanishes for all $v \in [0, 1]$. Since $x(0) = x_0 \geq 0$, and $x(i) = -b < 0$, there exists a smallest number v_1 in $[0, 1)$ such that $x(iv_1) = 0$. Suppose now that X is a solution of the minimum problem in $\mathcal{C}^*(\Gamma, S)$ which is of type III. Then, $0 < v_1 < 1$. Denote by B' the slit domain obtained by cutting the semidisk B along the imaginary axis from $w = 0$ to $w = iv_1$. Furthermore, let $w = \tau(\zeta)$ be the conformal mapping from B onto B', leaving the three points $w = +1, -1, i$ fixed. Then, $Y(\zeta) = X(\tau(\zeta))$ is again of class $\mathcal{C}^*(\Gamma, S)$ since $y(iv) = 0$ for all $v \in [0, 1]$. From the invariance of the Dirichlet integral with respect to conformal mappings we conclude that $Y(\zeta)$ is also a solution of the minimum problem in $\mathcal{C}^*(\Gamma, S)$, but of type I, by virtue of the Theorem in Section 1.4. By (1), $Y(\zeta) = (y^1(\zeta), y^2(\zeta), y^3(\zeta))$ possesses an expansion near $\zeta = 0$ of the form

$$(5) \qquad \begin{aligned} y^1(\zeta) &= \mathrm{Re}\{\kappa\zeta^2 + \cdots\}, \\ y^2(\zeta) &= \mathrm{Re}\{i\kappa\zeta^2 + \cdots\}, \\ y^3(\zeta) &= \mathrm{Re}\{\mu\zeta^{2n+1} + \cdots\}, \end{aligned}$$

where $\kappa > 0, \mu \neq 0$ and $n \geq 1$. Let $\zeta = \alpha + i\beta$. We infer from (5) that the images of suitable segments $(-\varepsilon, 0)$ and $(0, \varepsilon), \varepsilon > 0$, on I under the mapping $Y(\zeta)$ are different, that is, $y^3(-\alpha) \neq y^3(\alpha')$ if $0 < \alpha, \alpha' < \varepsilon$. On the other hand relation (5) in Section 1.5, $z(iv) = 0$ for $0 \leq v \leq 1$, implies that $y^3(\alpha) = 0$ for $0 \leq |\alpha| \leq \varepsilon', \alpha \in I$, if ε' is a sufficiently small positive number. Such a discrepancy is not possible, and X cannot be of type III. □

Finally we shall give *another proof of Theorem 1* without using the expansion formula. The symmetry theorem of the previous section shows that the minimum X in $\mathcal{C}^*(\Gamma, S)$ maps the interval $\{w = iv \colon 0 \leq v \leq 1\}$ onto the x-axis. If X is of type III, that is, if $x_0 > 0$, then also the value

$$v_1 := \inf\{v \geq 0 \colon x(iv) \leq 0\}$$

is positive. Now let τ be the conformal mapping from B onto the slit semidisk $B - \{iv \colon 0 \leq v \leq v_1\}$ mapping each of the points $i, 1, -1$ onto itself. Since the Dirichlet integral is conformally invariant, we conclude that

$$X \circ \tau =: Y = (y^1, y^2, y^3)$$

is another minimum for the Dirichlet integral in $\mathcal{C}^*(\Gamma, S)$, but Y is of type I. Because of formula (5) in Section 1.5, the third component $z(w)$ of the minimum X vanishes for $w = iv$, $0 \leq v \leq 1$. Therefore the third component $y^3(w)$ of $Y(w)$ satisfies

$$y^3(u, 0) = 0 \quad \text{and} \quad y^3_v(u, 0) = 0$$

on certain intervals $(-\delta, 0)$ and $(0, \delta), \delta > 0$, which are mapped by τ onto the slit $\{iv \colon 0 < v < v_1\}$. The reflection principle implies that $y^3(w) \equiv 0$ on B, which is impossible. □

1.7 Asymptotic Expansions for Surfaces of the Tongue/Loop-Type II

The aim of this section is the proof of the following

Theorem. *Let* $X(w) = (x(w), y(w), z(w))$ *be a stationary minimal surface in* $\mathcal{C}^*(\Gamma, S)$ *which is of type* II, *and let* $[u_1, u_2]$ *be its set of coincidence* $I_2, -1 <$

$u_1 < u_2 < 1$. (*It follows from formula* (2) *of Section* 1.5 *that* $u_2 = -u_1 > 0$.) *Then there are positive numbers* κ *and* μ, *and a real number* $z_1 \neq 0$, *such that*

$$
(1) \qquad \begin{aligned} x(w) &= \operatorname{Re}\{i\kappa(w-u_1)^{3/2} + \cdots\} \\ y(w) &= \operatorname{Re}\{-i\mu(w-u_1) + \cdots\}, \\ z(w) &= \operatorname{Re}\{z_1 - (\operatorname{sign} z_1)\mu(w-u_1) + \cdots\} \end{aligned} \qquad \textit{near } w = u_1,
$$

and

$$
(2) \qquad \begin{aligned} x(w) &= \operatorname{Re}\{\kappa(w-u_2)^{3/2} + \cdots\}, \\ y(w) &= \operatorname{Re}\{i\mu(w-u_2) + \cdots\}, \\ z(w) &= \operatorname{Re}\{-z_1 - (\operatorname{sign} z_1)\mu(w-u_2) + \cdots\}. \end{aligned} \qquad \textit{near } w = u_2,
$$

Moreover, no point on I *is a branch point of* $X(w)$.

Proof. Let $h(w)$ be the holomorphic function in a neighbourhood of $w = u_1$ in B satisfying $h(u_1) = 0$ such that $x(w) = \operatorname{Re} h(w)$, and

$$g(w) = h'(w) = x_u(w) - ix_v(w).$$

If $u \in I$ is close to u_1, we have $\operatorname{Re} g(u) = 0$ for $u > u_1$, and $\operatorname{Im} g(u) = 0$ for $u < u_1$. Consider the transformation $w = u_1 + \zeta^2$, and set $f(\zeta) = g(u_1 + \zeta^2)$. The function $f(\zeta)$ is holomorphic near $\zeta = 0$ in $\{\zeta\colon \operatorname{Re}\zeta > 0, \operatorname{Im}\zeta > 0\}$, and $\operatorname{Re} f(\zeta)$ vanishes on the positive real ζ-axis, while $\operatorname{Im} f(\zeta)$ is zero on the positive imaginary axis. The C^1-character of $x(w)$ in $\overline{B}$ allows us to extend $f(\zeta)$ by a twofold reflection analytically to a holomorphic function in a full neighbourhood of the point $\zeta = 0$, with an expansion

$$f(\zeta) = a_0 + a_1\zeta + a_2\zeta^2 + \cdots \quad \text{near } \zeta = 0.$$

The relations $x_u(u_1) = x_v(u_1) = 0$ imply that $a_0 = f(0) = 0$. For $v \geq 0$ we then get the expansion

$$g(w) = a_1(w-u_1)^{1/2} + a_2(w-u_1) + a_3(w-u_1)^{3/2} + \cdots .$$

(We choose the branch of the square root which is positive for large positive values of w.) An integration leads to the expansion

$$x(w) = \operatorname{Re}\{b_0 + b_1(w-u_1)^{3/2} + b_2(w-u_1)^2 + b_3(w-u_1)^{5/2} + \cdots\}$$

with complex coefficients $b_j = p_j + iq_j$. From the relation $x(u) = 0$ for $u > u_1$ it follows that $p_0 = p_1 = p_2 = \cdots = 0$; we may also assume that $q_0 = 0$. The condition $x_v(u) = 0$ for $u < u_1$ allows us to conclude that $q_2 = q_4 = \cdots = 0$. Denoting the first non-vanishing coefficient of the remaining ones by $i\kappa$, we arrive at

$$x(w) = \operatorname{Re}\{i\kappa(w-u_1)^{n+1/2} + \cdots\}$$

where $(-1)^n\kappa < 0$, and $n \geq 1$. By virtue of formula (2) in Section 1.5 we also have the expansion

$$x(w) = \mathrm{Re}\{\kappa(w-u_2)^{n+1/2} + \cdots\}$$

for $w \in \overline{B}$ near the value u_2. Arguments similar to those employed in the proofs in Section 1.4 show that we have $n = 1$ in the above expansions. Thus we obtain

$$\text{(3)} \qquad \begin{aligned} x(w) &= \mathrm{Re}\{i\kappa(w-u_1)^{3/2} + \cdots\} \quad \text{near } w = u_1, \\ x(w) &= \mathrm{Re}\{\kappa(w-u_2)^{3/2} + \cdots\} \quad \text{near } w = u_2. \end{aligned}$$

The harmonic function $y(w)$ vanishes on I as well as for $w = iv$, $0 \leq v \leq 1$, while $y(e^{i\theta}) < 0$ for $0 < \theta < \frac{\pi}{2}$ and $y(e^{i\theta}) > 0$ for $\frac{\pi}{2} < \theta < \pi$. Consider the two sets

$$Q^- = \{w\colon |w| < 1, u > 0, v > 0\}$$

and

$$Q^+ = \{w\colon |w| < 1, u < 0, v > 0\}.$$

Since $y(w) \geq 0$ for $w \in \partial Q^+$ and $y(w) \leq 0$ for $w \in \partial Q^-$, the maximum principle implies that $y(w) > 0$ for $w \in Q^+$ and that $y(w) < 0$ for $w \in Q^-$. It then follows from E. Hopf's lemma that $y_v(u) > 0$ for $-1 < u < 0$ and $y_v(u) < 0$ for $0 < u < 1$, and hence $y_v(u_1) > 0, y_v(u_2) < 0$. Because $y(u) = 0$ for all $u \in I$, the function $y(w)$ can be extended as a harmonic function into the lower half of the w-plane. Near $w = u_1$, the above relations lead to an expansion

$$y(w) = \mathrm{Re}\{-i\mu(w-u_1) + \cdots\}$$

with a constant $\mu > 0$.

The conformality relation $|X_u|^2 = |X_v|^2$ yields $z_u^2(u_1) = y_v^2(u_1)$ so that $z_u(u_1) = \pm\mu$, while $z_v(u_1) = 0$. We set $z_1 = z(u_1)$ and $z_2 = z(u_2)$. Since $u_1 = -u_2$, formula (4) of Section 1.5 implies that $z_1 = -z_2$. Hence,

$$z(w) = \mathrm{Re}\{z_1 \pm \mu(w-u_1) + \cdots\} \quad \text{near } w = u_1,$$

$$z(w) = \mathrm{Re}\{-z_1 \pm \mu(w-u_2) + \cdots - z_1\} \quad \text{near } w = u_2.$$

The conformality relation $|X_u|^2 = |X_v|^2$ also implies that

$$z_u^2(u) = x_v^2(u) + y_v^2(u)$$

for $u \in I_2$, because $x_u(u) = 0$ for $u \in I_2$ and $y_u(u) = z_v(u) = 0$ for $u \in I$. Assume that $x_v(u') = 0$ for some $u' \in (u_1, u_2)$. Since $x(w)$ can be extended as a harmonic function across I_2, we would then obtain an expansion of the form

$$x(w) = \mathrm{Re}\{\alpha(w-u')^n + \cdots\}, \quad n \geq 2,$$

valid in a full neighbourhood of the point $w = u'$. Arguments similar to those employed earlier in conjunction with the properties of the expansions (3) show that this is impossible. Thus, $x_v(u) \neq 0$ for $u_1 < u < u_2$; in fact, we see from (3) that $x_v(u) < 0$ for $u_1 < u < u_2$. It now follows that the derivative $z_u(u)$ cannot vanish in the interval of contact, so that $z_1 \neq 0$. Since $z_2 = -z_1$, we have $z_u(u) > 0$ for $u \in I_2$ if $z_1 < 0$, and $z_u(u) < 0$ for $u \in I_2$ if $z_1 > 0$.

This completes the proof of the theorem. □

1.8 Final Results on the Shape of the Trace. Absence of Cusps. Optimal Boundary Regularity

An inspection of the foregoing proofs shows that the relations

$$y_v(u) > 0 \quad \text{for } -1 < u < 0,$$

$$y_v(u) < 0 \quad \text{for } 0 < u < 1$$

hold in all three cases I, II, and III. In conjunction with the two expansion theorems of Sections 1.6 and 1.7 we obtain the following result about the shape of the trace of a stationary minimal surface in $\mathcal{C}(\Gamma, S)$. This result exactly corresponds to the experimental observations in Section 1.1.

Theorem 1. *Let X be a stationary minimal surface in $\mathcal{C}(\Gamma, S)$. In cases* I *and* III*, the trace $X(u), u \in I$, is a real analytic curve which is regular except for the branch point $w = 0$ of order* 1*. In case* II*, X has no branch points on I, and the trace curve $X(u), u \in I$, is a regular curve of class $C^{1,1/2}$.*

From the expansion formulas of Section 1.6, (1), and Section 1.7, (1) and (2), it is apparent that the three generic forms of the trace $X(u)$, $u \in I$, for a solution X of the minimum problem in $\mathcal{C}^*(\Gamma, S)$ look as depicted in Fig. 1.

In conclusion, let us describe a situation in which the trace curve $X(u)$, $u \in I$, is free of cusps.

Theorem 2. *Suppose that the open subarc of the arc Γ with the end points P_1 and P_3 lies in the half-space $\{z < 0\}$, and that the open subarc of Γ between*

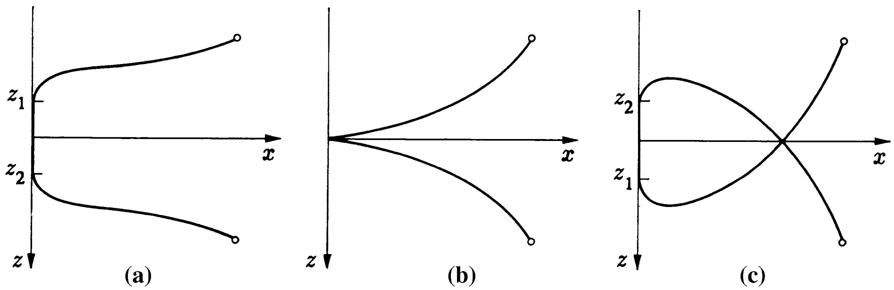

Fig. 1. (a) Case II, $z_1 < 0$ (tongue), **(b)** Case I (cusp), **(c)** Case II, $z_1 > 0$ (loop)

P_3 and P_2 is contained in the half-space $\{z > 0\}$. Then there exists exactly one stationary minimal surface X in $\mathcal{C}^*(\Gamma, S)$. This surface is of type II, *and its trace $X(u), u \in I$, on the half-plane S is a regular curve of class $C^{1,1/2}$ and has the form of a tongue.*

Remark 1. The expansions (1) and (2) of Section 1.7 show that the regularity class of a stationary surface of type II is exactly $C^{1,1/2}(B \cup I, \mathbb{R}^3)$ and no better on I, and Theorem 2 guarantees that there are surfaces of type II. Thus the principal regularity theorem from Chapter 2 of Vol. 2 cannot be improved.

Remark 2. The assumptions of Theorem 2 are satisfied if the z-component $p^3(s)$ of the representation $P(s)$ of Γ changes monotonously from $z = -c$ to $z = c$ as s moves from 0 to L; cf. Fig. 2. The situation is altered if Γ is deformed in such a way that $p^3(s)$ changes signs repeatedly (an odd number of times). After such a deformation, the trace may exhibit a cusp; see Fig. 3.

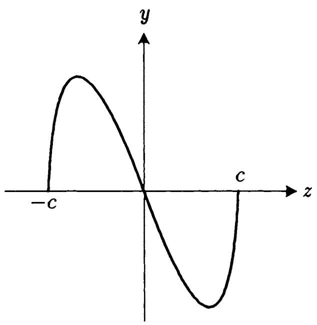

Fig. 2.

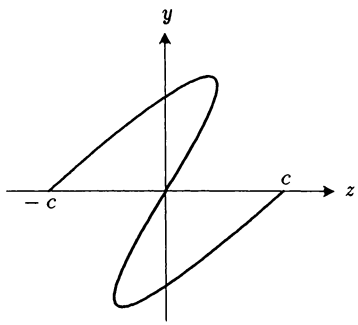

Fig. 3.

Proof of Theorem 2. We introduce the two arcs

$$C^+ := \left\{ w = e^{i\theta} : 0 < \theta < \frac{\pi}{2} \right\},$$

$$C^- := \left\{ w = e^{i\theta} : \frac{\pi}{2} < \theta < \pi \right\}.$$

Let $X(w) = (x(w), y(w), z(w))$ be the minimal surface under consideration. Then $z(w) > 0$ for $w \in C^+$ and $z(w) < 0$ for $w \in C^-$. Denote by Q^+ and Q^- the two components of the open set $Q = \{w \in B\colon z(w) \neq 0\}$ for which $C^+ \subset \partial Q^+$ and $C^- \subset \partial Q^-$ respectively. There cannot be further components of Q. In fact, if R were such a component different from Q^+ and Q^-, then $\partial R \subset B \cup I \cup \{i\}$. Moreover, $z(w) = 0$ at all boundary points of R in $B \cup \{i\}$. In view of the maximum principle, $z(w)$ cannot vanish everywhere on ∂R.

Hence, there is a point on I where $z(w)$ is different from zero, say, positive. Since Q^+ is adjacent to C^+ and Q^- is adjacent to C^-, the intersection $\partial R \cap I$ must be contained in a compact subinterval of I. Therefore, there is a point $u' \in I$ such that

$$z(u') = \max\{z(u) \colon u \in \partial R \cap I\} = \max\{z(w) \colon w \in \partial R\} > 0.$$

Clearly, a whole interval on I around u' is also contained in $\partial R \cap I$. Then, by E. Hopf's lemma, $z_v(u') < 0$, in contradiction to the relation $z_v(u) = 0$, which is valid for all $u \in I$.

Since Q^+ and Q^- are the only components of the set Q, we conclude from Section 1.5, (5) that

$$Q^+ = \{w \colon |w| < 1, u > 0, v > 0\}$$

and

$$Q^- = \{w \colon |w| < 1, u < 0, v > 0\}.$$

By means of arguments familiar from earlier occasions it is seen that $z_u(u)$ cannot vanish on the intervals $-1 < u < 0$ or $0 < u < 1$. In cases I or III the expansion (1) of Section 1.6 shows that a neighbourhood of $u = 0$ in B is divided into $2n + 2$ (and at least four) open sectors $\sigma_1, \sigma_2, \dots, \sigma_{2n+2}$ such that $z(w) > 0$ in $\sigma_1, \sigma_3, \dots, \sigma_{2n+1}$, and $z(w) < 0$ in $\sigma_2, \sigma_4, \dots, \sigma_{2n+2}$. From $Q = Q^+ \dot{\cup} Q^-$ we infer that this is impossible. Thus it follows from the above that the solution X must be of type II. Hence, be the uniqueness theorem of Section 1.5, the surface X is unique, and the description of the sets Q^+ and Q^- shows that the trace of X on the half-plane S has to be of the form of a tongue. This ends the proof of Theorem 2. □

1.9 Proof of the Representation Theorem

Now we want to supply the *proof of the representation theorem*, stated in Section 1.5, which is still missing. It will be based on a detailed discussion of the harmonic components $x(w), y(w), z(w)$ of the stationary minimal surface $X \in \mathcal{C}^*(\Gamma, S)$. For this purpose it is useful to recall the results of Sections 1.3 and 1.4 as well as the definitions of the subsets $B(\mu), B^+(\mu), B^-(\mu)$ of B and of the arcs $C_1^+(\mu), C_2^+(\mu), C^-(\mu)$ given in Section 1.4.

(i) We shall first pursue the discussion of case I assuming that $I_2 = \{u_0\}$. By Lemma 2 of Section 1.3, the functions $x(w), y(w)$, and $z(w)$ can be continued analytically as harmonic functions across the diameter I into the lower half of the w-plane. Since $x(u_0) = 0$ and $x(u) > 0$ for $u \neq u_0$, the function $x(w)$ must have an expansion

$$x(w) = \mathrm{Re}\{\kappa(w - u_0)^{2n} + \cdots\}$$

near $w = u_0$ where $\kappa > 0, n \geq 1$. A neighbourhood of $w = u_0$ in B is divided into $2n+1$ (and at least three) open sectors $\sigma_1, \sigma_2, \dots, \sigma_{2n+1}$ such that

$x(w) > 0$ in $\sigma_1, \sigma_3, \dots, \sigma_{2n+1}$, and that $x(w) < 0$ in $\sigma_2, \sigma_4, \dots, \sigma_{2n}$. Denote by $Q_1, Q_2, \dots, Q_{2n+1}$ the components of the set $B(0)$ which contain the sectors $\sigma_1, \sigma_2, \dots, \sigma_{2n+1}$, respectively. These components are mutually disjoint for topological reasons and because of the maximum principle. Then, by virtue of Lemma 2 of Section 1.4, it follows that $n = 1$ and that $B(0)$ consists of three different components Q_1, Q_2, Q_3. Clearly, $Q_2 = B^-(0)$. According to the remark following the same lemma we may assume that $C_1^+(0) \subset \partial Q_1, C_2^+(0) \subset \partial Q_3$. Since $x(u) > 0$ for $u \in I, u \neq u_0$, and since $x(1) = x(-1) = a > 0$, the interval $(u_0, 1)$ belongs to ∂Q_1 while the interval $(-1, u_0)$ is part of ∂Q_3. Then, by our standard reasoning, the gradient of $x(u)$ cannot vanish on I except for $u = u_0$. On the other hand, $x_v(u) = 0$ on I, so that $x_u(u) \neq 0$ for $u \neq u_0$. Therefore, the function $x(u)$ increases strictly from the value 0 to the value a as u increases from u_0 to 1, or decreases from u_0 to -1. Furthermore, $x_u(u) < 0$ for $-1 < u < u_0$, and $x_u(u) > 0$ for $u_0 < u < 1$. We observe finally that $x_u(u_0) = 0$, and that the expansion of $x(w)$ near the point $w = u_0$ must have the form

$$x(w) = \operatorname{Re}\{\kappa(w - u_0)^2 + \cdots\}, \quad \kappa > 0. \tag{1}$$

We assert that $|\nabla x(w)| > 0$ for all $w \in B$. Otherwise we would have $\nabla x(w_0) = 0$ for some $w_0 \in B$. Then, according to Radó's reasoning (cf. Lemma 2 of Section 4.9 in Vol. 1), the set $B(\mu)$ consists of at least four different components. This contradicts Lemma 2 in Section 1.4.

Next we consider the harmonic function $y(w)$. We have $y(e^{i\theta}) < 0$ for $0 < \theta < \frac{\pi}{2}$ and $y(e^{i\theta}) > 0$ for $\frac{\pi}{2} < \theta < \pi$, as well as $y(u) = 0$ for $-1 \le u \le 1$. As the angle θ increases from zero to π, the function $y(e^{i\theta})$ decreases from zero to its minimum value $y_{\min}$, then increases from $y_{\min}$ to its maximum value $y_{\max} = -y_{\min}$, and finally decreases again to zero. By conformality we have

$$x_u^2(u) + z_u^2(u) = y_v^2(u)$$

on I. Since $x_u(u) \neq 0$ for $u \neq u_0$, we see that $y_v(u) \neq 0$ for all $u \in I$, with the possible exception of $u = u_0$. It follows from the maximum principle that the open set $\{w \in B\colon y(w) \neq 0\}$ has exactly two components, Q^+ and Q^-, and that $y(w) > 0$ in $Q^+, y(w) < 0$ in Q^-. Applying once more Radó's argument, we infer that $|\nabla y(w)| > 0$ for all $w \in B$. Therefore, the two components Q^+ and Q^- are separated in B by an analytic arc $\mathcal{A}$ which has points in common with each horizontal line $v = \operatorname{Im} w = \text{const}$, $0 < v < 1$, considering that $y(w)$ changes signs in B along each such line. We claim that this arc, except for its end points, lies entirely in the domain $B^-(0)$, and that it has the end points $w = u_0$ on I and $w = i$ on ∂B.

As a first step we shall show that $y_v(1) < 0$ and $y_v(-1) > 0$. For this purpose recall that $X(w)$ can be extended to the full disk $\{w\colon |w| \le 1\}$ in such a way that $X(w)$ is the position vector of a minimal surface defined on $\{w\colon 1 - \delta_0 < |w| < 1\}$, for a suitable $\delta_0 > 0$.

In view of the boundary regularity results stated in Section 2.3 of Vol. 2, the surface X is of class $C^{1,\alpha}$ in $\{w \in \overline{B}\colon 1-\delta_0 \le |w| \le 1\}$. As the curve $X(e^{i\theta})$, $0<\theta<2\pi$, lies on a convex cylinder, the asymptotic expansion at boundary branch points (cf. Section 3.1 of Vol. 2) implies that our minimal surface cannot have branch points on the circular arc C. Hence it follows that

$$|X_u(e^{i\theta})|^2 = |X_v(e^{i\theta})|^2 > 0 \quad \text{for } 0 \le \theta \le \pi.$$

The arc Γ meets the half-plane S at right angles; therefore

$$X_v(e^{i\theta}) = (0, y_v(e^{i\theta}), 0) \quad \text{for } \theta = 0 \quad \text{and} \quad \theta = \pi.$$

Consequently, we have $y_v(\pm 1) \ne 0$; more precisely, $y_v(1) < 0$ and $y_v(-1) > 0$, since $y(e^{i\theta}) < 0$ for $0 < \theta < \frac{\pi}{2}$ and $y(e^{i\theta}) > 0$ for $\frac{\pi}{2} < \theta < \pi$. As we know, $y_v(u)$ cannot vanish on I for $u \ne u_0$. Therefore, $y_v(u_0) = 0$, and $y(w)$ has near $w = u_0$ an expansion

$$y(w) = \operatorname{Im}\{-\lambda(w-u_0)^n + \cdots\},$$

where $n \ge 2$, and λ is a real number different from zero. Since the set $\{w \in B : y(w) \ne 0\}$ has exactly two components, we see that $n = 2$ and $\lambda > 0$; that is, near $w = u_0$,

$$y(w) = \operatorname{Im}\{-\lambda(w-u_0)^2 + \cdots\}, \quad \lambda > 0. \tag{2}$$

The above results imply that the arc $\mathcal{A}$ which separates the components Q^+ and Q^- has as its end points (and only points on ∂B) the points $w = u_0$ and $w = i$.

Assume that $\mathcal{A}$, except for its end points, is not contained in $B^-(0)$. Then there is a point $w_1 \in B$ on this arc for which $x(w_1) \ge 0, y(w_1) = 0$. From the expansion (2) we see that near $u = u_0$, that is, for small positive values of ρ, the arc $\mathcal{A}$ has the representation

$$w = u_0 + \rho e^{i\theta(\rho)}, \quad \theta(\rho) = \frac{\pi}{2} + O(\rho).$$

It then follows from (1) that

$$x(w) = -\kappa\rho^2 + O(\rho^3)$$

for $w \in \mathcal{A}$ in a neighbourhood of $w = u_0$. Therefore, if we traverse the arc $\mathcal{A}$ from the point $w = u_0$ to the point $w = w_1$, we shall encounter a negative minimum for the function $x(w)$, restricted to $\mathcal{A}$. Assume that this minimum is attained at the point $w_2 \in B \cap \mathcal{A}$. Since $y(w) = 0$ on $\mathcal{A}$, and $\mathcal{A}$ is a regular arc, we have $x_u y_v - x_v y_u = 0$ at $w = w_2$. Thus, there exist numbers p and $q, p^2 + q^2 > 0$, satisfying the linear equations

$$p x_u(w_2) + q y_u(w_2) = 0, \quad p x_v(w_2) + q y_v(w_2) = 0.$$

In fact, $p \neq 0$ and $q \neq 0$, since $|\nabla x(w_2)| > 0$ and $|\nabla y(w_2)| > 0$. Consider the harmonic function

$$h(w) = p[x(w) - x(w_2)] + q[y(w) - y(w_2)] = px(w) + qy(w) + r,$$

where $r = -px(w_2)$. This function vanishes at $w = w_2$, together with its first derivatives. By Radó's lemma, $h(w)$ must have at least four distinct zeros on the boundary ∂B. On the other hand, since $pr = -p^2 x(w_2) > 0$, the straight line $px + qy + r = 0$ in the (x, y)-plane passes through the x-axis to the left of the origin and therefore intersects the boundary $\hat{\partial} D$ of the slit domain $D = D(0)$ in at most two points. Moreover, the functions $x(w), y(w)$ provide a topological mapping of ∂B onto $\hat{\partial} D$. Consequently, the function $h(w)$ vanishes on ∂B in at most two points. This is a contradiction to the previous statement. We have proved that the arc $\mathcal{A}$, except for its end points $w = u_0$, $w = i$, lies entirely in $B^-(0)$.

This fact will be used in the following way: Let $H(w)$ be a harmonic function in B of class $C^0(\overline{B})$ such that the open set $\{w \in B : H(w) \neq 0\}$ consists of exactly four components which are separated in B by four analytic arcs issuing from some point $w_1 \in B$. Suppose that two end points of these arcs lie on I, to the left and to the right of $w = u_0$, and two end points lie on C, to the left and to the right of $w = i$. Then, regardless of the location of the point $w = w_1$, the null set of the function $H(w)$ in B must contain two points w' and w'' in which

$$x(w') = 0, \quad y(w') > 0 \quad \text{and} \quad x(w'') = 0, \quad y(w'') < 0.$$

It can now be shown that the functions $x(w), y(w)$ provide a topological mapping from $\overline{B}$ to $D \cup \hat{\partial} D$. We already know that the relation between the boundaries ∂B and $\hat{\partial} D$ is a topological one and that interior points of B are mapped onto interior points of D. The bijectivity of the mapping follows from the monodromy principle once it has been shown that the Jacobian $\partial(x, y)/\partial(u, v)$ cannot vanish in B. Assume that $\partial(x, y)/\partial(u, v) = 0$ at some point $w_1 \in B$. Then, as before, there exist constants $p \neq 0$ and $q \neq 0$ satisfying the linear equations $px_u(w_1) + qy_u(w_1) = 0$ and $px_v(w_1) + qy_v(w_1) = 0$. It follows that the harmonic function

$$H(w) := p[x(w) - x(w_1)] + q[y(w) - y(w_1)] = px(w) + qy(w) + r,$$

$$r := -px(w_1) - qy(w_1),$$

and its first derivatives vanish at $w = w_1$. Radó's lemma implies that $H(w)$ must have at least four different zeros on ∂B. On the other hand, any straight line $px + qy + r = 0$, $p \neq 0$, in the x, y-plane intersects $\hat{\partial} D$ in at most four points. The case of four distinct points is only possible for $pr < 0$. Because of the bijectivity of the relation between ∂B and $\hat{\partial} D$ we conclude that $H(w)$ possesses exactly four different zeros on ∂B if $pr < 0$. Under the circumstances, the set

$$\{w \in B : H(w) \neq 0\}$$

consists of exactly four components which are separated in B by four analytic arcs issuing from $w = w_1$. Two end points of these arcs lie on I, to the left and to the right of $w = u_0$, and two end points lie on C, to the left and to the right of $w = i$. The observation formulated earlier implies that there are two points w', $w'' \in B$ for which

$$qy(w') + r = 0, \quad qy(w'') + r = 0, \quad y(w') > 0, \quad y(w'') < 0.$$

These relations are incompatible with the inequality $q \neq 0$, and we have proved that the functions $x(w)$, $y(w)$ furnish a topological mapping from $\overline{B}$ to $D \cup \hat{\partial} D$.

Let $w = \omega(x, y)$ be the inverse map, and set

$$Z(x, y) = z(\omega(x, y)), \quad (x, y) \in D \cup \hat{\partial} D.$$

The function $Z(x, y)$ provides a nonparametric representation

$$\{z = Z(x, y) : (x, y) \in D \cup \hat{\partial} D\}$$

of our minimal surface $X = X(w)$, $w \in \overline{B}$. $Z(x, y)$ is real analytic in D and on both shores of the open segment $0 < x < a$ of the x-axis (having of course different limits $\lim_{y \to +0} Z(x, y)$ and $\lim_{y \to -0} Z(x, y)$), continuous in $D \cup \hat{\partial} D$, and of class C^1 in $D \cup \hat{\partial} D$ except at the points $(0, 0)$ and $(a, 0)$. Given that

$$x_v(u) = z_v(u) = 0 \quad \text{on } I,$$

and that $x_u(u) \neq 0$, $y_v(u) \neq 0$ for $u \neq u_0$, $u \in I$, it also follows from the relation

$$\frac{\partial}{\partial y} Z(x, y) = \left. \frac{z_v x_u - z_u x_v}{x_u y_v - x_v y_u} \right|_{\omega(x,y) = w}$$

that

$$\lim_{y \to \pm 0} \frac{\partial}{\partial y} Z(x, y) = 0 \quad \text{for } 0 < x < a.$$

Thus, the proof of the theorem is completed for case I.

(ii) We turn now to a discussion of case II, assuming that I_2 is a closed interval $u_1 \leq u \leq u_2$, where $-1 < u_1 < u_2 < 1$. We know that

$$y(u) = y_u(u) = z_v(u) = 0 \quad \text{for } |u| < 1$$

as well as

$$x(u) > 0, \quad x_v(u) = 0 \quad \text{for } -1 \leq u < u_1 \quad \text{and} \quad u_2 < u \leq 1$$

and

$$x(u) = x_u(u) = 0 \quad \text{for } u_1 \leq u \leq u_2.$$

The functions $y(w)$ and $z(w)$ can be continued as harmonic functions across the diameter I into the disk $\{w\colon |w| < 1\}$. For the function $x(w)$ such a continuation is possible across the intervals $u_1 < u < u_2$, $-1 < u < u_1$ and $u_2 < u < 1$, but the resulting extended function will have isolated singularities at the points $w = u_1$ and $w = u_2$. Recall that $B^-(0)$ is connected and that $B^+(0)$ can have at most two components. From the situation at hand it follows that $B^+(0)$ consists of exactly two components and that we have the expansions

$$x(w) = \text{Re}\{i\kappa_1(w - u_1)^{3/2} + \cdots\}, \quad \kappa_1 > 0, \quad \text{near } w = u_1,$$

and

$$x(w) = \text{Re}\{\kappa_2(w - u_2)^{3/2} + \cdots\}, \quad \kappa_2 > 0, \quad \text{near } w = u_2.$$

The derivation of these expansions is based on the arguments employed for the proof of Section 1.7, (3), except that we are at the present stage not able to conclude that $u_1 = -u_2$ and $\kappa_1 = \kappa_2$.

From here on, we can follow the line of reasoning used in part (i) of the proof. We find that, as u decreases from u_1 to -1 or increases from u_2 to 1 the function $x(u)$ increases strictly from the value zero to the value a, and also that $x_u(u) \neq 0$ for $-1 < u < u_1$ and $u_2 < u < 1$. Since $x_u(u) = 0$ on (u_1, u_2) and since $x(w) \not\equiv \text{const}$, we also see that $x_v(u) \neq 0$, and therefore $x_v(u) < 0$ for $u_1 < u < u_2$. It can furthermore be proved again that $|\nabla x(w)| > 0$ for all $w \in B$.

As for the function $y(w)$, we see as in (i) that both sets $Q^+ = \{w \in B : y(w) > 0\}$ and $Q^- = \{w \in B : y(w) < 0\}$ are connected, and that $y_v(u) > 0$ near $w = -1$, and $y_v(u) < 0$ near $w = 1$. On $(-1, u_1) \cup (u_2, 1)$, we have $x_u^2(u) + z_u^2(u) = y_v^2(u)$, and $x_u^2(u) > 0$. Hence, $y_v(u) > 0$ for $-1 < u < u_1$, and $y_v(u) < 0$ for $u_2 < u < 1$. We claim that $y_v(u_1) \neq 0$, $y_v(u_2) \neq 0$. The assumption $y_v(u_1) = 0$ leads to $X_u(u_1) = X_v(u_1) = 0$, so that $w = u_1$ would have to be a branch point of X. However, the asymptotic expansion of $X(w)$ near a branch point does not allow for terms containing the power $(w - u_1)^{3/2}$. A similar contradiction arises from the assumption $y_v(u_2) = 0$. It follows that the derivative $y_v(u)$ must vanish somewhere in the interval (u_1, u_2). Since the set $\{w \in B : y(w) \neq 0\}$ has only two components, our standard reasoning shows that there exists exactly one point $u_0 \in (u_1, u_2)$ such that $y_v(u_0) = 0$. The expansion of $y(w)$ near $w = u_0$ is

$$y(w) = \text{Re}\{i\lambda(w - u_0)^2 + \cdots\}, \quad \lambda > 0.$$

It follows as in (i) that $|\nabla y(w)| > 0$ for $w \in B$ and that the Jacobian $\partial(x, y)/\partial(u, v)$ cannot vanish in B.

The functions $x = x(w)$, $y = y(w)$ provide a mapping between the boundaries ∂B and $\hat{\partial} D$. (Here $D = D(0)$ is the slit domain in the (x, y)-plane defined in the statement of the theorem.) This mapping is topological, except on the interval $[u_1, u_2]$ of I which corresponds wholly to the point $(0, 0)$ on $\hat{\partial} D$. From the non-vanishing of the Jacobian $\partial(x,y)/\partial(u,v)$ in B it follows that $x(w)$, $y(w)$ furnish a homeomorphism between B and D. A repetition of the further discussion of part (i) leads to the conclusion that the minimal surface $X = X(w)$, $w \in \overline{B}$, admits a nonparametric representation $z = Z(x,y)$. The function $Z(x,y)$ has the properties stated in the theorem.

(iii) Case III can easily be reduced to case I. For the purpose of this reduction, let $x_0 = \min\{x(u) \colon u \in I\}$, and suppose that $x_0 = x(u_0)$. Then, $x(u) > x_0$ for $u \neq u_0$, $u \in I$. We choose a new Cartesian coordinate system with coordinates ξ, η, ζ, defined by the relations $\xi = x - x_0$, $\eta = y$, $\zeta = z$; see Fig. 1 of Section 1.3. Introduce $\Gamma_0 := \Gamma \setminus (x_0, 0, 0)$ and the functions

$$\xi(w) := x(w) - x_0, \quad \eta(w) := y(w), \quad \zeta(w) := z(w),$$

and the surface $Y(w) = (\xi(w), \eta(w), \zeta(w))$. Furthermore let S be the half-plane $\{(\xi, \eta, \zeta) \colon \xi \geq 0, \eta = 0\}$. Then $Y(w)$ is a stationary minimal surface of type I in $\mathcal{C}^*(\Gamma_0, S)$. Applying part (i) of this proof to $Y(w)$, we may deduce the desired properties of X from those of Y by going back to the old coordinates x, y, z.

This completes the proof of the representation theorem. □

1.10 Scholia

1. *Remarks about Chapter 1*

Except for minor modifications and the second proof of Theorem 1 in Section 1.6, the results of this chapter and their proofs are taken from the paper [3] of Hildebrandt and Nitsche.

There remains the challenging problem to extend the results of this section to non-planar supporting surfaces S and, more generally, to arbitrary configurations $\langle \Gamma_1, \dots, \Gamma_k, S_1, \dots, S_l \rangle$. Experimental evidence indicates that it should be possible to prove similar results in the general case. A certain generalization is given in the following Chapter 2.

2. *Numerical Solutions*

So far we have not touched upon the problem of numerical solutions of boundary value problems for minimal surfaces. Both the nonparametric minimal surface equation

$$\operatorname{div} \frac{\nabla z}{\sqrt{1 + |\nabla z|^2}} = 0 \tag{1}$$

and the parametric equations

$$\Delta X = 0, \quad |X_u|^2 = |X_v|^2, \quad \langle X_u, X_v \rangle = 0 \tag{2}$$

have been treated. Here we mention that the partially free boundary problem for (2) with a planar support surface can effectively be solved by means of the finite element method, a comprehensive presentation of which can be found in the treatise of Ciarlet [1]. A numerical approach to partially free problems was given by Wohlrab [2,3]. (However, his formulae are in part faulty. For corrections, see e.g. A. Pape [1].) In addition we want to give some (rather incomplete) references to the literature concerning the numerical treatment of minimal surfaces.

The nonparametric equation (1) was dealt with by Concus [1–4] using a *finite difference scheme* and solving the resulting finite difference equations by a nonlinear *successive overrelaxation method.*

The *finite element method* was applied to minimal surfaces by many numerical analysts. We only mention the work of Mittelmann [1–6], Jarausch [1], Wohlrab [2,3], and Dziuk [9,10]. Whereas the first three authors used the variational formulation as a point of departure, Dziuk applied an iteration procedure suggested by the *mean curvature flow* of surfaces. Furthermore we refer to the work of Dziuk and Hutchinson [1–3], Hutchinson [1], Polthier [5,6], Dörfler and Siebert [1], and Hinze [1]. We also mention the purely computational work by Wagner [1,2] and Steinmetz [1].