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JEAN LERAY AND ALGEBRAIC TOPOLOGY

Armand Borel

I. 1933–1939

1. Leray's first contributions to mathematics belong to fluid dynamics. ([1931a] to [1933a]). The joint paper with Juliusz Schauder [1933b] marks his first involvement with algebraic topology. It follows the same pattern as earlier work of Schauder: proof of a theorem of algebraic topology in Banach spaces and applications to the existence of solutions of certain P.D.E. We first summarize briefly the parts of it most relevant here.

Schauder's results in algebraic topology pertain to a transformation of a Banach space B into itself of the form

$$(1) \quad \Phi(x) = x - F(x)$$

where F is a completely continuous map (i.e. transforms bounded sets into relatively compact ones) defined on B or sometimes only on a bounded subset. He extends to this situation two fundamental results of L.E.J. Brouwer in the finite dimensional case, namely

a) *A fixed point theorem:* if F is defined on, and maps into itself, the closure of an open non-empty bounded convex set, then it has a fixed point in it.

This is first proved in [13] (in a slightly different form) and used to show the existence of solutions for certain hyperbolic equations.

b) *Invariance of the domain:* if F is defined on B and Φ is bijective, then Φ maps any open set onto an open set, hence is bicontinuous.

It is applied to certain elliptic equations. The type of theorem obtained is roughly the following: if for some initial choice of data there is only one solution and if for nearby data there is at most one solution, then there is indeed one solution for data sufficiently close to the initial ones (*see* [14] for the precise statements).

In both cases, the proof of the topological theorem is a reduction to Brouwer's case by means of suitable finite-dimensional approximations.

2. The first part of the joint paper [1934c] with J. Schauder, announced in [1933b], is devoted to the definition and basic properties of a "topological degree" of Φ , again in analogy with Brouwer's work. Schauder had already used

the Brouwer index, but here the goal is to have a definition valid in a Banach space, for a transformation of the type (1) above, when F is defined on the closure $\bar{\omega}$ of an open bounded set ω (but does not necessarily leave it stable). Given a point b not on the image of the boundary ω' of $\bar{\omega}$, they define a topological index $d(\Phi, \omega, b)$ with the following natural properties:

1) If $\omega = \omega_1 \cup \omega_2$ where ω_1 and ω_2 are two disjoint bounded open sets, then

$$d(\Phi, \omega, b) = d(\Phi, \omega_1, b) + d(\Phi, \omega_2, b).$$

(ii) If $d(\Phi, \omega, b) \neq 0$, then $b \in \Phi(\omega)$.

(iii) The degree remains constant when ω and F vary continuously in such a way that b never meets $\Phi(\omega')$.

They also introduce an index $i(\Phi, a)$ of Φ at a point a which is isolated in its fiber $\Phi^{-1}(b)$, where $b = \Phi(a)$. It is an integer which, under some further technical assumptions on F , is equal to ± 1 . If $\Phi^{-1}(b)$ consists of finitely many points a_j , then $d(\Phi, \omega, b)$ is the sum of the $i(\Phi, a_j)$.

As in [13], [14], these results are proved by reduction to the finite-dimensional case. They are applied to a family of transformations

$$(2) \quad \Phi(x, k) = x - F(x, k)$$

depending on a parameter k varying in a closed interval K of the real line, where for each $k \in K$, the transformation $F(x, k)$ is as above, defined on $\overline{\omega(k)}$, and the union of the $\overline{\omega(k)}$ is assumed to be bounded in $B \times K$. The goal is to investigate the fixed points of $F(x, k)$, i.e. to find those x and k for which $\Phi(x, k) = 0$. To this end, the index $d(\Phi(x, k), \omega(k), 0)$ is examined. It is assumed that for some value k_0 of k in K , $F(x, k_0)$ has finitely many isolated fixed points, all in $\omega(k_0)$ and that $d(\Phi(x, k_0), \omega(k_0), 0)$ is not zero. If it is known on the other hand that $F(x, k)$ has at most finitely many fixed points, contained in some bounded set independently of k , then, under some further technical assumptions which I shall not state, it is shown that $F(x, k)$ has at least one fixed point for every $k \in K$, and that some of these fixed points form a family depending continuously on k . This result is then applied to a variety of functional or partial differential equations.

3. In [1935a] Leray brings a complement to the topological part of [1934c]. He gives a formula for the topological degree of the composition of two maps (of type (1)) and deduces from it first the invariance of domain, under assumptions somewhat more general than those of Schauder, and second a theorem about the number of bounded connected components of the complement of a bounded closed subset: it is the same for two closed bounded subsets F_1, F_2 if there exists a homeomorphism φ of F_1 onto F_2 such that the differences $\varphi(x) - x$ ($x \in F$) belong to some relatively compact subset.

The paper [1934c] reduces the proof of the existence of solutions to *a priori* majorations and also shows that, under suitable circumstances, local uniqueness implies global uniqueness. Leray's publications until 1939 provide many applications of these principles to fluid dynamics and P.D.E., for which I refer to Part II of these *Selecta*.

II. 1940–1945

4. The Second World War broke out in 1939 and J. Leray was made prisoner by the Germans in 1940. He spent the next five years in captivity in an officers' camp, Oflag XVIII A in Austria. With the help of some colleagues, he founded a university there, of which he became the Director (“recteur”). His major mathematical interests had been so far in analysis, on a variety of problems which, though theoretical, had their origins in, and potential applications to, technical problems in mechanics or fluid dynamics. Algebraic topology had been only a minor interest, geared to applications to analysis⁽¹⁾. Leray feared that if his competence as a “mechanic” (“mécancien”, his word) were known to the German authorities in the camp, he might be compelled to work for the German war machine, so he converted his minor interest to his major one, in fact to his essentially unique one, presented himself as a pure mathematician and devoted himself mainly to algebraic topology⁽²⁾.

The first major outcome of this work is the series of three papers [1945a,b,c], the three parts of a “course in algebraic topology taught in captivity”, announced in part in [1942a, b, c, d].

In describing them and subsequent work, I shall use the current terminology, which has been standard for the last 45 years or so, but also indicate the one proposed by Leray. This should not hide the fact that many of these concepts were completely new at the time and underwent some variations before the present formulations were arrived at.

As was pointed out, the theorems of algebraic topology used by Schauder and Leray-Schauder reviewed above were all proved by reduction to the finite-dimensional case by suitable approximations. The first main goal of Leray was to build up a theory of equations and transformations directly applicable to more general topological spaces. This required a new definition of homology. Leray was also keen not to use any subdivision of complexes, simplicial approximations, orientability assumptions and not to assume the spaces to be quasilinear ([1945a], p.97–98). Before getting to these papers, I shall first describe Leray's starting point, as outlined in *loc.cit.*

5. Until about 1935, the basic objects of algebraic topology were the homology groups, usually defined for simplicial complexes, though more general concepts had been introduced by Vietoris and E. Čech, mainly for compact spaces. Around 1935, it was discovered by several people (J. Alexander, E. Čech, Kolmogorov, H. Whitney) that a product adding degrees could be introduced on complexes dual to those defining homology, without assuming the underlying

space to be a manifold (in which case a product could be defined by Poincaré duality from the intersection product in homology). The de Rham theorems, expressing the homology of compact smooth manifolds in terms of differential forms, had already shown that this product could be defined directly by means of the exterior product of differential forms. With this example in mind, Alexander realized that also in the general case these new complexes could be defined directly, not as duals to some pre-existing ones and that homology groups could in turn be viewed as dual objects to the new groups. His definitions of the new complex, boundary operator and product were indeed inspired by exterior differential calculus [1].

Leray adopted Alexander's point of view and minimized, almost suppressed from 1946 on, the use of the traditional homology groups. They occur mainly for comparison purposes or for the treatment of a generalization of manifolds and of Lefschetz numbers. He never lost sight of the analogy with E. Cartan's exterior differential calculus, of which he had acquired first-hand knowledge by writing up for publication the notes of a course given by E. Cartan [5]. In the introduction to [1945b], he points out that his "forms on a space" (*see 6*) obey most of the rules of the calculus of Pfaffian forms and states that the main interest of that paper seems to him to be its treatment of a problem in topology, alien to any assumption of differentiability, by computations of that nature.

Early on, H. Whitney had proposed to call cohomology groups and cup product the new groups and the product [17], a suggestion which was soon rather widely adopted, but not by Leray until 1953. Prior to that, the word cohomology occurs only in two C.R. Notes. Leray kept to homology otherwise, prefacing several of his later papers, including the main ones, by stating that he would call homology what is usually referred to as cohomology since he will deal exclusively with the latter. I shall use cohomology.

Another requirement for the new cohomology ring to be defined was that it should allow one to carry over to more general spaces the proof of the theorem of H. Hopf, to the effect that a compact connected manifold endowed with a continuous product satisfying certain conditions (for instance, defining a group structure) has the same rational homology as a product of odd-dimensional spheres.

6. In [1945a] Leray first defines a notion of *abstract* complex over a ring L (either \mathbb{Z} , or $\mathbb{Z}/m\mathbb{Z}$ or \mathbb{Q}): a graded free finitely generated L -module, with a coboundary operator d increasing the degree by one (n° 7). It is called a simplex if it is acyclic. A *concrete* complex K on a space E (I shall simply say complex on E) is an abstract complex, to each basis element X of which is assigned a non-empty subset $|X|$ of E , its support. The support of a linear combination is then, by definition, the union of the supports of the basis elements occurring in it and it is required that $|dX| \subset |X|$. If F is a subspace of E , the intersection $F.K$ of F and K is the quotient of K by the submodule of elements with support not meeting F , the support of the image $F.X$ of X being $F \cap |X|$. Let K' be another complex on E . For $x \in E$ there is a natural homomorphism $r_x : K \otimes K' \rightarrow xK \otimes xK'$. The support of $Z \in K \otimes K'$ is then the set of $x \in E$

for which $r_x(Z) \neq 0$. The *intersection* $K \circ K'$ of K and K' is, by definition, the quotient of $K \otimes K'$ by the submodule of elements with empty support, endowed with the obvious supports.

Next, Leray introduces (the first version of) an important notion in all of his work in topology, that of “couverture”: a complex K on E with *closed* supports such that xK is acyclic for all $x \in E$ and the sum of the basis elements of degree zero is a cocycle, called the unit cocycle of K . A linear combination of elements of K , with coefficients in L , as above, is a “form on E ”. I do not know of any translation of *couverture* in the mathematical literature. In later presentations of the theory, beyond Leray’s work, it appears in such a disguised form that I shall neither need nor venture a translation, and simply use the French word [in sheaf theory, the sheaf associated to a *couverture* would be a resolution of the constant sheaf with stalk L].

The notion of *couverture* is stable under product, intersection, intersection with a closed subset and inverse image. The stability under product or intersection is a consequence of an algebraic argument we shall come to in **9**, which is fundamental for the whole paper and later developments.

Let E be a normal space. The union of the *couvertures* on E , with coefficients in L , is shown to be an L -algebra, with respect to sum and a product defined via the above intersection product. Its cohomology is, by definition, the cohomology ring of E , with coefficients in L , to be denoted here $H^*(E; L)$. It is not compared in this paper with the definitions of Alexander and Kolmogorov. [In [1945b], Leray states he cannot do so for lack of documentation.] It is mostly used for compact spaces (with an extension to differences of such, *see 7*).

A *cover* of E is a collection of subsets, the union of which is E . To a finite closed cover of E is associated an abstract complex, its nerve, which is made into a complex on E by assigning to a simplex as support the intersection of the subsets represented by its vertices. This is a *couverture*, the *couverture* generated by the given cover. To compute $H^*(E; L)$ it suffices to consider a cofinal family of finite closed covers and all the *couvertures* obtained by iterated intersections from those defined by their nerves. More precisely, the constructions in nos 16 to 18 present the cohomology ring as a direct limit of cohomology rings of *couvertures* associated to the nerves of suitable finite closed covers. They could be replaced by slightly bigger open finite open covers with the same nerves, so that, for compact spaces, the cohomology is essentially equivalent to Čech cohomology. For normal, not locally compact spaces, it does not seem to me that this type of cohomology has been considered elsewhere. It will also not occur later in Leray’s work. Except on one point (*see 9*) I shall not discuss technical details at all, since modifications of the definitions often led to simpler and more powerful arguments. Leray then establishes many properties of his cohomology ring. A first immediate consequence of the construction is Theorem 12, p. 122, according to which every cohomology class of strictly positive dimension is nilpotent. A compact space is said to be simple if it is acyclic. A fundamental result (Theorem 6, p. 126) asserts that a compact space which is a deformation retract of one of

its points is simple. If a *couverture* has simple supports, then the cohomology groups of E are those of the underlying abstract complex (Theorem 12, p. 138). This shows in particular that the cohomology of a finite polyhedron is dual to the usual homology. If the non-empty intersections of the elements in a cover are simple, in which case the cover is said to be “convexoïd”, then there is a finite procedure to determine the cohomology *ring* (n° 37).

§22 is devoted to a Künneth rule and §§23 to 25 to generalizations of the theorems of Hopf mentioned earlier and of Samelson. Theorem 2 quoted above allows one to adapt to compact connected spaces with a product the argument of Hopf’s, to the effect that a homogeneous indecomposable element (Leray says “maximal cycle”) has odd dimension.

Let E be a compact connected space endowed with a convexoïd finite cover by closed subsets, and ξ a continuous map of E into itself (a representation of E into itself in Leray’s terminology). Chapter III associates to ξ a Lefschetz number Λ_ξ . It is defined simplicially, using the *couverture* defined by the given cover and its dual complex. It is the same for two homotopic continuous maps (Theorem 16, p. 162). The space E is said to be convexoïd if it has a fundamental set of closed neighborhoods which are simple as well as all their non-empty intersections [this is the definition given in [1953a], the one here is slightly different in formulation, but equivalent.] It is shown that if $\Lambda_\xi \neq 0$, then ξ has at least one fixed point (Theorem 17, p. 163). If E is moreover a topological group, then n° 44 generalizes a theorem of Hopf on the degree of the k -th power map.

7. The first part of [1945b], Chapter IV, is concerned with relations between the cohomology of a normal space E , a closed subspace F and the difference $U = E - F$. By definition, the latter is the “cohomology of the interior of U ”, i.e. the cohomology of the sub-complex of the complex defining $H^*(E; L)$ consisting of elements with support in U . It is not quite a topological invariant, since the supports are subsets of U which are closed in E . However, it is in the (main) case where E is compact, because these subspaces are just the compact subsets of U so they have an internal characterization. He then shows that $H^*(E; L)$, $H^*(F; L)$ and $H^*(U; L)$ are related by a long exact cohomology sequence (not in this language, but by proving three times that a kernel is an image). This leads to a generalization of Alexander duality, modulo an identification of the duals of Leray’s cohomology groups with the usual homology groups, discussed in sections 35 of [1945a] and 66 of the present paper. Next, n°s 56–59 consider the case where E is the union of two closed subsets and establish the existence of a Mayer–Vietoris sequence.

For a locally compact space, the cohomology introduced by Leray will turn out to be equivalent to the Alexander–Spanier cohomology with compact supports. As Leray points out, it is non-trivial for the line and so it is not true that a non-compact space and a deformation retract have in general the same cohomology. To remedy this, he introduces in section IV another type of cohomology groups, based on the notion of “pseudo-cycle”. As far as I know it plays a role in the 1945 paper but disappears from his later treatment, so, again, I shall just

keep his terminology. A “pseudo-cycle” on the normal space E is an operator which assigns to each compact subset of B of E a cohomology class of B , the assignment being compatible with intersection by a closed subset. They can be added, intersected and form a graded L -algebra. [It is in fact the projective limit of the cohomology of the compact subsets of E , with respect to inclusions.] If any two elements of E are contained in a compact connected subset, then $H^0 \cong L$. Leray also defines the “pseudocycles in the interior of U ”, in case U is open in E and proves the existence of a long exact cohomology sequence. With respect to that cohomology, Euclidean space is acyclic and more generally E and a deformation retract have isomorphic cohomology rings.

Chapter V is devoted to manifolds, Poincaré duality, computes the cohomology of projective spaces and discusses the relations between the cohomology of a closed subset of the n -sphere and of its complement, in particular proves the Jordan–Brouwer theorem.

8. The first part of [1945c], Chapter VI, generalizes the Leray-Schauder theory in the framework of the cohomology theory developed in [1945, a, b].

Let E be a convexoid space (see 6), O an open subset of E and ξ a continuous map in E of a closed subset F of E . Leray defines an index $i(O)$, related to the fixed points of ξ contained in O . It is an integer, equal to the Lefschetz number Λ_ξ of ξ if $O = E$. It is defined if the closure \bar{O} of O belongs to F and there is no fixed point in $\bar{O} - O$. It is zero if there is no fixed point in O and is invariant under continuous deformation (Theorem 22, p. 212). If E is moreover acyclic, it depends only on the restriction of ξ to $\bar{O} - O$. In fact, n° 88 provides a direct definition using only the restriction of ξ to $\bar{O} - O$.

Next the definition and properties of $i(O)$ are extended to different, or apparently different, situations, in particular to the following one: E is a topological space (not necessarily convexoid), F a closed subspace, T a convexoid space and $\xi = \varphi \circ \tau$ is a composition of continuous maps

$$\xi : F \xrightarrow{\tau} T \xrightarrow{\varphi} E.$$

The total index $i(O)$ of ξ is then by definition the index $i(\varphi^{-1}(O))$ of $\tau \circ \varphi$ (nos 81, 82, pp. 223-225).

As is pointed out on p. 213, these results contain the Leray-Schauder theory, and the applications include the theorems of existence and/or uniqueness of solutions in [1934c] as well as in [1933c], [1936a], [1939].

Some of the main results on $i(O)$ and on its relation with the Lefschetz number were reproved and generalized in [1959c]. Notably, the assumption made several times in the present paper that the cohomology of F is finitely generated is dropped⁽³⁾.

Chapters VII and VIII are devoted to algebraic topology. In particular, the following situation is considered: E and E' are topological spaces, T an acyclic

convexoid space of homeomorphisms of E into E' , F a closed subspace of E and $\tau : F \rightarrow T$ a continuous map. Let F' be the set of points $\tau(x).x$ ($x \in F$). Then it is shown that the group of pseudocycles of $E - F$ and $E' - F'$ are isomorphic (Theorem 35, p. 245). If $E = E'$ is euclidean space and $\tau(x)$ is the translation bringing x to $\tau(x).x$, this yields the Alexander theorem and the invariance of the domain. Finally, a generalization of the Fredholm alternative is also proved.

9. These three papers first of all fulfill Leray's initial main goals, namely, to set up a cohomology theory (Chapters I, II, IV) and use it as a framework for a theory of equations encompassing the one of Leray-Schauder (Chapters III, V, VI). In addition they prove and generalize a number of theorems in algebraic topology, mostly known in some form, though sometimes derived in quite novel ways and greater generality (chapters VII, VIII). However, Leray realized that he could go much further, as hinted in a footnote to [1945c] p. 201. This led to the work announced in [1946a, b], which broke entirely new ground.

The starting point is an argument which occurs repeatedly in [1945a]. Its first goal was to prove that the "forms on a space" (see **6**) obey some of the rules of exterior differential calculus (cf. the introductory remarks in [1945b] quoted above in **5**). According to [1950a] p. 9 or [1959c], p.10, it is the analysis of this argument which led Leray to the cohomological invariants of a continuous map, described initially in [1946b]. Its first occurrence is in lemma 2, n^o 4: given two abstract complexes C and C' , where C is acyclic and has a unit cocycle U , it asserts that the cohomology of the product is naturally isomorphic to that of C' . Let z be a cochain of degree $m > 0$. It is a sum of expressions $u^a \times v^b$, where u^a is a cochain of degree a of C and v^b a cochain of degree b of C' ($a + b = m$). It is said to be of weight q if q is the maximum of a . Assume $q \geq 1$. If z is a cocycle, then the sum of the terms of weight $q + 1$ in dz is zero, and it follows that z is cohomologous to a cocycle of weight $q - 1$ hence, by induction on q , to a cocycle of weight zero. As to those, it is easily seen that the map $c' \mapsto u \times c'$ induces an isomorphism of $H^*(C')$ onto the space of cocycles of weight zero modulo coboundaries, which proves the lemma. It is first used to prove that a product of couvertures is again one and then applied to more complicated situations ([1945a], n^{os} 17, 27, 32). Theorem 12, p. 138, quoted in **6**, is also an application of that principle, to be referred to as the fundamental argument or fundamental lemma.

Prop. 10.4 in [1950a] provides a translation in terms of spectral sequences and it is only in that form that it is used there⁽⁴⁾. For the reader familiar with the notion of filtration and spectral sequences, we sketch the proof of lemma 2 in those terms (see also remark 2, p. 8, 9 of Exp. VI in [3]).

Let $S = C \times C'$. Filter S by $S^{-p} = \bigoplus_{q \geq p} C^q \times C'$.

Then $E_{-1}^p = S^{-p}/S^{-p-1}$, the differential d_{-1} is the partial differential with respect to C . The induction on the weight shows that $E_0^p = 0$ for $p \neq 0$, hence $E_0^* = E_\infty^* = GH(S) = E_0^0 = H(C')$, and the second part.

III. 1946–1950

10. We now come to the two seminal Notes [1946a, b]. The first one introduces sheaves, cohomology with respect to a sheaf and the cohomology ring of a continuous closed map $\pi : E \rightarrow E^*$ of normal spaces. Leray wanted to associate to π a cohomology ring of E^* with respect to the “variable coefficients” $H^*(\pi^{-1}y)$, ($y \in E^*$)⁽⁵⁾. This led formally to the notions just listed.

A *sheaf* \mathcal{B} on E is a functor which associates to every *closed* subset F of E a module (or ring, as the case may be) $\mathcal{B}(F)$, which is zero if F is empty, and to each inclusion $F' \subset F$ a homomorphism $r_{F',F} : \mathcal{B}(F) \rightarrow \mathcal{B}(F')$ with the usual transitivity properties. It is called normal if $\mathcal{B}(F)$ is the inductive limit of the $\mathcal{B}(F')$ ($F \subset F'$).

Let $b \in \mathcal{B}(F)$. It is said to be reducible if there exists a finite cover $\{F_i\}_{i \in I}$ of F such that $r_{F_i,F}b = 0$ for $i \in I$. The reduced sheaf defined by \mathcal{B} associates to F the quotient of $\mathcal{B}(F)$ by the module of reducible elements.

In a way, sheaves had already implicitly occurred in [1945a]: let K be a complex on E . To a closed subspace F , there is associated the section $F.K$ of K by F (see **6**), i.e. the quotient of K by the submodule of elements with support not meeting F . Clearly, the map $F \mapsto F.K$ defines a sheaf, which is normal. The stalk $x.K$ at $x \in E$ already played a considerable role in the theory, as we saw. Another important example of a normal sheaf is the q -th cohomology sheaf $\mathcal{B}^q : F \mapsto H^q(F; L)$.

A form on E with coefficients in \mathcal{B} is a finite linear combination $\sum_i b_i X_i$, where the X_i 's are basis elements of some *couverture* and $b_i \in \mathcal{B}(|X_i|)$. If \mathcal{B} is the constant sheaf $\mathcal{L} : \mathcal{B}(F) = L$, these are the forms on E of [1945a] and it is asserted that the constructions and results there generalize, whence the definition of the cohomology group (or ring) $H^*(E; \mathcal{B})$ of E with respect to the normal sheaf \mathcal{B} of modules (or rings).

These two examples show why it is natural in the present setup to view sheaves as functors from closed rather than open subspaces. Recall that the cohomology here is with compact supports, so that the assignment to open sets of their cohomology would lead to a “cosheaf”, where the natural maps go in the same direction as the inclusions.

By definition the sheaf $\pi(\mathcal{B})$ on E^* associates $\mathcal{B}\pi^{-1}(F^*)$ to $F^* \subset E^*$.

The last part of [1946a] introduces the cohomology ring of π . Let again \mathcal{B}^q be the q -th cohomology sheaf of E . The transform $\pi(\mathcal{B}^q)$ associates to F^* the q -th cohomology ring of $\pi^{-1}(F^*)$. Then the (p, q) -cohomology group of π is $H^p(E^*; \pi(\mathcal{B}^q))$ and the cohomology ring of π is the direct sum of these groups, endowed with the product inherited from those in the cohomology of E and of the closed subsets of E ⁽⁶⁾.

11. The Note [1946b] is devoted to the “structure of the cohomology ring of π ”. By that is meant a construction allowing one to relate it to the cohomology

of E . It is a first version of what became later known as the spectral sequence of π . Not all the features of the latter appear explicitly, but several essential ingredients already do.

We let $P_1^{p,q} = H^p(E^*; \pi(\mathcal{B}^q))$, and P_1 be the direct sum of the $P_1^{p,q}$. We shall call p the base-degree. The structure in question is defined by a sequence of submodules

$$(1) \quad 0 = Q_{-1}^{p,q} \subset Q_0^{p,q} \subset \dots \subset Q_{p-1}^{p,q} \subset P_{p+1}^{p,q} \subset \dots \subset P_2^{p,q} \subset P_1^{p,q}$$

of $P_1^{p,q}$, of submodules

$$(2) \quad 0 = E^{-1,p+1} \subset E^{0,p} \subset \dots \subset E^{p-1,1} \subset H^p(E; L) = E^{p,0}$$

of $H^p(E; L)$, and isomorphisms

$$(3) \quad \Delta_r : P_r^{p,q} | P_{r+1}^{p,q} \xrightarrow{\sim} Q_r^{p-r,q+r+1} / Q_{r-1}^{p-r,q+r+1}$$

$$(4) \quad \Gamma_{p,q} : P_{p+1}^{p,q} / Q_{q-1}^{p,q} \xrightarrow{\sim} E^{p,q} / E^{p-1,q+1}$$

allowing to get information on the successive quotients of the composition series (2) by successive approximations, starting from P_1 .

These modules and Δ_r are defined in terms of couvertures and of the action of the coboundary operator on these. The construction is quite intricate, only sketched there and I can only try to give some idea of it.

Fix $r \in [2, p+1]$. Let $x \in P_1^{p,q}$ represented by a form on E^* , i.e. a finite sum $\sum_{\alpha} z^{p,\alpha} C^{q,\alpha}$ where $z^{p,\alpha}$ is a cocycle on $\pi^{-1}(|X^{q,\alpha}|)$. Assume there is a form $L^{p,q}$ on E which can be written

$$L^{p,q} = \sum_{\alpha} L'^{p,\alpha} \pi^{-1}(X^{q,\alpha}) + \sum_{s>0; \lambda} L^{p-s,\lambda} \pi^{-1}(X^{q+s,\alpha})$$

such that $L'^{p,\alpha} \pi^{-1}(|X^{q,\alpha}|) \sim z^{p,\alpha}$ and that $dL^{p,q}$ can be written similarly, but with p replaced by $p-r$ and q by $q+r+1$, where the terms of base degree $p-r$ in $dL^{p,q}$ represent a class y in $P_1^{p-r,q+r+1}$. By definition, $P_r^{p,q}$ is generated by those x 's, $Q_r^{p-r,q+r+1}$ by the y 's and Δ_r associates y to x . By definition, Δ_r annihilates $P_{r+1}^{p,q}$. There is an ambiguity in the choice of $L^{p,q}$, which leads to view Δ_r as a map of $P_r^{p,q} / P_{r+1}^{p,q}$ onto $Q_r^{p-r,q+r+1} / Q_{r+1}^{p-r,q+r+1}$.

If $L^{p,q}$ can be chosen to be a cocycle, then $x \in P_{p+1}^{p,q}$ (and conversely), the class $[x]$ belongs to $E^{p,q}$ by definition and $\Gamma_{p,q}x$ is its image in $E^{p,q} / E^{p-1,q+1}$.

In short, $E^{p,q} / E^{p-1,p+1}$ is a subquotient of P_1 . It is arrived at by successive approximations, by means of a descending induction on the base-degree. This is reminiscent of the fundamental lemma, of course, but going from the latter to this construction is obviously a "giant step". It is only sketched in this Note,

was never described in more detail, so that it is hardly possible, at least for this writer, to see how it could effectively be used. The results announced in this and the following Note, proved before Leray arrived at the next, and final, formulation of his theory show that he did.

12. The last part of [1946b] gives some applications. First an analog in this context of a theorem of Vietoris: if E^* is compact and $\pi^{-1}(x)$ is acyclic for all $x \in E^*$, then π induces an isomorphism of $H^*(E^*; L)$ onto $H^*(E; L)$. If π is the projection of a locally trivial fiber bundle, with typical fiber F and E^* is simply connected, the Betti numbers of E are majorized by those of $E^* \times F$. Finally, if E is a compact simply connected Lie group, F a closed one-parameter subgroup, L a field of characteristic zero, then $H^*(E/F; L)$ is obtained from $H^*(E; L)$, which is an exterior algebra on odd degree generators, by replacing one factor $\wedge x_{2i+1}$ by $L[x]/(x^{i+1})$, where x has degree two. (In fact, $i = 1$, [11b]).

13. In the following year, the theory underwent a number of changes, partly under the influence of contributions by J-L. Koszul and H. Cartan. In [11a], Koszul gives a purely algebraic definition of the construction underlying [1946b], introducing what is now known as the spectral sequence of a filtered differential algebra A (in the case of a decreasing filtration). A (decreasing) filtration on an algebra A is defined by a sequence of two-sided ideals A^p ($p \in \mathbb{Z}$) such that

$$(1) \quad A^p \supset A^q \text{ if } p \leq q, \cup_p A^p = A, \cap_p A^p = \{0\}.$$

$$(2) \quad A^p \cdot A^q \subset A^{p+q}$$

and, if A is differential, $dA^p \subset A^p$ ($p \in \mathbb{Z}$). In that case, $H(A)$ is endowed with the filtration defined by the $H(A)^p$, where $H(A)^p$ denotes the subgroup of cohomology classes represented by a cocycle in A^p . The spectral sequence relates the grading ring $Gr A = \bigoplus_p A^p/A^{p+1}$ to $Gr H(A) = \bigoplus_p H(A)^p/H(A)^{p+1}$ by means of a sequence of graded algebras E_r ($r \geq 0$), where $E_0 = Gr A$ and E_{r+1} is the cohomology of E_r with respect to a differential d_r . If A is graded and d increases the degree by one, then E_r is bigraded and d_r increases the filtration degree by r , decreases a complementary degree by $r - 1$. Moreover E_r tends to $Gr H(A)$, and is equal to it if the filtration is bounded.⁽⁷⁾

In the notation of **11**, $E_r^{p,q} = P_{r-1}^{p,q}/Q_{r-2}^{p,q}$ [11b].

Early in 1947, H. Cartan noticed a formal similarity between the fundamental lemma of [1945a] and a proof of the de Rham theorems contained in a letter of A. Weil [15].⁽⁸⁾ This was his starting point towards an axiomatic cohomology theory, quite different from his previous approach (Comm. Math. Helv. **18** (1945), 1–15) which was much more in the mainstream of algebraic topology at the time.

14. Cartan and Leray lectured at a Colloquium in Paris, June 26–July 2, 1947, but the Proceedings, published in 1949 only, do not contain their original communications. The article of Leray [1949a] is “different in title and contents”

from the oral lecture and summarizes a lecture given in November 1947 and a course given at the Collège de France in 1947-48 (*see* the footnote on p. 61). Cartan withdrew his communication and replaced it by a short text, written in 1949, stating that his views had changed considerably, partly under the influence of [1946a] and of Leray's lecture at the Colloquium and he was preparing a full-fledged exposition (the subject matter of [7]).

The theory outlined in [1949a] is basically the final form, as can be seen from the systematic exposition [1950a]. It starts with algebraic notions: differential algebra, filtered ring and spectral sequence attached to a differential filtered ring, essentially as introduced in [11a], or [6a] with some technical differences, though: the subgroups defining the filtration need not be two-sided ideals and, in the differential case, are not necessarily stable under the differential. The spectral sequence (E_r) may therefore have non-trivial terms with negative index, which in a sense tend to $Gr A$ as $r \rightarrow -\infty$. If the filtration and grading are bounded, then $E_r = Gr A$ (resp. $E_r = GrH(A)$) for r sufficiently small (resp. big). The fundamental lemma is now embedded in some spectral sequence statements. [Initially, Leray allows a filtration by the real numbers \mathbb{R} , but uses only \mathbb{Z} when defining the spectral sequence and in his subsequent papers. Filtrations and spectral sequences indexed by \mathbb{R} were considered later by R. Deheuvels (Annals of Math. **61** (1995), 13–72), upon Leray's suggestion, in connection with the calculus of variations.]

Spaces are always locally compact. The notion of sheaf is as in [1946a], except that the condition “normal” is replaced by “continuous”. It is the same if the space X is compact, but stronger otherwise⁽⁹⁾. For instance, given a ring L , the constant sheaf which assigns L to every closed subset F , the transition homomorphisms being the identity, is continuous if and only if X is compact. If it is not, the sheaf associating L to compact subsets, and zero to non-compact ones, is continuous; it is called the “sheaf identical to L ”. The notion of sheaf is further extended to that of differential filtered sheaf.

Next complexes are defined. The original definition (*see* **6**), in which the supports are now assumed to be closed, is modified in two ways, proposed by H. Cartan in his lecture: a complex is not necessarily a free module and is moreover assumed to be endowed with a product adding the degrees, i.e. it is a differential graded ring (with closed supports). The most important innovation however is the introduction of *fine* complexes. This was done at the Colloquium lecture already, while Cartan proposed a similar notion in his own (*see* the already quoted footnote on p. 61): a complex K on X is fine if, given a finite open cover $\{U_\alpha\}$ of X by subsets which are relatively compact or with compact complements, there exists endomorphisms r_α of K , for the additive structure only, such that $\text{supp } r_\alpha k \subset U_\alpha$ for all $k \in K$ and the sum of the r_α is the identity. This then replaces a union of complexes with arbitrarily small supports (*see* **6**). The intersection $K \circ \mathcal{B}$ of a complex K with the sheaf \mathcal{B} is defined. It is a complex, which is fine if K is so. Similarly, the intersection $K \circ K'$ of two complexes K, K' is fine if one of them is so. The sheaf is assumed to be

continuous, which forces the elements of $K \circ \mathcal{B}$ to have compact supports. The cohomology ring $H^*(K \circ \mathcal{B})$ of X with respect to \mathcal{B} is by definition $H^*(K \circ \mathcal{B})$, where K is a fine couverture. It has of course to be shown to be independent of the choice of K , up to natural isomorphisms. To this effect the fundamental lemma, or some variant, is used to show that if K and M are fine couvertures, then $K \circ M$ is also one and the natural maps

$$K \circ \mathcal{B} \rightarrow K \circ M \circ \mathcal{B}, \quad M \circ \mathcal{B} \rightarrow K \circ M \circ \mathcal{B}$$

induce isomorphisms in cohomology. The construction of Alexander, modified by Čech, the initial inspiration for Leray (*see* 5) gives rise to a fine couverture, showing that when \mathcal{B} is the “sheaf identical to a ring L ” (*see* above), the cohomology $H^*(X \circ \mathcal{B})$ is the Alexander-Čech, also called Alexander-Spanier, cohomology of X with compact supports, coefficients in L .

The case of a differential filtered sheaf is also considered (*n^o* 23) and the homology sheaf $\mathcal{H}\mathcal{B} : F \mapsto H(\mathcal{B}(F))$, denoted there $\mathcal{F}\mathcal{B}$, is introduced. The group $H^*(K \circ \mathcal{B})$, computed with respect to a total differential (the “hypercohomology” with respect to \mathcal{B} in case the filtration is associated to a grading), is naturally filtered and is the abutment of a spectral sequence in which one term is $H^*(K \circ \mathcal{H}\mathcal{B})$. In the present set up, this is the *fundamental theorem of sheaf theory*. Again, the hypercohomology and the spectral sequence are independent of the fine couverture \mathcal{K} and define topological invariants.

A familiar consequence, not drawn here or in [1950a], but apparently in the original text⁽¹⁰⁾, pertains to homomorphisms of differential filtered sheaves. Let $\mu : \mathcal{B} \rightarrow \mathcal{C}$ be one. If it induces an isomorphism of $H(\mathcal{B}(x))$ onto $H(\mathcal{C}(x))$ for all $x \in X$, then it induces an isomorphism of $H^*(K \circ \mathcal{B})$ onto $H^*(K \circ \mathcal{C})$.

All this is valid in fact only under suitable boundedness assumptions on the degrees and filtrations under consideration, which I have ignored (e.g., it suffices that they be bounded in both directions).

Let $\pi : X \rightarrow Y$ be a continuous map and K_X (resp. K_Y) a fine couverture of X (resp. Y). Then $\pi^{-1}K_Y \circ K_X$ is a fine couverture of X . The spectral sequence of $Z = \pi^{-1}K_Y \circ K_X \circ \mathcal{B}$ with respect to the filtration defined by the degree in K_Y is by definition the spectral sequence of π . It relates the (hyper)cohomology of Y with respect to $\pi_*(K_X \circ \mathcal{B})$ to the cohomology of $H^*(X \circ \mathcal{B})$. In fact, given integers ℓ, m , Leray defines a filtration of Z , using m -times the degree in K_Y and ℓ times the degree in K_X , whence a spectral sequence for each choice of ℓ and m , but they are not essentially different. In studying fibre bundles Leray uses mostly the ones corresponding to $\ell = 0, m = 1$, or $\ell = -1, m = 0$. In the sequel, I shall always stick to the former. The r -th term of a spectral sequence is denoted \mathcal{H}_r by Leray. The index depends on the filtration: \mathcal{H}_r for the filtration $\ell = 0, m = 1$ is \mathcal{H}_{r-1} for the filtration $\ell = -1, m = 0$. [The construction of [1946b], *see* 11, is a precursor of the spectral sequence assigned to $\ell = -1, m = 0$.] As before, I shall use E_r .

The spectral sequence replaces the construction of [1946b]. The underlying idea is the same, but more easily described, notably because it starts with a complex on X , rather than the equivalent of the E_2 -term. Let

$$Z^{p,q} = \pi^{-1}K_Y^p \circ K_X^q \circ \mathcal{B} \text{ and } F_p Z = \sum_{i \geq p, q \geq 0} Z^{i,q}.$$

The $F_p Z$ define the filtration $\ell = 0, m = 1$. Let $z \in Z$. Its filtration degree is the biggest p such that $z \in F_p Z$. Cocycles are arrived at by successive approximations: one looks at $z \in F_p Z$ such that $dz \in F_{p+r} Z$ ($r \geq 0$). These elements form C_r^p . The latter contains C_{r-1}^{p+1} and $D_r^p := dC_r^{p-r}$; by definition

$$E_r = C_r^p / (C_{r-1}^{p+1} + D_r^p)$$

and d_r is induced by d . As r gets bigger, z is closer to a cocycle, and its actually one if r is greater than its total degree.

Except in [1950b], no groundring is specified and Leray speaks of filtered rings and spectral rings. He shifts to filtered algebras and spectral algebras in [1950c].

15. Assume now that a discrete group G acts freely and properly on X and that $Y = X/G$ is the quotient space. The map π has discrete fibres and the spectral sequence cannot give much information. However, Leray indicated in his lecture how to associate a spectral sequence to that situation when G is finite. This led to a joint paper with H. Cartan [1949b], which defines the spectral sequence of a finite regular covering of a locally compact space, relating Eilenberg-MacLane cohomology of G with coefficients in $H^*(X; L)$ to the cohomology of Y , in cohomology with compact supports, and to two C. R. Notes of Cartan, where the restrictions that G be finite and the supports be compact are lifted [6].

16. The paper [1950a], based on courses given at the Collège de France in 1947-48 and 1949-50, provides a comprehensive exposition of the theory. The overall plan is the same as that of [1949a] with many technical refinements I shall not go into, depending for instance on various assumptions made on the complexes under consideration and on whether filtrations are bounded or not. I content myself to mention some items not occurring in [1949a]. If X has finite cohomological dimension, then it carries a fine couverture with degrees bounded by the dimension n°40. The Mayer-Vietoris sequence attached to a cover by two closed sets is established in n°49. n°67 considers the effect of retractions on cohomology and discusses homotopic maps. n°69 to 73 are devoted to locally constant systems in Steenrod's sense and their relations with the fundamental group. Determination of the cohomology when X has a finite convexoid cover n°74, or is more particularly a finite polyhedron n°75, spectral sequence of a simplicial map between polyhedra n°77. The last two sections show that the spectral sequence of a map is not necessarily a homotopy invariant and give some indications on how to define homotopy invariants by means of these constructions.

This paper is the last one devoted by Leray to his theory of cohomology with compact supports of a locally compact space with respect to a sheaf and to the general properties of the spectral sequence of a continuous map. The former was considerably generalized by H. Cartan [9]⁽¹¹⁾. From the start, Leray applied the latter to fibre bundles, in particular to the study of the relations between the cohomology rings of a compact connected Lie group, a closed subgroup U and the quotient G/U . We now turn to these applications, backtracking a little since they began in 1946 already.

17. The work of Leray on fibre bundles and homogeneous spaces is contained in six C.R. Notes and two papers. Four of the C.R. Notes announce without proofs results established, often in greater generality, in one of the two papers. I shall therefore treat them rather briefly. The cohomology is usually with respect to a field C , of characteristic zero when homogeneous spaces are discussed.

Given a space with finitely generated cohomology, its Poincaré polynomial $P(E, t)$ (with respect to C) is, by definition

$$P(E, t) = \sum_{i \geq 0} \dim H^i(E; C) \cdot t^i$$

[1946c] considers first a map $\pi : E \rightarrow E^*$ as in [1946b] and describes some relations between Poincaré polynomials of E, E^* and the invariants of π defined in [1946b].

The remaining part of the Note is concerned with a locally trivial fibre bundle (E, B, F, p) with total space E , base B , typical fibre F and projection p . First some consequences of Poincaré duality are drawn when E, B, F are orientable compact connected manifolds. The last section gives sharper relations between the Poincaré polynomials of E, B, F when the cohomology rings of the fibres form a constant system. [1946d] describes the Poincaré polynomial of G/T , when G is a simple compact connected classical Lie group and T a maximal torus of G . This is pursued further in [1949e], where the rational cohomology of G/U is determined when G is locally isomorphic to a product of classical groups by a torus and U a closed subgroup of maximal rank. If U is connected $P(G/U, t)$ is given by a formula conjectured by G. Hirsch. If it is not and U^o is its identity component, then $H^*(G/U; C)$ may be identified to the invariants of U/U^o in $H^*(G/U; C)$, the operation of U/U^o being defined by right translations.

From 1949 on, the cohomology ring of a compact space X , with respect to coefficients which are clear from the context, is denoted \mathcal{H}_X . The Note [1949f] has three parts: the first one extends some of the results of the previous one to a compact space X which is a principal bundle for G . In particular, the projection $X/T \rightarrow X/G$ induces an isomorphism of $H^*(X/G)$ onto the invariants of the Weyl group $\mathcal{N}T/T$ in $H^*(X/T)$. If X is a group containing G as a subgroup, this reduces the study of $H^*(X/G)$ to that of $H^*(X/T)$. The second one (Theorem 2) determines the cohomology of G/S , where G is simple, classical, and S a

singular subtorus of codimension one in a maximal torus T of G . It is the tensor product of an algebra of even dimensional elements by an exterior algebra with one generator of degree equal to the maximum of the degrees of the primitive generators of $H^*(G; \mathbb{Q})$. The third part is devoted to sphere fibrations and describes how the Gysin exact sequence relates to the spectral sequence of the projection.

18. The two Notes [1949c] and [1949d] are somewhat apart and concerned with a topic Leray did not come back to (but is taken up again in [4]). There homology and cohomology (in characteristic zero) do occur and the standard terminology is used. By theorems of Hopf and Samelson, $H^*(G)$ and the homology algebra $H_*(G)$, where the product is the Pontrjagin product, are exterior algebras $\wedge P^*$ and $\wedge P$ over spaces of primitive elements, in natural duality. Assume G operates on a locally compact space X by means of a map $q : G \times X \rightarrow X$ and let $m : G \times G \rightarrow G$ be the product map. Then, by definition

$$q \circ m = q \circ q : G \times G \times X \rightarrow X.$$

By consideration of the corresponding maps in cohomology it is shown that a primitive homogeneous element $x \in P_a$ induces a differential δ_x of $H^*(X)$ decreasing the degree by a and this assignment extends to a homomorphism of $\wedge P^*$ into the algebra of graded endomorphisms of $H^*(X)$. If X is a compact orientable manifold and $c \in H^p(X)$ is dual to a submanifold of codimension p , then a submanifold dual to $\delta_x.c$ is described geometrically.

In the case $X = G/U$, where G is a compact connected Lie group, U a closed connected subgroup, $X = G/U$, this construction yields a new proof of a theorem of Samelson asserting that $\pi^*H^*(G/U)$ is a subalgebra generated by primitive elements, where $\pi : G \rightarrow G/U$ is the canonical projection. If U is of maximal rank, the Euler-Poincaré characteristic $\chi(G/U)$ of U is $\neq 0$, according to a theorem of Hopf and Samelson, hence π^* annihilates $H^i(G/U)$ for all $i > 0$.

The next Note considers more generally a projection $\pi : X \rightarrow Y$, where G acts on X and Y and commutes with π . Then the operations of $\wedge P^*$ on X and Y extend to differentials of the terms E_r of the spectral sequence of π , commuting with the differentials. Various consequences are drawn.

Let G and U be as in the previous Note, but U not necessarily of maximal rank. Let $\mathcal{N}U$ be the normaliser of U in G and M its identity component. Then $\mathcal{N}U/M$ operates freely by right translations on G/U . The Lefschetz number of $n \in \mathcal{N}U$ is $\chi(G/U)$ if $n \in M$, and is zero if $n \notin M$. Therefore the representation of $\mathcal{N}U/M$ induced in $H^*(G/U)$ is a multiple of the regular representation if the Betti numbers of G/U in odd degrees are all zero.

19. The paper [1950b] is devoted first to general properties of the spectral sequence of a fibre bundle (E, B, F, π) : structure of E_2 , case where the cohomology algebras of the fibres form a constant system, interpretation of π^* and of the restriction $r^* : H^*(E) \rightarrow H^*(F)$ in the spectral sequence, triviality of the spectral sequence in case r^* is surjective (i.e. F is totally non-homologous

to zero), various inequalities relating to Poincaré polynomials of E, B, F , etc. The last chapter discusses several special cases: F is a sphere, where generalizations of results of Gysin and of Chern-Spanier are obtained, B is a sphere, where the H.C. Wang exact sequence is proved, F is a product of even dimensional spheres and 2 is divisible in the coefficient ring, in which case F is totally non-homologous to zero, E, B, F are compact orientable manifolds.

20. The paper [1950c] uses the filtration $\ell = -1, m = 0$, for spectral sequences therefore E_r here stands for \mathcal{H}_{r-1} there. Cohomology is always with respect to a field of characteristic zero. G, T and U are as before, and $W = \mathcal{N}T/T$ is the Weyl group of G .

In the first part, U has the same rank as G . Results stated earlier for classical groups are now proved in general. The new ingredients are the theorem of Chevalley on invariants of finite reflection groups and an argument, supplied by this writer, showing that the Betti numbers of G/T vanish in odd degrees. The Hirsch formula giving $P(G/U, t)$ when U is connected is established. Let \mathcal{P}_T be the symmetric algebra over $H^1(T)$, where all the degrees are doubled. The group W operate on it. Let \mathcal{R}_G be the ideal generated by the invariants of W without constant term. It is shown that $H^*(G/T) = \mathcal{P}_T/\mathcal{R}_G$ and that $E_3 = E_\infty$ in the spectral sequence of the projection $G \rightarrow G/T$. The next section of the paper is devoted to the situation considered in [1949f] and establishes without restriction on G the theorems stated there. This reduces the study of $H^*(G/U)$ to that of $H^*(G/S)$, where S is a maximal torus of U . It may be assumed to be contained in T and the last section provides a theorem on the E_2 -term of the spectral sequence of the projection $G/S \rightarrow G/T$, with fibre T/S . It is of course equal to $H^*(G/T) \otimes H^*(T/S)$. However, using the results of the first part, Leray shows that it is isomorphic, as a differential algebra, with $\mathcal{P}_S \otimes H^*(G)$, endowed with an explicitly given differential d . Unpublished computations to prove Theorem 2 of [1949f] indicate that Leray had that picture in mind already then. Here it is particularly interesting because a theorem announced by Cartan in [8] implies that $H^*(G/S) = H^*(\mathcal{P}_S \otimes H^*(G))$. As a consequence $E_3 = E_\infty$, a fact which is clear if $\dim T/S = 1$, but not otherwise.

This is the last paper devoted by Leray to algebraic topology, a topic which had played in his work a minor role in the thirties, a major one in the forties, occurred only incidentally in it after 1950 and was profoundly influenced by Leray's contributions.

Notes

(¹) J. Schauder once wrote to Leray that he did not view himself as a topologist per se and commented in another letter: “I am, as you are, a man of the applications” (“Ich bin, so wie Sie, ein Mann der Anwendungen”), a remark quoted by Leray in [1979].

(²) The only exceptions where a course in analysis, based on the Notes of a Cours d’Analyse at the Ecole Polytechnique, brought by some prisoners who had been students there, and a course on special relativity (where “Einstein” became “Albert” whenever some member of the German staff was passing by during a lecture).

(³) In [10], A. Deleanu extends the theory to neighborhood retracts of convexoid spaces (which are not always convexoid), so that it also includes Lefschetz’s fixed point theorem for absolute neighborhood retracts.

(⁴) In [3], this lemma (Exp. I, Théorème 6), the main argument of which is called induction on the weight, is also used to give a first proof, without spectral sequences, of a main uniqueness theorem of [1949a] or [1950a], and to compare Leray’s cohomology ring with others (Exp. III, IV). It is also a main tool in Cartan’s first two versions of the theory (*see* Note⁽¹¹⁾).

(⁵) In the comments to [16], p. 526-27, Vol. II, of his Collected Papers, A. Weil recalls a short conversation in June 1945 with Leray, just back from captivity, in which Leray spoke of a homology theory with variable coefficients depending on the point, an idea he found quite striking and communicated shortly afterwards to H. Cartan.

(⁶) $\pi(\mathcal{B})$ and $\pi(\mathcal{B}^q)$ are the analogues of the direct image $\pi_*\mathcal{B}$ of \mathcal{B} and of the q -th right derived functor $R^q\pi_*\mathcal{B}$ of the direct image functor in the now standard sheaf theory.

(⁷) The term filtration is not used there. It was proposed later by H. Cartan, in print for the first time in [6a]. As to the E_r ’s, Koszul speaks of a sequence of homologies, Leray of a spectral ring, from [1949a] on, and of a spectral algebra in [1950c]. I shall use spectral sequence.

(⁸) We try here to compare them. Let M be a smooth connected manifold, N the nerve of the open cover constructed in [15] (or [16]), such that all non-empty intersections U_σ (σ simplex of N) are contractible (the analog of a convexoid cover in [1945a]). Let $A^{p,q}$ be the space of p -cochains of N which assign to a p -simplex σ the smooth differential q -forms on U_σ . The direct sum A^{\cdots} of the $A^{p,q}$ is a bigraded algebra, endowed with two commuting differentials

$$d : A^{p,q} \rightarrow A^{p,q+1} \quad \delta : A^{p,q} \rightarrow A^{p+1,q}$$

stemming from exterior differentiation and from the coboundary operator in N . Let $E^{p,q}$ (resp. $H^{p,q}$) be the subspace of $A^{p,q}$ spanned by the elements annihilated by $d\delta$ (resp. d or δ). (Weil's notation is different, his two superscripts are the total degree $m = p + q$ and q). Weil establishes isomorphisms

$$F^{0,m}/H^{0,m} = H_{DR}^m(M), \quad F^{m,0}/H^{m,0} = H^m(N),$$

$$F^{p,q}/H^{p,q} = F^{p+1,q-1}/H^{p+1,q-1} \quad (0 \leq q \leq m)$$

where $H_{DR}^m(M)$ refers to de Rham cohomology, which, by composition, yield an isomorphism of $H_{DR}^m(M)$ onto $H^m(N)$. Each step is quite similar to the key argument in the fundamental lemma, though there is no reason to believe that Weil was aware of it. On the other hand, it seems rather plausible (also to Weil) that the definition of the $A^{p,q}$ had been suggested in part by the idea of cohomology with variable coefficients. In fact, apart from the fact that Weil deals with an open rather than closed cover, $A^{p,q}$ is, in the framework of [1946a], the space of p -forms of the couverture N with coefficients in the sheaf of differential q -forms. However, the global strategy of the proof is different from that of Leray to establish uniqueness theorems, which amounts to compare the two objects under consideration to a third one, their intersection, rather than directly to one another. The algebra A'' admits a total differential $d - \delta$ (which Weil does not consider explicitly, but the sequences of cocycles of total degree m satisfying (I) in [16] are cocycles with respect to it). Then a descending induction on p and q would show that $H^m(A'')$ is isomorphic to $H_{DR}^m(M)$ and to $H^m(N)$. In [16], written later, the argument is further simplified by the use of homotopy operators, which even allow one to define directly maps in both directions between simplicial cochains and differential forms. Weil also shows in the same way that $H^m(N)$ is isomorphic to the m -th cohomology space of M in singular cohomology.

⁽⁹⁾ Let X be not compact and \bar{X} its one-point compactification. Given the sheaf \mathcal{B} on X , define the sheaf $\bar{\mathcal{B}}$ on \bar{X} by the rule $\bar{\mathcal{B}}(F) = \mathcal{B}(F \cap X)$, (F closed in \bar{X}). Then, by definition \mathcal{B} is continuous on X if $\bar{\mathcal{B}}$ is normal on \bar{X} .

⁽¹⁰⁾ In [2], n° 2, I state this is so. I do not remember whether I had seen the original text or had only been informed by Leray.

⁽¹¹⁾ This is the third version of Cartan's work on this topic. The first one [12], which is likely to be rather close to the oral lecture at the 1947 Colloquium, is also based on the notion of complex with supports. The main change with respect to the definition in [1945a] is the introduction of differential graded complexes, called gratings in analogy with a terminology of J. Alexander, which are graded algebras and fine. There are no sheaves as such but, as in [1945a], given a complex K , the functor assigning to a closed subset F the complex $F.K$ plays an important role. There are no spectral sequences. The uniqueness theorem is established for compact spaces, by means of an analogue of the consequence

of the fundamental theorem of sheaf theory mentioned in **14**: a homomorphism $K' \rightarrow L'$ of fine differential graded gratings which induces an isomorphism of $H^i(xK')$ onto $H^i(xL')$ for all $x \in X$ induces an isomorphism in cohomology (again under suitable boundedness conditions), in the special case where $H^i(xK)$ and $H^i(xK')$ are zero for $i > 0$.

The main argument to establish it, p.159–165, is patterned after the fundamental one of [1945a], outlined here in **9**. [No reference is indicated there, but this is acknowledged in the next version [7], Exp. XV, n° 7.] Applications to the de Rham theorems and the singular cohomology of HLC spaces are also given.

The second stage [7] is still devoted to locally compact spaces, but cohomology with closed supports is included (if the space is also paracompact). The basic notion in [7] is that of sheaf, defined as in [1946a], a condition similar to normality being embedded into the definition. Those sheaves correspond in fact to “presheaves” in current terminology. The distinction between presheaves and sheaves becomes important if cohomology with closed supports of non-compact spaces is to be included, and Cartan introduces the *completion* of the given sheaf, which would now be called the sheaf associated to, or defined by, a presheaf.

The notion “fine” is carried over to sheaves. A sheaf in which all transition homomorphisms are surjective (which is in fact the sheaf associated to the complex of sections on the whole space) is called a carapace, and the cohomology is defined by means of fine carapaces, in which $H^i(B(x))$ is acyclic for all $x \in X$, the counterpart of a fine couverture. Numerous examples are given and, once the uniqueness theorem is proved, many consequences are drawn, including Poincaré duality on manifolds, for cohomology with closed supports or with compact supports.

In [9] the theory is developed in much greater generality, with a stronger use of homological algebra, which Cartan was developing at the time with S. Eilenberg. X is only assumed to be regular. A sheaf is now defined as a functor on open subsets and injective resolutions are introduced. Cohomology is defined with respect to a family Φ of supports and the spectral sequence of a continuous map is also defined in that context. The fundamental theorem of sheaf theory (XIX, Thm. 3) is proved in full generality.

This exposition and Cartan-Eilenberg’s “Homological Algebra” (Princeton University Press, 1956) paved the way for the treatment of sheaf theory and spectral sequences in the framework of homological algebra by A. Grothendieck: *Sur quelques points d’algèbre homologique*, Tôhoku M. J. **9**, 1957, 119–221 and R. Godement: “Topologie algébrique et théorie des faisceaux”, Hermann, Paris 1958.

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Jean Leray's Contributions to the Theory of Partial Differential Equations

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Jean Leray is one of the leading mathematicians of the 20th century. A large part of his interests center on partial differential equations, especially those arising in mathematical physics. His investigations, some of them going back more than 60 years, still set the agenda of research in the fields in which he worked. The methods he has introduced have found their uses in far-flung areas of mathematics.

Leray's papers are well organized; each distinct result has a chapter of its own, and the chapters are divided into short sections devoted to particular technical aspects of the argument. Since a priori estimates lie at the heart of most of his arguments, many of Leray's papers contain symphonies of inequalities; sometimes the orchestration is heavy, but the melody is always clearly audible.

Leray has studied both stationary problems, mostly governed by elliptic equations, and time-dependent problems, governed by parabolic and hyperbolic equations. His 1933 dissertation, in the *Journal de Mathématiques Pures et Appliquées*, deals with stationary problems, using an abstract and extended version of Erhardt Schmidt's method of deformation and bifurcation. A wealth of applications are presented:

1) Carleman has shown that the equation $\Delta u = g(x)$ in a domain D , subject to the nonlinear boundary condition $du/dn = F(u)$, has a solution when F is an increasing function of u ; Leray removed this restriction on F .

2) Leray showed that the Dirichlet problem for the equation $\Delta u + hu^p = 0$ in the unit ball in 3 dimensions, u prescribed constant on the boundary, has a finite number of solutions when $p \leq 4$, but can have infinitely many solutions when $p \geq 6$. This foreshadows the notion of a critical exponent.

3) Chapters II and III of the dissertation are devoted to the study of steady rotating fluids in three dimensions that satisfy the Navier-Stokes equation. On the boundary of the domain containing the fluid the velocity is prescribed as the boundary values of some given divergence free vector field. The key to the proof of existence of such flows is an a priori estimate of the square integral of the first derivatives of the velocity field. Surprisingly, this estimate was obtained

by an indirect argument, that used the weak compactness of bounded sets of square integrable functions, as well as the strong L^2 compactness of functions whose first derivatives are uniformly bounded in L^2 . From this estimate Leray deduced uniform boundedness and equicontinuity of velocity and its first partial derivatives; this argument uses essentially the three-dimensionality of space. The results described here are extended to unbounded domains.

The remaining chapter of the dissertation is devoted to showing that the initial value problem for the Navier-Stokes equation in two dimensions has a unique regular solution in the whole space. In another lengthy paper in the same journal the following year, he proved the existence of regular solutions in bounded domains in the plane.

In the same year, 1934, Leray and Schauder devised the epoch-making method bearing their name, using deformations to prove the existence of solutions. This method extends Brouwer's notion of the degree of a mapping to identity plus compact mappings of infinite-dimensional spaces. Like its finite-dimensional counterpart, the degree remains invariant under continuous deformations at every point that is not the image of a boundary point. To apply this principle in a concrete situation, two sets of a priori estimates have to be made: one showing the compactness of the one-parameter family of mappings employed, the other showing that all points on a sphere of radius R are mapped into points outside of a sphere. In addition, one has to verify for a particular value of the parameter that the degree of the mapping is nonzero. Leray and Schauder gave a number of applications of their method to solve the Dirichlet problem for various classes of quasilinear second order elliptic equations; the norm they employ is the Hölder norm.

Leray returned to elliptic problems again and again; in a technically formidable paper in 1939 he showed how to use degree theory to construct solutions of boundary value problems for second order fully nonlinear elliptic equations in two variables, including the Monge-Ampère equation. In the sixties, in collaboration with J.L. Lions, he examined results of Vishik, and of Minty and Browder, from the point of view of degree theory in finite-dimensional space. In the seventies, he and Y. Choquet-Bruhat used a fixed point theorem to solve the Dirichlet problem for second-order elliptic equations in divergence form.

In a 1935 paper in *Commentarii Mathematici Helvetici*, Leray used degree theory to construct steady ideal fluid flow in the plane around an obstacle and its wake. The complex potential $f(z)$ maps the exterior $z = x + iy$ of the obstacle and its wake conformally onto the complex plane split along the positive real axis. According to Bernoulli's law the flow speed is constant, say $= 1$, along the wake; therefore $|df/dz| = 1$ there. It is useful to reformulate the problem in terms of the inverse mapping $z(f)$ from the split plane to the exterior of the obstacle and its wake. It is convenient to represent the split plane as the conformal image $f(\zeta)$ of the upper half of the unit disc, so that the two arcs of the wake are the images of the intervals $(-1, 0)$ and $(0, 1)$ respectively. Such a

mapping is given by

$$f(\zeta) = a \left[\frac{\zeta + \zeta^{-1}}{2} - \cos s_0 \right]^2,$$

a and s_0 real, $a > 0$. Since the flow velocity df/dz is nonzero except at the stagnation point on the obstacle, one can write $df/dz = e^{-iw}$. Regard w as a function of ζ in the half disc; then

$$\frac{dz}{d\zeta} = \frac{dz}{df} \frac{df}{d\zeta} = e^{iw} \frac{df}{d\zeta}.$$

Since the flow speed is 1 on the wake, $w(\zeta)$ is real on $[-1, 1]$. By the Schwarz reflection principle, $w(\zeta)$ can be extended to the whole unit disc, and can be represented by the Poisson-Schwarz formula in terms of the real part $\theta(e^{is})$ of w on the upper half of the unit circle $\zeta = e^{is}$, $0 \leq s \leq \pi$. This function $\theta(e^{is}) = \phi(s)$ has to be chosen so that $z(\zeta)$ maps the upper half of the unit circle onto the prescribed obstacle. This leads to an integral equation for $\phi(s)$, due to Henri Villat. The given obstacle can be deformed into a single vertical line segment; the flow around a line segment is known explicitly. Leray shows, using the Leray-Schauder theory, that Villat's integral equation has a solution for all obstacles during this deformation. The proof relies on a priori estimates in the Hölder norm, and uses some geometric restrictions on the original obstacle.

Ever since its appearance, the Leray-Schauder degree has been one of the most powerful methods for dealing with nonlinear problems. A quick search of the Mathematical Reviews disclosed 591 references to papers that make use of it.

We turn now to Leray's studies of time-dependent problems. In a paper that appeared in *Acta Mathematica* in 1934, Leray investigates the existence, uniqueness and smoothness of solutions of the initial value problem for the Navier-Stokes equation in three-dimensional space. Physicists sometimes deride such existential pursuits by mathematicians, saying that they stop just when things are getting interesting; but what Leray found about existence, smoothness and uniqueness of solutions was far more interesting for the physics of fluids than anything thought of before. He showed that in three space dimensions, smooth initial data give rise to solutions that are smooth for a finite time; these solutions may be continued beyond this time only as generalized (weak) solutions of the Navier-Stokes equations. Leray calls these *turbulent* solutions. He shows that if two solutions, one regular and the other turbulent, have the same initial values, then they are equal; but it is not known if turbulent solutions are uniquely determined by their initial data.

Leray shows that in order for a solution to become turbulent at time T the maximum velocity $V(t)$ must blow up like $const/\sqrt{T-t}$ as t approaches T . No such solutions have been found so far. Leray has suggested that there may be singular similarity solutions of the form

$$u_i(x, t) = (T - t)^{-1/2} U_i((T - t)^{-1/2} x),$$

u_i denoting the components of velocity. Clearly, a solution of this form becomes singular as t approaches T . However, recently Nečas, Růžička and Šverák have shown that the equations that must be satisfied by the functions U_i have no solution of class L^3 in the whole three-dimensional space. Even more recently, Tai-peng Tsai has shown that no similarity solution, unless identically zero, has locally finite energy and locally finite rate of energy dissipation.

Leray's results suggest a scenario for the occurrence of turbulence in fluid flow as the breakdown of smooth solutions, as well as the possibility of the branching of weak solutions into different time histories; of course in the latter case the Navier-Stokes equations would have to be augmented, possibly by a statistical theory involving all branched generalized solutions with given initial data.

In the course of constructing his possibly turbulent solutions Leray has used a host of concepts and methods of functional analysis that have since become an indispensable part of the arsenal of analysts: the weak compactness of bounded sequences in L^2 , and that a weakly convergent sequence is strongly convergent if and only if the limit of the norms is the norm of the limit. Leray defined the weak derivative of an L^2 function in the modern sense, as well as the concept of an L^2 vector field that is divergence free in the weak sense. He used mollifiers to show that a weak derivative is a strong derivative.

The basic construction in Leray's existence proof replaces the problematic quadratic terms $u_k \frac{\partial u^i}{\partial x_k}$ in the Navier-Stokes equation by $\bar{u}_k \frac{\partial u^i}{\partial x_k}$, where $\bar{u}_k = u_k * j_\epsilon$, j a mollifier. Leray observed that if the vector field u is divergence free in the weak sense, \bar{u} is divergence free, so that the energy dissipation relation holds for the modified equations. Leray showed that these modified equations have regular solutions. Using weak compactness he extracted a subsequence of ϵ so that $u_\epsilon(x, t)$ converges weakly for all rational t . Using strong compactness of sequences of functions whose first derivatives are bounded in the L^2 norm he showed that $u_\epsilon(x, t)$ converges strongly except for a set of t of measure zero; here he also made use of an ingenious estimate on the propagation of energy. The resulting objects solve the Navier-Stokes equations in the weak sense; they are regular except for a closed set of values of t that form a set of measure zero.

Despite of much effort remarkably little has been learned in the last 60 years about the smoothness of the weak solutions constructed by Leray. Scheffer was the first to study the size of the singular set in space-time; subsequently Caffarelli, Kohn and Nirenberg have shown that the one-dimensional Hausdorff measure of the singular set is zero. In particular, the singularities cannot lie along a smooth curve. Very recently, simplified derivations of the CKN result have been given by Fang-Hua Lin and Chun Liu, as well as by Gang Tian and Zhouping Xin.

There has been some advance in existence theory. In 1951 Eberhardt Hopf showed that the Navier-Stokes equations have weak solutions with prescribed initial values in smoothly bounded domains in three-dimensional space, with zero velocity at the boundary. Hopf's proof makes use of the same functional analytic machinery as Leray's, but it is simpler in some details; in particular, instead of mollification he uses a Galerkin procedure to construct approximate

solutions. A different approach to existence theory was taken by Fujita and Kato; they used fractional powers of operators, and the theory of semigroups.

Our knowledge of smooth solutions has advanced. Leray had shown that if the initial data are sufficiently smooth and tend to zero sufficiently fast near infinity, then a unique smooth solution exists in a time interval $[0, T]$; the size of this interval may depend on the viscosity γ . Ebin and Marsden, Swann, and Kato have shown that in domains without boundaries T may be chosen to be independent of the size of viscosity, and that as γ tends to zero, these solutions with fixed initial data tend to the solution of the inviscid incompressible Euler equations. No comparable result is known for flows in a domain with boundaries.

Leray has shown that in the absence of a driving force in the interior or on the boundary, solutions of the Navier-Stokes equation tend to zero as t tends to ∞ , and that they regain regularity after a finite time. Much work has been done since on the behavior of driven viscous flows as $t \rightarrow \infty$, such as the finiteness of the Hausdorff dimension of the so-called attractor set, see e.g. Babin and Vishik, Constantin, Foias, Temam, Ladyzhenskaya and the literature quoted there.

Major effort has been devoted to devising and implementing effective computational schemes for calculating Navier-Stokes flows, steady and time-dependent. Curiously, although for many classes of partial differential equations computations have, in von Neumann's prophetic words, "provided us with those heuristic hints which are needed in all parts of mathematics for genuine progress", computations have so far failed to shed much light on whether there are regular solutions that become turbulent.

After the war Leray turned his attention to time dependent hyperbolic partial differential equations. Second-order hyperbolic partial differential equations, of which the prototype is the wave equation

$$u_{tt} - \Delta u = 0 ,$$

were well understood. As pointed out long ago by Friedrichs and Lewy, the key to the initial value problem is furnished by energy inequalities. These are derived by multiplying the equation $a(x, D)u = 0$ by mu , where $m(x, D)$ is a first-order differential operator, equal to $\partial/\partial t$ for the wave equation; the product $m(u)a(u)$ is integrated over a domain in x, t space-bounded by an initial and a final surface. Integration by parts produces integrals over the bounding surfaces whose integrands are quadratic forms in the first derivatives of u . If the bounding surfaces are spacelike, these quadratic forms are positive definite. In this case the integrals are interpreted as energy, and the integral relation obtained is the conservation of energy.

To extend this beautiful scheme to hyperbolic equations of order n greater than two, three obstacles had to be overcome: 1) How to recognize as positive an integral of a quadratic form of derivatives of order $n - 1 > 1$? 2) How to choose the operator $m(x, D)$ appearing in the factor mu ? 3) What to make of a truly incomprehensible paper of Petrowsky from 1937, where energy estimates for higher order equations are derived?

The first problem was solved by Gårding, who showed that the relevant criterion is the positivity of an associated form of order $2(n-1)$. The second problem was solved by Leray by the requirement that the characteristics of the operator m separate those of the operator a . The third obstacle was overcome by Leray when he observed that Petrowsky's construction requires a continuous choice of bases in the tangent space of the sphere, possible only in special dimensions.

We remark here that in 1958 Calderon showed how energy estimates can be derived by employing singular integral (pseudodifferential) operators as symmetrizers of hyperbolic operators. Once this was accomplished, Petrowsky's work could be interpreted in retrospect as constructing pseudodifferential operators based on Fourier series, instead of the Fourier integral.

Leray's derivation of energy inequalities, and their application to prove existence and uniqueness of solution of the Cauchy problem for hyperbolic equations with variable coefficients are outlined in a note included in this volume. They are described in detail in the second part of his Princeton IAS Lecture Notes, 1953/54. The first part discusses explicit formulas for the solution of Cauchy's problem for hyperbolic operators with constant coefficients and containing no lower order terms. This material recapitulates and extends previous results of Herglotz and Petrowsky; the methods used are Fourier analysis, theory of analytic functions of many variables, and algebraic geometry. Further results along these lines have been derived by Atiyah, Bott and Gårding.

For the success of the energy method as described above it is essential that the hyperbolic operator $a(x, D)$ in question have *distinct* characteristics. In the sixties Leray became interested in hyperbolic equations with multiple characteristics. A typical example is

$$u_{tt} + u_x = 0 ;$$

this equation has solutions of the form $u = e^{-inx + \sqrt{int}}$, which shows that solutions do not depend boundedly in the C^N norm on their initial data at $t = 0$, no matter how large N is. It follows that the initial value problem cannot be solved for all C^N initial data. The same conclusion holds for all hyperbolic operators $a(x, D)$ with multiple characteristic, unless restrictions, called the Levi-Lax condition, are placed on the allowable lower order terms, see Mizohata. In the sixties Ohya had discovered that if the coefficients of $a(x, D)$ and the prescribed initial data are not only C^∞ but in an appropriate Gevrey class, then the initial value problem has a solution that belongs to a Gevrey class. We recall that $f(x)$, $x = x_1, \dots, x_n$ belongs to the Gevrey class $\alpha > 1$ if f is C^∞ , and if its derivatives satisfy inequalities of the form

$$|\partial^\beta f| \leq c^k (k!)^\alpha ,$$

for $|\beta| \leq k$. Here $\partial^\beta = \partial_1^{\beta_1} \dots \partial_n^{\beta_n}$, and $|\beta| = \beta_1 + \dots + \beta_n$; c is some constant. The importance of Gevrey classes in this context is that they are *not* quasi-analytic, i.e. that they contain functions with arbitrarily prescribed compact support. Therefore it is possible to define domains of dependence and domains of influence for Gevrey class solutions.

Leray, in collaboration with Ohya, generalized Ohya's result considerably, including even quasilinear equations and systems of n^{th} order equations. A prototype result is as follows:

Let $a(x, D) = a_1(x, D) \dots a_p(x, D)$ be a product of p hyperbolic operators a_j , each with distinct characteristics. Denote by m the order of a . Let b denote some partial differential operator of order $m - p + q$, $q < p$. Suppose that the coefficients of the operator a belong to Gevrey class $\alpha > 1$, and suppose furthermore that the L^2 norm of the space derivatives of the initial values $\partial_t^\gamma u$, $\gamma = 0, \dots, m - 1$ satisfy a Gevrey type estimate at $t = 0$, $\int |\partial_x^\beta \partial_t^\gamma u|^2 dx \leq c^{\gamma+|\beta|} (\gamma + |\beta|)!^\alpha$ for all multiindices β and for $\gamma < m$. Under the condition that $\alpha < p/q$, Leray and Ohya show that the initial value problem has a unique solution of Gevrey class for $0 \leq t \leq T$.

Note that for $q = 0$ there is no restriction on α ; in this case the energy method can be used to construct solutions of $[a + b]u = 0$ with initial data prescribed as arbitrary functions of class C^N provided that N is large enough.

The technique used to construct solutions in Gevrey class is a modification of the method of majorants; it employs formal series. We refer to the article by Ohya and Tarama for a review of this field of inquiry.

According to the now classical principle of Holmgren, if a linear equation has solutions for a dense set of initial data, then the only solution of the adjoint equation with zero initial values is zero. Consequently, nonuniqueness for the initial value problem implies that for the adjoint equation the initial value problem can be solved only for a nondense set of initial values. De Giorgi has given examples of nonzero solutions of hyperbolic equations with multiple characteristics whose initial values are zero. By extending De Giorgi's examples Leray showed that the conditions he and Ohya have found to be sufficient for solving the initial value problem in Gevrey classes are very nearly necessary.

Hyperbolic operators with multiple characteristics appear in some problems of differential geometry, and in magneto-hydrodynamics. Recently, Oseledets, and somewhat later Buttke, have reformulated the incompressible Euler equations as a hyperbolic system of quasilinear, pseudodifferential equations with multiple characteristic; perhaps the method described above shed some light on these equations.

In a joint paper with Hamada and Wagschal, Leray investigated the propagation of singularities of solutions of hyperbolic equations. The singularities in question are of analytic data, and the authors show that they propagate along characteristic hypersurfaces.

The rest of the papers included in this collection are on diverse topics. Two of them deal with the bi- and N-harmonic equation in a strip in the plane. Leray used the representation of N-harmonic functions in terms of analytic functions, and a theorem of Hans Lewy on the continuations by reflection of solutions of partial differential equations, to construct Green's function in a strip, subject to various boundary conditions.

The source of problems about the biharmonic equation is the theory of elasticity; in a 1908 prize-winning Mémoire for the Académie des Sciences, Hadamard

discussed the biharmonic equation and posed this question: if a clamped flat plate is subject to a distributed force on its surface acting in the same direction, does the resulting deflection take place in the same direction? In the standard model for the elastic bending of thin flat plates this amounts to the following question: if a function $u(x, y)$ satisfies the inequality $\Delta^2 u \geq 0$ in a domain, and if u and its first derivatives vanish on the boundary of the domain, is $u \geq 0$ in the domain? In 1948 Duffin showed that the answer is *no* for a strip. This is equivalent to the statement that Green's function for a strip, with clamped boundary conditions, changes sign. It should be interesting to deduce this from Leray's formula for Green's function.

Leray's formulation of analytical problems in geometric terms is very much in the spirit of Poincaré, although for Poincaré function spaces were a promised land he saw but did not enter. Like Poincaré, Leray chose to work mostly on problems that came from physics. In marked contrast, the founding members of the Bourbaki movement, most of them Leray's contemporaries, sought inspiration not in nature but in mathematics itself. That Leray remained faithful to nature had a profound effect on postwar French mathematics. For it was his achievements, prestige and influence that assured a rightful place for his outlook; he was the intellectual guide of the present distinguished French school of applied mathematics. More than that, he provided that balance between the concrete and the abstract that is so essential for the health of mathematics.

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Jean Leray and Several Complex Variables

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Introduction

Complex analysis occupies a special place in the scientific work of J. Leray.

On the one hand, for Leray the problems of complex analysis have never been the main purpose of his research. On the other hand, in his works on differential equations and mathematical physics Leray systematically and with great success used not only various methods of algebraic topology and partial differential equations but also methods of contemporary complex analysis (Behnke-Thullen (1934), Bochner-Martin (1948)). Moreover, if the applications required it, he developed the new branches of complex analysis. As a result of such a seemingly utilitarian approach to complex analysis, the Leray works, concerned with the problems of differential equations, mathematical physics and algebraic topology as they were, enriched very significantly the whole complex analysis theory.

Without exaggeration one can say that during the fifties-sixties the ideas of Leray twice radically changed the direction of the development of contemporary complex analysis.

The Leray sheaf theory was the main tool for the great breakthrough in complex analysis in the early fifties.

Indeed, before the fifties the theory of functions of several complex variables was based, in general, on the traditional constructive methods of analysis.

One should mention: A. Weil's (1935) work, who in the thirties obtained the Cauchy integral formula for polynomial polyhedra in n -dimensional complex space and proved for such polyhedra an analog of the Runge approximate theorem; E. Martinelli's (1938) and S. Bochner's (1943) works, which by integral representation methods have proved a general Hartogs type theorem about extension of holomorphic functions; and a series of K. Oka's and H. Cartan's

works, who in the period of 1936–1950, using the Weil formula, solved “the fundamental problems” (the problems of P. Cousin, K. Weierstrass, H. Poincaré, E. Levi and A. Weil).

At the same time in the forties Leray, in connection with the study of the topology of continuous mappings and fiber spaces, developed the so called “sheaf theory” (1946a; 1950a, b). This theory, including the theory of cohomology with values in a sheaf and the theory of spectral sequences, was obtained by Leray, apparently, without any thoughts about complex analysis.

However, sheaf cohomology has been present implicitly in several complex variables at least since the appearance of the results of Cousin and the Cousin problems (1895). Based on the ideas of Leray (1950a, b) and Oka, (1950) H.Cartan (1950) introduced coherent analytic sheafs. In these terms, the Oka result (1950), which gives a deep extension of the Weierstrass division theorem, means that the structure sheaf of every complex space is coherent.

Immediately after this, the Leray sheaf theory found fundamental applications to complex analysis (Cartan’s seminar (1951–54)). Namely, it was found that the methods of the Leray sheaf theory allow not only to reduce constructive methods (integral formulas of the Cauchy-Weil type) to a minimum in the Oka-Cartan theory, but to give far-reaching generalization of this theory. Namely, in the theory of coherent analytic sheaves (Cartan-Serre) and in the theory of complex spaces (Grothendieck-Grauert-Remmert), the results of Oka and of Cartan on “fundamental problems” found their full completion. “By sheafifying one suddenly was able to obtain results one had not to dream of in 1950” – noted R.Remmert in the encyclopedic article (1994).

Thus, in the fifties the constructive analytic methods of integral representations were practically driven out of multidimensional complex analysis and were replaced by algebraic methods of the Leray sheaf theory. The weakness of sheaf theory is that it does not provide quantitative estimates for solutions of “fundamental problems”.

At the same time in the fifties, Leray while systematically engaged in a systematic study of the Cauchy problem (1952,53a,56b,57b,58a,59b), sharply advanced the development of necessary analytic methods, in particular, the residue theory on complex manifolds. In this connection he introduced the highly general Cauchy-Leray integral formula. This formula on the one hand incorporated, as particular cases, formulas of Weil, Bochner-Martinelli and others, and on the other hand led to progress not only for the Cauchy problem, but also for a series of other important problems of complex analysis and differential equations, which it seems could not be solved by the non-constructive methods of sheaf theory.

Thus, in the sixties, thanks to Leray, the constructive methods of residue theory and of integral representations occupied once again a first rank position in the complex analysis of several variables.

The most fundamental contribution of Leray to several complex variables is his theory of the holomorphic Cauchy problem (1957b,58a,62b,76e,85a). Both

due to its results as well as its methods, this theory, to great extent, constitutes more a section of complex analysis than of partial differential equations.

Thus, among the theories developed by Leray and fundamentally influenced by contemporary complex analysis, one can distinguish the following three:

- sheaf cohomology and spectral sequences;
- theory of residues and Cauchy-Leray formulas;
- the holomorphic Cauchy problem.

The main results of Leray on sheaf theory and spectral sequences, having a purely topological nature, are located in volume I *Selecta* of J. Leray. For their fundamental influence on topology, algebraic geometry and complex analysis, we refer to the monograph P. Griffiths and J. Harris (1978), to the encyclopedic articles (A. Onishchik (1990), R. Remmert (1994)), and to the review of A. Borel (1997).

The present survey is concerned with the closely interconnected works of Leray on the theory of multidimensional residues, the generalized Laplace transforms and the holomorphic Cauchy problem. These works of Leray have complex analysis *per se* as theme of investigation.

We note here, that elements of the theory of multidimensional residues first appeared in the form of the Cauchy formula for polydisc and the first results on the holomorphic Cauchy problem in works of Cauchy in 1841. His results became well known only in 1875, when they were rediscovered and clarified by S. Kowalewska (1875) and G. Darboux (1875).

The Cauchy-Kowalewska theorem states that, for the non-characteristic Cauchy problem for a system with analytic coefficients and analytic initial data, there are unique local solutions in the class of analytic functions.

The connection between multidimensional complex analysis and the Cauchy problem has been more apparent in formulas for elementary solutions of elliptic and hyperbolic equations with constant coefficients, found in increasing generality in works of I. Fredholm (1900), N. Zeilon (1911,21), G. Herglotz (1926,28), L. Fantappiè (1943) and I. Petrowski (1945).

Namely, these formulas express elementary solutions $u(x)$ for homogeneous hyperbolic operators $P(-i\frac{d}{dx})$ of arbitrary order in terms of abelian integrals on the surface $\{\xi \in \mathbb{C}P^n : P(\xi) = 0, x \cdot \xi = 0\}$.

The idea of Leray to develop and to apply the theory of residues to the Cauchy problem had already appeared in (1952,53a), when he obtained in this way the elegant Herglotz-Petrowski-Leray formula for fundamental solutions of hyperbolic differential equations with constant coefficients.

Further, the development of this formula and also of ideas of H. Poincaré (1887) and L. Fantappiè (1943) led Leray to the theory of residues, to the Cauchy-Fantappiè-Leray formula and to fundamental results on the holomorphic Cauchy problem.

1. The Herglotz-Petrowski-Leray formulas for elementary solutions of hyperbolic equations with constant coefficients

From the formal point of view, Leray (1952,53a), in his own words, did the following:

“The calculation, which Herglotz (1926,28) began and Petrowski (1945) continued, gives an elementary solution for $P(-i\frac{d}{dx})$ with help of periods of abelian integrals. We will clarify and finish its calculation. As has been outlined by Fl. Bureau, they without justification transpose non absolutely convergent integrals (Herglotz (1926) II, p.290). We take off their assumption that the cone $P(\xi) = 0$ has no singular points, and we give an invariant expression for their results. Using the Schwartz distribution we define an elementary solution everywhere but not only there where it is a function.”

However, from a conceptual point of view Leray in (1952a,62b) did much more. We give here one of the most characteristic results of Leray from (1952a,62b) supposing for simplicity, that $P(\xi)$ is a strictly hyperbolic polynomial.

Let $P(\xi)$ then be a homogeneous polynomial of degree m . Suppose, that the polynomial $P(\xi)$ is strictly hyperbolic with respect to $\eta \in \mathbb{R}^n \setminus \{0\}$, i.e. for each $\xi \in \mathbb{R}^n$, non-proportional to η , the equation $P(\xi + \lambda\eta) = 0$ has m different roots as a function of λ . Let $A = \{\xi \in \mathbb{C}^n : P(\xi) = 0\}$ and Γ be a connected component of the set $\mathbb{R}^n \setminus \text{Re } A$, containing η . Let K be a cone dual to Γ .

With the help of the operational calculus, it follows (J. Hadamard (1932), M. Riesz (1949), L. Schwartz (1950,51)) that there exists a unique fundamental solution or the strictly hyperbolic (with respect to η) operator $P(-i\frac{\partial}{\partial x})$ with support in the cone K and this solution has the form :

$$(1.1) \quad u(x) = L\left(\frac{1}{P(\xi)}\right) = \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{e^{i(x, \xi - i\eta)}}{P(\xi - i\eta)} d\xi, \quad P(-i\frac{\partial}{\partial x}) u(x) = \delta(x),$$

where $d\xi = d\xi_1 \wedge \dots \wedge d\xi_n > 0$ on \mathbb{R}^n , and $\delta(x)$ is the Dirac function.

From this expression of the Laplace transform $L(\frac{1}{P(\xi)})$, it is difficult to draw a conclusion about the behavior of the function $u(x)$. So, first G. Herglotz (1926,28) and then in more complete form J. Petrowski (1945) and then in more invariant form Leray (1953a) transformed the integral $L(\frac{1}{P(\xi)})$ into an integral of a holomorphic differential form with respect to a compact cycle.

Let

$$X^* = \{\xi \in \mathbb{C}P^{n-1} : (x, \xi) = 0\}, \quad x \in \mathbb{R}^n.$$

$$A^* = \{\xi \in \mathbb{C}P^{n-1} : P(\xi) = 0\}.$$

Let $\eta^\pm(\xi)$ be vector fields in \mathbb{R}^n , satisfying the conditions $\pm(x, \eta^\pm(\xi)) < 0$ for $\xi \in \mathbb{C}P^{n-1}$ and $\frac{(dP(\xi), \eta^\pm(\xi))}{(dP(\xi), \eta)} > 0$ for $\xi \in \mathbb{C}P^{n-1}$, $P(\xi) = 0$. We denote by β^\pm the cycle $\mathbb{C}P^{n-1} \setminus (A^* \cup X^*)$, obtained from $\mathbb{R}P^{n-1} \subset \mathbb{C}P^{n-1}$ as a result of small shift along the field $-i\eta^\pm(\xi)$. The cycle $\beta^* = \beta^+ - \beta^-$ in $\mathbb{C}P^{n-1} \setminus (A^* \cup X^*)$ is called the Leray cycle.

Theorem (Leray, 1952a, 62b). *The derivatives of the fundamental solution $u(x)$ are represented by the following integrals of rational forms*

$$(1.2) \quad \frac{\partial^{|\nu|} u(x)}{\partial x^\nu} = \frac{(-q-1)!(-1)^\nu i^{n-m}}{(2\pi)^n} \int_{\beta_x^*} \frac{\xi^\nu \omega'(\xi)}{(x \cdot \xi)^{|\nu|+n-m} P(\xi)},$$

where

$$\omega'(\xi) = \sum_{k=1}^n (-1)^{k-1} \xi_k \wedge_{j \neq k} d\xi_j, \quad |\nu| + n - m > 0.$$

The above formulas of Leray coincide, after calculation of residues of the integral (1.2), with classical formulas of Herglotz-Petrowski.

These formulas led to the essential development and refinement of the fundamental Petrowski theory of lacunas in the works of Atiyah-Bott-Gårding (1970, 73) and V. Vassiliev (1995).

On the other hand, and this is the most important, these formulas showed that the investigation of the Cauchy problem to a great extent depends on the development of the theories of multidimensional residues, Laplace transforms, and multidimensional Cauchy formulas.

2. The Cauchy problem for linear equations with polynomial coefficients

Starting from the Herglotz-Petrowski-Leray formula, Leray began in (1956b) the study of the Cauchy problem for equations with variable coefficients. He stated his program of investigations in the following way in the introduction to Leray (1957b):

“Nous nous proposons d’étudier globalement le problème linéaire de Cauchy dans le cas complexe, puis dans le cas réel et hyperbolique, en supposant les données analytiques. Notre principal but est la proposition suivante: les singularités de la solution appartiennent aux caractéristiques issues des singularités des données ou tangentes à la variété qui porte les données de Cauchy. C’est l’extension aux équations aux dérivées partielles de la propriété fondamentale des solutions des équations différentielles ordinaires, linéaires et analytiques: leurs singularités sont des singularités des données”.

However, the global Cauchy problem (both in the complex and the real domain) turned out to be a theme so large, difficult and interesting that, in spite of the efforts of Leray himself and his successors (see Y. Hamada (1969,70), D. Schiltz, J. Vaillant, C. Wagschal (1982), C. Wagschal (1983), E. Leichtman (1990), D’Agnolo-Schapira (1991), Sternin-Shatalov (1994)), this problem is not yet completely solved.

One of the most brilliant and uncompleted ideas of Leray is contained in (1956b). Namely, the main statement of Leray (1956b) is the following .

Let $X \subset \mathbb{C}^n$, $x = (x_1, \dots, x_n)$ be a point of X and $S = \{x \in X : s(x) = 0\}$ a smooth hypersurface in X . Let $a(\xi, x)$ be a polynomial of $(\xi, x) \in (\mathbb{C}^n)^* \times \mathbb{C}^n$

of the degree m with respect to ξ . Let $-l$ be the smallest integer such that $x_0^{-l} a(x_0 \xi, \frac{x}{x_0}) = A(\xi, x_0, \dots, x_n)$ is a polynomial in $x_0, \dots, x_n, \xi_1, \dots, \xi_n$. Let $h(\xi, x)$ be the principal part of $a(\xi, x)$ with respect to ξ . Let $v = v(x)$ be a function holomorphic on X . Suppose, that the hypersurface S is not characteristic for $a(\xi, x)$, i.e. suppose that $h(\frac{\partial s}{\partial x}, x) \neq 0 \quad \forall x \in S$.

Theorem (Leray, 1956b). *The solution of the holomorphic Cauchy problem*

$$(2.1) \quad a\left(\frac{\partial}{\partial x}, x\right)u(x) = v(x), \quad v(x) = O((s(x))^m)$$

in the neighbourhood of S can be given by the formula

$$u(z) = J[U^*(\xi, x)v(x)],$$

where the function $U^*(\xi, x)$ and the operator J are defined by the following way. Let $\xi \cdot x = \xi_0 + \xi_1 x_1 + \dots + \xi_n x_n$ and.

$$\omega^*(\xi) = \sum_{k=0}^n (-1)^k \xi_k d\xi_0 \wedge \dots \wedge d\xi_{k-1} \wedge d\xi_{k+1} \wedge \dots \wedge d\xi_n.$$

There exists a generator of relative homologies α of domains $\{(\xi, x) \in (\mathbb{C}P^n)^* \times X\}$ with respect to three hypersurfaces: $s(x) = 0$, $\xi \cdot x = 0$, $\xi \cdot z = 0$ such that for $l > n$

$$J(U^*(\xi, x)v(x)) = \frac{1}{(2\pi i)^{n-1}} \int_{\alpha} \frac{(\xi \cdot z)^{l-n-1}}{(l-n-1)!} U^*(\xi, x)v(x) \times \omega^*(\xi) \wedge \omega(x).$$

For $l \leq n$ the operator $J(U^*(\xi, x)v(x))$ is represented by the residue integral

$$(2.2) \quad \frac{(-1)^{l-n-1}}{(2\pi i)^{n-1}} \int_{(b\alpha) \cap \{\xi \cdot z = 0\}} \frac{d^{n-l}[U^*(\xi, x)v(x)\omega^*(\xi) \wedge \omega(x)]}{(d\xi \cdot z)^{1+n-l}},$$

where $b\alpha$ is the boundary of α . The function $U^*(\xi, x)$ is the solution of the Cauchy problem: $A(\xi, -\frac{\partial}{\partial \xi})U^*(\xi, x) = 1$, $U^*(\xi, x) = 0((\xi \cdot x)^{m-l})$. The function $U^*(\xi, x)$ is homogeneous in $\xi \in \mathbb{C}^{n+1}$ of degree $-l$ and is called the homogeneous unitary solution of $A(\xi, -\frac{\partial}{\partial \xi})$.

The above formulas are the remarkable development of the Herglotz-Petrowski-Leray and Fantappie (1943, 56) formulas corresponding to the case of constant coefficients, when

$$a(\xi, x) = a(\xi), \quad l = m, \quad U^*(\xi, x) = \frac{1}{a(\xi)}.$$

The optimistic plan, to obtain from the above formulas explicit information about singularities of the solution u of the Cauchy problem (2.1) with singular data v was again formulated in the survey Leray (1963b).

Several profound steps in the realisation of this program were taken in the fundamental series of Leray's papers subtitled "Problème de Cauchy I, II, III, IV, VI". The Leray paper with subtitle "Problème de Cauchy V" has not been published, but in Leray (1956b) and Leray (1962b, 63b, 64a) there are some indications on the ideas of this work.

3. Uniformization and asymptotic expansions

In the introduction to the article “Problème de Cauchy I” (1957b), Leray describes his idea of uniformization of the solution of the Cauchy problem in the following brief and expressive way

“Ce premier article étudie la solution $u(x)$ du problème de Cauchy près de la variété S qui porte les données de Cauchy. Si S n’est caractéristique en aucun de ses points, alors, $u(x)$ est holomorphe près de S , vu le théorème de Cauchy-Kowalewski, et nos théorèmes n’énoncent rien de neuf. Mais nous admettons que S soit caractéristique en certains de ses points: il s’agit d’un cas sans analogue en théorie des équations différentielles ordinaires, en théorie des équations aux dérivées partielles ce cas joue un rôle fondamental, parce qu’il est celui où $u(x)$ présente les singularités les plus simples: $u(x)$ peut être uniformisé et, sauf des cas exceptionnels, est algébroïde”.

In the simplest form Leray’s uniformization result has the following appearance. Let $a(x, \frac{\partial}{\partial x})$ be a differential operator of the degree m with holomorphic coefficients in the domain $\Omega \subset \mathbb{C}^n$. Let $g(x, \frac{\partial}{\partial x})$ be the principal part of the operator $a(x, \frac{\partial}{\partial x})$. Let $u(x), v(x)$ and $s(x)$ be holomorphic functions of $x \in \Omega$ and let $\xi \in \mathbb{C}$. Assume that $S_\xi = \{x \in \Omega : s(x) = \xi\}$. Denote by $\text{char } S_\xi$ the characteristic points S_ξ , i.e. $\text{char } S_\xi = \{x \in S_\xi : g(x, \frac{\partial s}{\partial x}) = 0\}$. Suppose, that S_ξ is a smooth hypersurface and $\dim \text{char } S_\xi < \dim S_\xi$. Let K_x denote the characteristic conoid with vertex at x .

The point $x \in \text{char } S_\xi$ is called exceptional, if it possesses one of two properties:

either S_ξ and K_x are tangent at an infinite number of points in a neighbourhood of x ,

or a bicharacteristic curve of the surface S_ξ , emanating from $(x, \frac{\partial s}{\partial x})$ consists of a unique point x .

Consider the Cauchy problem:

to find a function $u_\xi(x) = u(\xi, x)$ in the neighbourhood of the surface S_ξ satisfying the conditions

$$(3.1) \quad \begin{aligned} a(x, \frac{\partial}{\partial x}) u(\xi, x) &= v(x) \\ u(\xi, x) &= O((s(x) - \xi)^m). \end{aligned}$$

Due to the Cauchy-Kowalewska theorem there exists a local solution of this problem in the neighbourhood of any point $x \in S_\xi \setminus \text{char } S_\xi$.

We define a function $\xi = \xi(t, x)$, $x \in \Omega$, $t \in \mathbb{C}$, $|t| < \varepsilon$, as a solution of the nonlinear Cauchy problem for the first order equation

$$\frac{\partial \xi}{\partial t} + g(x, \frac{\partial \xi}{\partial x}) = 0, \quad \xi(0, x) = s(x).$$

This problem is solved by the well known method of characteristics. Consider a composition $u(\xi(t, x), x)$ of solutions of the Cauchy Problem (3.1) in the neighbourhood of noncharacteristic points of the surface S_ξ with a function $\xi(t, x)$.

Uniformization theorem (Leray, 1957b). *The composed functions*

$$\frac{\partial^j u(\xi(t, x), x)}{\partial \xi^j}, \quad j = 0, 1, \dots, m - 1$$

are holomorphic for small t and $x \in \Omega$. On the image of the domain of holomorphy of the function $\xi(t, x)$ by the mapping $(t, x) \mapsto (\xi(t, x), x)$ the support of the singularities of the multi-valued function $u(\xi, x)$, satisfying (3.1), belongs to the set K of those values (ξ, x) , for which the hypersurface S_ξ is tangent to the characteristic conoid K_x .

In the neighbourhood of any non-exceptional point of the surface S_ξ the function $(\xi, x) \mapsto u(\xi, x)$, satisfies (3.1) and all its derivatives up to the order $m - 1$ inclusively are algebraic functions.

This result of Leray has important refinements:

In the work of Gårding-Kotake-Leray (1964a), developing Leray (1957b), an asymptotic expansion for the solution of the Cauchy problem in the neighbourhood of singular points was obtained.

Namely, through quadratures along bicharacteristics generating K one can calculate an asymptotic expansion of u near K ; its first terms determine the values of $(\frac{\partial}{\partial \xi})^j u(\xi, x)$, $j = 0, 1, 2, \dots, m - 1$, on K .

It is interesting to note that at the points, where one can not apply the Cauchy-Kowalewska theorem, nevertheless the solution can be calculated and, in addition, in explicit and simple form.

Moreover, in the work of Gårding-Kotake-Leray (1964a) the results on uniformization and asymptotic expansions are generalized to systems of equations, which is important for applications. The invariants introduced in this work, namely, the bicharacteristic function for one equation (3.1) and the bicharacteristic matrix for a system of equations, were developed in J. Vaillant's work (1968) and applied to the analytic Goursat problem.

The Leray uniformization method was applied with success to nonlinear systems in the work of Y.Choquet-Bruhat (1966).

4. Unitary solutions, elementary solutions and the generalized Laplace transform

A fundamental concept in the Leray program (1957b, 63b) is a so called unitary solution of the Cauchy problem. Let ξ be a complex linear function of $x \in \Omega$, i.e. $\xi(x) = \xi \cdot x = \xi_0 + \xi_1 \cdot x_1 + \dots + \xi_n \cdot x_n$. Denote by ξ^* the hyperplane in Ω , or the point in $(\mathbb{C}P^n)^*$, defined by the equation $\xi^* : \xi \cdot x = 0$. Let $a(x, \xi)$ be a polynomial of degree m with respect to ξ , independent of ξ_0 , whose coefficients are holomorphic with respect to $x \in \Omega$. Let

$$a(x, \xi) = g(x, \xi) + g'(x, \xi) + \sum_{j_1 + j_2 + \dots + j_n \leq m - 2} a_{j_1, \dots, j_n} \xi_1^{j_1} \xi_2^{j_2} \dots \xi_n^{j_n},$$

where g, g' are homogeneous polynomials of degree (m) and $(m-1)$ respectively.

A unitary solution for the operator $a(x, \frac{\partial}{\partial x})$ is by definition a solution $U(\xi, y)$ of the following Cauchy problem

$$a(y, \frac{\partial}{\partial y}) U(\xi, y) = 1$$

where the function $U(\xi, y)$ has a zero of order m on the surface $\xi \cdot y = 0$. Due to zero homogeneity with respect to ξ , the function $U(\xi, y)$ is a function of $y \in \Omega$ and $\xi^* \in (\mathbb{C}P^n)^*$. Let $a^*(x, \frac{\partial}{\partial x})$ be the adjoint operator for $a(x, \frac{\partial}{\partial x})$ and $U^*(\xi, y)$ be a unitary solution corresponding to $a^*(x, \frac{\partial}{\partial x})$.

The above cited uniformization result of Leray (1957b) can be applied to describing the singularities of $U(\xi, y)$ in the neighbourhood of characteristic points y of ξ^* , i.e. of points $(y, \xi) : \xi \cdot y = 0$ and $g(y, \xi) = 0$.

Namely, consider the solution $x(\tau, \eta, y)$, $\xi(\tau, \eta, y)$ of the ordinary differential system

$$(4.1) \quad \begin{aligned} \frac{dx_j}{d\tau} &= \frac{\partial g(x, \xi)}{\partial \xi_j}, \quad \frac{d\xi_j}{d\tau} = -\frac{\partial g(x, \xi)}{\partial x_j}, \quad j = 1, 2, \dots, n \\ \frac{d\xi_0}{dt} &= \sum_{j=1}^n x_j \frac{\partial g(x, \xi)}{\partial x_j} - g(x, \xi) \end{aligned}$$

with initial values

$$x(0, \eta, y) = y, \quad \xi(0, \eta, y) = \eta, \quad \eta \cdot y = 0.$$

The mapping $\xi(\tau, \eta, y)$ has an important homogeneity property

$$\xi(\theta^{1-m}\tau, \theta\eta, y) = \theta\xi(\tau, \eta, y), \quad \theta \in \mathbb{C}.$$

Leray's uniformization result shows that, in general, the multivalued functions $U(\xi, y)$ and $U^*(\xi, y)$ are uniformized together with their derivatives of order $< m$ by the mapping $\xi(\tau, \eta, y)$, i.e. the functions $U(\xi(\tau, \eta, y), y)$ and $U^*(\xi(\tau, \eta, y), y)$ are holomorphic in $\eta \in \mathbb{C}^{n+1}$, $y \in \Omega$ and small $\tau \in \mathbb{C}$.

On the image of the domain of holomorphy of the function $\xi(t, \eta, y)$ by the mapping $(t, \eta, y) \rightarrow (\xi(t, \eta, y), y)$, the support of the singularities $U(\xi, y)$ belongs to the set K of those (ξ, y) for which the characteristic cone K_y with vertex at y is tangent to the plane $\xi \cdot y = 0$.

Moreover, the following expression for the principal singularity of $U(\xi, x)$ can be given:

$$(4.2) \quad \begin{aligned} \left(\frac{\partial}{\partial \xi_0}\right)^m U(\xi(\tau, \eta, y), y) &= \frac{1}{g(y, \eta)} e^{\lambda(\tau, \eta, y)} \left[\frac{D(\xi_1(\tau, \eta, y), \dots, \xi_n(\tau, \eta, y))}{D(\eta_1, \dots, \eta_n)} \right]^{-1/2} \\ &+ \text{holomorphic function of } \tau, \eta, y; \end{aligned}$$

when τ is small, where

$$[D(\xi_1, \dots, \xi_n)/D(\eta_1, \dots, \eta_n)]^{-1/2} = 1 \quad \text{for } \tau = 0$$

and $\lambda(\tau, \eta, y)$ is a solution of

$$\frac{d\lambda}{d\tau} = \left[g'(x, \xi) - \frac{1}{2} \sum_j \frac{\partial^2 g(x, \xi)}{\partial x_j \partial \xi_j} \right] \quad \text{with} \quad \lambda(0, \eta, y) = 0.$$

Example (Leray (1958a)). Let us consider the Tricomi operator

$$a\left(x, \frac{\partial}{\partial x}\right) = g_0\left(\frac{\partial}{\partial x}\right) + \sum_{j=1}^m x_j g_j\left(\frac{\partial}{\partial x}\right) + g'\left(\frac{\partial}{\partial x}\right),$$

where g_0, \dots, g_n are homogeneous functions of degree m and g' is a homogeneous function of degree $m - 1$. For such an operator the formula (4.2) for $\left(\frac{\partial}{\partial \xi_0}\right)^m U(\xi, y)$ is exact, i.e. does not contain additional holomorphic terms of τ, η, y .

For the proof of this result Leray uses the following simple and important reciprocity result. Let $a\left(x, \frac{\partial}{\partial x}\right)$ be linear differential operator of degree m with polynomial coefficients. Denote by l the smallest integer number such that $x_0^l a\left(\frac{x}{x_0}, x_0 \xi\right)$ is a polynomial in $x_0, \dots, x_n, \xi_1, \dots, \xi_n$. Suppose, that $A(x_0, \dots, x_n, \xi) = x_0^l a\left(\frac{x}{x_0}, x_0 \xi\right)$. The operator $A\left(-\frac{\partial}{\partial \xi}, \xi\right) = A\left(-\frac{\partial}{\partial \xi_0}, \dots, -\frac{\partial}{\partial \xi_n}, \xi\right)$ is called the Laplace transform of $a\left(x, \frac{\partial}{\partial x}\right)$.

Let U_{-l}^* be a solution of Cauchy's problem $A\left(-\frac{\partial}{\partial \xi}, \xi\right) U_{-l}^*(\xi, y) = 1$ and $U_{-l}^*(\xi, y)$ vanishes $(l + m)$ times for $\xi \cdot y = 0$. Then

$$\left(-\frac{\partial}{\partial \xi_0}\right)^m U^*(\xi, y) = \left(-\frac{\partial}{\partial \xi_0}\right)^{m+l} U_{-l}^*(\xi, y).$$

The results of Leray (1957b, 58a) on uniformization of unitary solutions of the Cauchy problem are used in an essential in the fundamental work of Leray (1962b) for defining the singular part of the elementary solution for a hyperbolic operator.

An operator $a\left(x, \frac{\partial}{\partial x}\right)$ of degree m with holomorphic coefficients and with real principal part $g\left(x, \frac{\partial}{\partial x}\right)$ is said to be hyperbolic, if for any $x \in \mathbb{R}^n$ the characteristic polynomial $g(x, \xi)$ is hyperbolic with respect to ξ .

Let $\Gamma(x)$ be the (convex) cone of all directions p , such that for any line $\xi = p \cdot \lambda + \eta$, with η not proportional to p , the equation $g(x, p\lambda + \eta) = 0$ has m real and distinct roots. The polynomial $g(x, \xi)$ is called hyperbolic in $x \in X$ if the cone $\Gamma(x)$ is not empty.

The famous theorem of J. Hadamard (1923), J. Schauder (1935) and I. Petrowski (1937) states the global existence and uniqueness of an (elementary) solution $E(x, y)$ for the equation

$$a\left(x, \frac{\partial}{\partial x}\right) E(x, y) = \delta(x, y)$$

with condition $\text{supp } E \subset \mathcal{E}(y)$, where $\mathcal{E}(y)$ is the union of all time-like paths, originating from y . A vector at a point $x \in \mathbb{R}^n$ is called time-like if it belongs to the cone $K(x)$ dual to the cone $\Gamma(x)$.

This existence and uniqueness result gives no precise information about the singularities of $E(x, y)$. Such information can be obtained from the following.

Theorem (Leray, 1962b). *The following formula for the elementary solution $E(x, y)$ is valid*

$$(4.3) \quad E(x, y) = \mathcal{L}(U^*(\xi, y)),$$

where $U^*(\xi, y)$ is a unitary solution of the operator a^* adjoint to a .

The generalized Laplace transform, introduced by Leray (1962b), is denoted by \mathcal{L} .

Formula (4.3) is a deep analog of the famous F. John formula (1950) for an elementary solution for an elliptic operator with non-constant coefficients.

Since in the survey of Leray (1963b) opening the present volume, the transform \mathcal{L} is carefully defined and its properties are described in detail, we give here only the following main formula

$$(4.4) \quad \mathcal{L}(U^*(\xi, y)) = \frac{(-1)^{m-n-1}}{2(2\pi i)^{n-1}} \int_h \frac{d^n[U^*(\xi, y)\omega^*(\xi)]}{(d\xi \cdot x)^{n+1}},$$

where the multi-valued $(n-1)$ form under the integral sign is uniformized by a mapping $\xi = \xi(\tau, \eta, y)$ of the form (4.1).

For more complete description and understanding formula (4.4), it is necessary to consider the n -form

$$(4.5) \quad (\xi \cdot x)^{-n-1} U^*(\xi, y) \omega^*(\xi).$$

With the uniformization $\xi = \xi(\tau, \eta, y)$, the form (4.5) is a meromorphic form in the domain

$$\psi = \{(\tau, \eta) \in \mathbb{C}^{n+1} : |\tau| |\eta|^{m-1} < \text{const}\},$$

depending on parameters x, y and invariant under the group of transformations

$$(\tau, \eta) \rightarrow (\theta^{1-m}\tau, \theta\eta), \quad \theta \in \mathbb{C}.$$

So, it is possible (and necessary) to consider the form (4.5) as a form on the manifold $\tilde{\psi}$, which is the quotient of ψ by the above group of transformations.

Furthermore, the $(n-1)$ -form

$$\frac{d^n U^*(\xi, y) \omega^*(\xi)}{(d\xi \cdot x)^{n+1}}$$

appearing under the integral sign (4.4) is determined as the residue-form of n -form (4.5) on the manifold

$$\tilde{x} = \{(\tau, \eta) \in \tilde{\psi} : \xi(\tau, \eta, y) \cdot x = 0\} \subset \tilde{\psi}.$$

The complex manifold $\tilde{\psi}$ contains a unique compact submanifold, namely, the projective subspace

$$y^* = \{(\tau, \eta) \in \tilde{\psi} : \tau = 0\} \subset \tilde{\psi}.$$

It is interesting to note, that this submanifold y^* is exceptional in the sense of H. Grauert (1962).

The cycle under the integral sign (4.4) is an appropriate compact homology class of \tilde{x} relatively to y^* . The class $h = h(\tilde{x}, y^*)$ varies continuously with x, y .

From the Leray formula (4.3), it follows that $E(x, y)$ as a function of x is holomorphic outside of the conoid $K(y)$. Moreover, the principal part of the singularity of $E(x, y)$ can be computed on the conoid $K(y)$.

It is important to note that the fundamental definitions (4.3),(4.4) of the generalized Laplace transform of multi-valued functions use in an essential way properties of the residue of a meromorphic form on a complex manifold with an exceptional submanifold.

The previous result of Leray (1962b), applied to the Tricomi operator of the degree $m \leq n$, gives the following explicit formula for the elementary solution

$$E(x, y) = \frac{(-1)^{n-m}}{(2\pi i)^{n-1}} \int_h \frac{d^{n-m}}{(d\xi \cdot x)^{1+n-m}} \left[e^{-\lambda} \frac{D(\xi_1, \dots, \xi_n)}{D(\eta_1, \dots, \eta_m)} \tilde{\omega} \right],$$

where $\xi = \xi(\tau, \eta, y)$ and $\lambda = \lambda(\tau, \eta, y)$ are functions determined in (4.1) and (4.2),

$$\tilde{\omega}(t, \eta) = (1 - m)td\eta_1 \wedge \dots \wedge d\eta_m - dt \wedge \omega'(\eta).$$

Formula (4.3), applied to the homogeneous operator with constant coefficients $a(\frac{\partial}{\partial x})$, turns into the Herglotz-Petrowski-Leray formula (1.2) from section 1, since for this case $U^*(\xi, y) = \frac{(\xi \cdot y)^m}{m!a(\xi)}$. Namely, this reduction together with equality (1.1) gives the reason for calling (4.3) the generalized Laplace transform.

The work of Leray (1962b) was generalized to the case of non strictly hyperbolic equations in works of Atiyah-Bott-Gårding (1970,73) and was used by them for developing the Petrowski theory of lacunas for hyperbolic differential operators.

We note further, that recently H. Shapiro (1992) and B. Sternin, and V. Shatalov (1994) have successfully applied Leray's (1957b,62b,63b) theory to classical problems of potential theory.

5. Residue theory

The program of investigations and the results of Leray on the Cauchy problem turned out to be closely connected to the (multidimensional) residue theory. The multidimensional residue theory started actually with H. Poincaré's (1887) work, where he introduced a residue 1-form for any rational 2-form $\frac{P(z,w)}{Q(z,w)} dz \wedge dw$, $(z, w) \in \mathbb{C}^2$ and where the following remarkable theorem was proved: an integral of the rational 2-form with respect to any two-dimensional cycle in the domain $\{(z, w) \in \mathbb{C}^2 : Q(z, w) \neq 0\}$, can be expressed through periods of abelian integrals on the Riemann surfaces determined by the equation $Q(z, w) = 0$.

Poincaré's work gave rise on the one hand to notions of cohomology, necessary in the general Leray residue theory (H. Poincaré, 1895, S. Lefschetz, 1924,

G. de Rham, 1955) and on the other hand led successively I. Fredholm (1900), N. Zeilon (1911), G. Herglotz (1926), I. Petrowski (1945), J. Leray (1953) to remarkable formulas, expressing elementary solutions of (hyperbolic) equations with constant coefficients through periods of abelian integrals.

The obtention of deeper results (Problème de Cauchy IV) and the realization of the program (Problème de Cauchy V) for equations with non constant coefficients required a more complete and careful residue theory, achieved by Leray himself (Problème de Cauchy III).

We give here some key results of this theory. Suppose, that X is a complex manifold of dimension n and S is a complex submanifold of codimension 1.

Proposition and definition (de Rham, 1954; Leray, 1959). If φ is d -closed $C^{(\infty)}$ regular differential form on $X \setminus S$, which has a pole of the first order on S , then in the neighbourhood U_a of any arbitrary point $a \in S$ the form φ is represented as follows:

$$\varphi = \frac{ds_a}{s_a} \wedge \psi + \theta_a,$$

where $s_a(X)$ are determining functions of the manifold S in U_a and ψ_a, θ_a are forms regular in U_a . Moreover, the form $\psi|_S$ is globally and uniquely determined and closed on S . If φ is holomorphic in $X \setminus S$ then the form $\psi|_S$ is holomorphic on S . The form $\psi|_S$ is called the residue-form on S and is denoted by $\text{res}[\varphi]$.

Example (Poincaré, 1887). The definition of the Leray residue-form generalizes the mapping residue, introduced by Poincaré for the following important special case.

Let $X = \mathbb{C}^n$, $S = \{z \in \mathbb{C}^n : s(z) = 0\}$ be a smooth complex hypersurface and $\varphi = \frac{f(z) dz_1 \wedge \dots \wedge dz_n}{s(z)}$, where $f(z), s(z)$ are holomorphic functions, $z \in \mathbb{C}^n$.

Thus,

$$(5.1) \quad \text{res}[\varphi] = (-1)^j f(z) \frac{dz_1 \wedge \dots \wedge \hat{dz}_j \wedge \dots \wedge dz_n}{\frac{\partial s}{\partial z_j}}$$

at points $z \in S$, where $\frac{\partial s}{\partial z_j} \neq 0$.

In applications, including applications to the Cauchy problem, it is impossible to avoid the situation where the submanifold S , where the residue must be considered, has singularities. Already for the simplest but very important case, when $S \subset X$ has isolated double quadratic points and φ is a holomorphic p -form on $X \setminus S$ with a pole of the first order on S , de Rham (1954) and Leray (1959b) remark with regard to formula (5.1) that the residue-form $\text{res}[\varphi]$ is holomorphic at singular points $\text{Sing } S \subset S$ iff either $p < n = \dim_{\mathbb{C}} X$, or $p = n$ and the coefficient of n -form $\varphi(x) \cdot s(x)$ vanishes on $\text{Sing } S$.

Here we encounter the emergence of non-trivial facts of the theory of residue-currents theory, started already in works of L. Schwartz (1953), P. Lelong (1954), M. Atiyah, W. Hodge (1955) and continued under the influence of Leray's work (1959b) by M. Herrera, P. Liberman (1971), P. Dolbeault (1971), N. Coleff, M. Herrera (1978) and by their successors (A. Dickenstein, C. Sessa (1985), M. Passare (1988), A. Yger (1987), A. Tsikh (1988)).

The central fact of this theory is the following. To each meromorphic n -form $\frac{f}{g}$ on X corresponds a current $[\frac{f}{g}]$, defined by the formula

$$\langle [\frac{f}{g}], \varphi \rangle = \lim_{\varepsilon \rightarrow 0} \int_{|g| > \varepsilon} \frac{f}{g} \wedge \varphi,$$

where φ is a regular $(0, n)$ -form with compact support in X . The existence of the latter limit, for the general case, is provided by the theorem of Herrera-Lieberman (1971), using the theorem on resolution of singularities of H. Hironaka (1964). From the given result it follows that for any regular $(0, n - 1)$ -form ψ with compact support in X , we have

$$(5.2) \quad \langle \bar{\partial}[\frac{f}{g}], \psi \rangle = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{|g| = \varepsilon} \frac{f}{g} \wedge \psi.$$

Suppose, further, that the form f/g has a pole of the first order on $S = \{x \in X : g(x) = 0\}$. Then the last limit can be calculated through the Leray residue-form $\text{res}[\frac{f}{g}]$. Namely, using the Stokes formula and the Coleff-Herrera theorem (1978) one can obtain the equality

$$(5.3) \quad \lim_{\varepsilon \rightarrow 0} \frac{1}{2\pi i} \int_{\{x \in X : |g| = \varepsilon\}} \frac{f}{g} \wedge \psi = \lim_{\delta \rightarrow 0} \int_{\{x \in S : |\frac{\partial g}{\partial z_j}| \geq \delta\}} \text{res}[\frac{f}{g}] \wedge \psi, \quad 1 \leq j \leq n.$$

If a manifold S has no singularities, then $\text{res}[\frac{f}{g}]$ is a holomorphic $(n - 1)$ -form on S and formula (5.3) is equivalent to the Leray (1959) residue formula for holomorphic n -form with a pole of the first order on S .

If a manifold S has singularities $\text{Sing } S$ then the form $\text{res}[\frac{f}{g}]$ is, in general, meromorphic on S (with poles on $\text{Sing } S$). However, the form $\text{res}[\frac{f}{g}]$ is nevertheless always dual holomorphic in the sense that the current $\text{res}[\frac{f}{g}] \wedge \frac{i}{\pi} \partial \bar{\partial} \log |g|$ is $\bar{\partial}$ -closed. If X is the Stein manifold then the latter property characterizes those meromorphic forms on a closed hypersurface $S \subset X$, which can be residues of forms $\frac{f}{g}$ on $X \setminus S$. In particular, for the case $\dim_{\mathbb{C}} S = 1$, only abelian differentials in the sense of M. Rosenlicht (1954) can be residues on S .

Example (Poincaré, 1899; Lelong, 1954). Developing the Poincaré construction, Lelong proved the following important special case (5.2),(5.3). For any meromorphic function f on X the Poincaré-Lelong formula

$$\langle \bar{\partial} \frac{df}{f}, \psi \rangle = \pi i \sum_{\nu} \gamma_{\nu} \int_{S_{\nu}} \psi = \pi i ([S], \psi),$$

where ψ is a regular $(n - 1, n - 1)$ -form with compact support, $S = \cup_{\nu} \gamma_{\nu} S_{\nu}$ is a divisor on X , consisting of components $\{S_{\nu}\}$ of zeros and poles of the function f with multiplicities $\{\gamma_{\nu}\}$.

The following proposition of Leray (1959b) allows for residues, having a purely cohomological nature to reduce all calculations to the case of forms with poles of the first order.

Proposition-Definition (Leray, 1959b). For each closed regular differential form φ on $X \setminus S$, there exists a form $\tilde{\varphi}$ cohomologous to φ and having on S a pole of the first order. Moreover, the cohomology class of the form $\text{res}[\tilde{\varphi}]$ depends only on the cohomology class of the form φ . The cohomology class of the form $\text{res}[\tilde{\varphi}]$ is called the residue-class of the form φ and is denoted by $\text{Res}[\varphi]$. Thus, a homomorphism of cohomology spaces is determined

$$\text{Res } H^p(X \setminus S) \rightarrow H^{p-1}(S).$$

The Leray proposition is well supplemented by the following algebraic de Rham theorem (A.Grothendieck (1966)).

Let X be a compact algebraic manifold and $S \subset X$ be a positive divisor. Then the ring of de Rham cohomologies of the affine algebraic manifold $U = X \setminus S$ is generated by rational forms on X with poles on S .

Example (Leray, 1959b). Let $X = \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1}$, $S = \{(\xi, z) \in \mathbb{C}P^{n-1} \times \mathbb{C}P^{n-1} : \xi \cdot z = 0\}$, $\varphi = \frac{\omega'(\xi) \wedge \omega'(z)}{(\xi \cdot z)^n}$.

The residue-class of this meromorphic form does not contain holomorphic forms. This is a fundamental observation implying, for example, from the Cauchy-Fantappiè-Leray formula, that in the general residue theory of holomorphic forms on analytic manifolds, one is forced to consider functions and forms not necessarily analytic.

This example of a holomorphic form φ on an affine algebraic manifold $X \setminus S$ with a singularity of finite order on S and non cohomologous to a holomorphic form with a pole of the first order on S is rather a rule than an exception.

Developing this phenomenon, using the above formulated the Grothendieck result, P. Griffiths (1969) has obtained a deep residue-theoretic interpretation of the Hodge theory for a smooth algebraic hypersurface $M \subset \mathbb{C}P^n$. In particular, for any rational n -form φ having a pole of order $\leq p+1$ on M the corresponding residue-class $\text{Res}[\varphi]$ belongs to the space

$$\bigoplus_{q \leq p} H^{n-q-1, q}(M)$$

and conversely any element of the latter space (besides $[\mathbb{C}P^{(n-1)/2} \cap M]$ from $H^{(n-1)/2, (n-1)/2}(M)$, if n is odd) can be obtained as the residue of an appropriate rational n -form with pole of order $\leq p+1$ on M .

In order to formulate the general Leray residue-formula it is necessary to introduce the definition of the Leray co-boundary.

Definition (Leray, 1959b). Consider a tubular neighbourhood V of submanifold $S \subset X$, which is a locally trivial bundle with base S and fiber V_a , $a \in S$, homeomorphic to the disc. To each $(p-1)$ -dimensional element of a $(p-1)$ -chain σ_{p-1} in S corresponds p -dimensional chain in $X \setminus S$ of the form

$$\delta\sigma_{p-1} = \cup_{a \in \sigma_{p-1}} \delta a, \quad \text{where } \delta a = \partial V_a,$$

homeomorphic to $\partial V_a \times \sigma_{p-1}$ with a natural orientation. Since $\partial\delta = -\delta\partial$, we have the co-boundary Leray homomorphism of homology groups

$$\delta : H_{p-1}(S) \rightarrow H_p(X \setminus S).$$

If a family S_1, \dots, S_m of submanifolds of co-dimension 1 is in general position, then a multiple co-boundary homomorphism is well defined

$$\delta^m : H_{p-m}(S_1 \cap \dots \cap S_m) \rightarrow H_p(X \setminus (S_1 \cup \dots \cup S_m)),$$

and anti-commutative relative to the order S_1, \dots, S_m . The homomorphism δ^{-1} was firstly defined by Poincaré (1887).

The residue-formula (Leray, 1959b). *Under the conditions of the previous definition, for an arbitrary closed p -form φ on $X \setminus (S_1 \cup \dots \cup S_m)$ and for a cycle $\sigma \in Z_{p-m}(S_1 \cap \dots \cap S_m)$ we have the residue formula*

$$(5.4) \quad \int_{\delta^m \sigma} \varphi = (2\pi i)^m \int_{\sigma} \text{Res}^m[\varphi].$$

In applications of the residue formula, the following very simple algorithm for reducing a polymeromorphic form φ to a cohomologous form having a pole of the first order is useful.

Proposition (Leray, 1959b, Z. Shapiro, 1958). Let submanifolds S_j be represented in a neighbourhood V of the set $S_1 \cap \dots \cap S_m$ be represented by equations $s_j(z) = 0$, where s_j are functions holomorphic in V , $j = 1, \dots, m$. If a form $\varphi \in Z^p(X \setminus (S_1 \cup \dots \cup S_m))$ in V is represented as

$$\varphi = \frac{ds_1 \wedge \dots \wedge ds_m \wedge \omega}{s_1^{r_1+1} \dots s_m^{r_m+1}},$$

then

$$\text{Res}^m[\varphi] \ni \frac{1}{r_1! \dots r_m!} \frac{\partial^{r_1+\dots+r_m} \omega}{\partial s_1^{r_1} \dots \partial s_m^{r_m}} \Big|_{S_1 \cap \dots \cap S_m},$$

where $\frac{\partial^{r_1+\dots+r_m} \omega}{\partial s_1^{r_1} \dots \partial s_m^{r_m}}$ is recurrently found from the equality

$$d\omega = ds_1 \wedge \frac{\partial \omega}{\partial s_1} + \dots + ds_m \wedge \frac{\partial \omega}{\partial s_m},$$

resulting from the condition that φ be closed:

$$d\omega \wedge ds_1 \wedge \dots \wedge ds_m = 0.$$

Useful generalization of the Leray residue-formula for the case of complex (respectively real-analytic) submanifolds of arbitrary co-dimension were obtained by F. Norguet, J. King and J. Poly.

Let S be complex (respectively real-analytic) submanifolds of co-dimension q in the complex (respectively real-analytic) manifold X , i.e. for any $a \in S$ there exists a neighbourhood U_a such that

$$S \cap U_a = \{z \in U_a : s_1(z) = \dots = s_q(z) = 0\},$$

where s_j are holomorphic (respectively real-analytic) functions in U_a and $ds_1 \wedge \dots \wedge ds_q \neq 0$ on U_a .

A differential form $\tilde{\varphi}$, regular in $X \setminus S$, is called simple if there exist forms ψ and θ , regular in neighbourhood $V(S)$, such that

$$(5.5) \quad \tilde{\varphi}|_{V(S)} = \omega' \left(\frac{\bar{s}}{|s|^2} \right) \wedge \psi + \theta,$$

where

$$\bar{s} = (\bar{s}_1, \dots, \bar{s}_q), \quad |s|^2 = \sum_{j=1}^q |s_j|^2.$$

Let $\delta : H_{p-2q+1}(S) \rightarrow H_p(X)$ be the co-boundary homomorphism induced by the mapping $a \mapsto \delta a = \partial V_a$, $a \in S$, where V_a is $2q$ -dimensional ball-fiber of the locally-trivial neighbourhood bundle $V(S) \rightarrow S$ with base S .

The residue-formula (Norguet (1964, 1971), King (1970), Poly (1972)). *Any closed p -form φ on $X \setminus S$ is cohomologous to some closed simple form $\tilde{\varphi}$ on $X \setminus S$ of the type (5.5). Moreover, for any cycle $\sigma \in Z_{p-2q+1}(S)$ we have the residue-formula*

$$(5.6) \quad \int_{\delta\sigma} \varphi = \int_{\delta\sigma} \tilde{\varphi} = \frac{(2\pi i)^q}{(q-i)!} \int_{\sigma} \psi|_S.$$

Example (Martineau, 1960-61, Tsuno, 1982). The residue-formula (5.6), extended to the case $X = \mathbb{C}^n$, $S = \mathbb{R}^n$, and φ a $\bar{\partial}$ -closed $(0, n-1)$ -form on $X \setminus S$ leads to the clearest interpretation of hyperfunctions of M. Sato (1959-60). Namely, for this case the mapping

$$\text{Res} : [\varphi] \mapsto \Phi(x) = \text{Res} [\varphi](x) = \lim_{\varepsilon \rightarrow 0} \int_{\{\theta \in \mathbb{R}^n : |\theta|=1\}} \varphi(x + i\varepsilon\theta)$$

realizes an isomorphism of cohomologies $[\varphi] \in H^{0, n-1}(\mathbb{C}^n \setminus \mathbb{R}^n)$ and of hyperfunctions Φ on \mathbb{R}^n .

For the present case, φ is $\bar{\partial}$ -cohomologous to the $\bar{\partial}$ -closed simple form $\tilde{\varphi}$, iff $\text{Res} [\varphi] = \Phi$ is a smooth function on \mathbb{R}^n .

If Φ has compact support then a simple form $\tilde{\varphi}$, equivalent to φ , can be taken of the following type

$$\tilde{\varphi}(z) = \frac{(n-1)!}{(2\pi i)^n} \int_{x \in \mathbb{R}^n} \Phi(x) \omega' \left(\frac{\bar{z-x}}{|z-x|^2} \right) \wedge dx, \quad z \in \mathbb{C}^n \setminus \mathbb{R}^n.$$

We describe some important applications of the Leray residue theory.

Leray (1959b, 62b) applied the residue theory to the investigation of concrete integrals depending on parameters arising from solving the Cauchy problem (section 4).

The techniques of Leray (1959, 62) in combination with the Grothendieck algebraic de Rham theorem allowed Atiyah-Bott-Gårding (1970, 73) to clarify and to develop the Petrowski theory of lacunas for hyperbolic differential operators.

F. Pham (1967) developed Leray's investigation in a more general context. Namely, one can consider the integral $I(t) = \int_{x \in \gamma} \omega(x, t)$ of a rational (algebraic) differential p -form $\omega(x, t)$ depending algebraically on the parameter $t \in T$, over a p -cycle γ of an algebraic manifold X , where γ does not intersect the singularity $S(t)$ of the form $\omega(x, t)$.

It was proved that an integral $I(t)$ is a (multivalued) analytic function of the parameter t outside of an analytic manifold $L \subset T$, called the Landau manifold. L. Landau (1959) considered such manifolds ("apparent contours") while investigating singularities of Feymann integrals. The nature of the singularity of the integral $I(t)$ near the Landau manifold was examined. For the case considered by Leray, the singularities $\omega(z, t)$ have the form of poles on the hypersurface $S(t) = \{x \in X : s(x, t) = 0\}$ depending linearly on t . To the Landau manifold corresponds the manifold L of those values t , where S_t has a singular (double quadratic) point. For this case Leray (1959b), applying the Picard-Lefschetz formula and a residue-formula, prove the following .

Let $p = n = \dim_{\mathbb{C}} X$. Then going around the manifold L along a simple loop, beginning and ending in a point $t_0 \in T \setminus L$, the integral $I(t_0)$ turns into

$$I(t_0) + (-1)^{\frac{(n-1)(n-2)}{2}} (2\pi i) N \int_e \text{Res } \omega(x, t),$$

where e is a so called $(n-1)$ -dimensional "vanishing" cycle on S_{t_0} and N is the linking index of e with γ . Hence, Leray (1959b) obtains explicit formulas for the singular part $I(t)$ in the neighbourhood of L . As singularities of this integral, only poles, algebraic singularities of the second order or logarithmic singularities can appear.

Further, Leray (1967a), generalizing the work of N. Nilsson (1964), applied the residue theory to the investigation of singularities of integrals of the large class of multivalued analytic forms, whose singularities form algebraic submanifolds.

More precisely, Nilsson (1964) and Leray (1967a) consider p -forms $\omega(z, t)$, $z \in \mathbb{C}P^n$, depending on the parameter $t \in \mathbb{C}^N$ and having algebraic singularities, i.e. admitting a holomorphic extension to simply connected coverings of complements to algebraic hypersurfaces. Moreover, in $\mathbb{C}P^n \times \mathbb{C}^N$ they consider algebraic hypersurfaces

$$V_j(t) = \{(z, t) \in \mathbb{C}P^n \times \mathbb{C}^N : v_j(z, t) = 0\}, \quad j = 1, \dots, m,$$

hyperplanes

$$W_k(t) = \{(z, t) \in \mathbb{C}P^n \times \mathbb{C}^N : w_k(t) \cdot z = 0\}, \quad k = 1, \dots, l,$$

algebraically depending on $t \in \mathbb{C}^N$, and relative $(p-m)$ -cycles $\gamma^{p-m}(t)$ on

$$V(t) = V_1(t) \cap \dots \cap V_m(t) \quad \text{respectively} \quad W(t) = \cup_{k=1}^l w_k(t),$$

depending continuously on $t \in \mathbb{C}^N$

Further, they investigate integrals of the type

$$J(t) = \int_{\gamma^{p-m}} \operatorname{Res} \frac{\omega(z, t)}{v_1^{p_1}(z, t) \dots v_m^{p_m}(z, t)}.$$

For the simplest and important case $p = n$ the main result of this investigation is that the function $J(t)$ has algebraic singularities on \mathbb{C}^N , i.e. admits holomorphic extension to the universal covering of the complement to some algebraic hypersurface in \mathbb{C}^N . If, in addition, the branches ω are linear combinations of a finite number of branches then the branches $J(t)$ are also linear combinations of a finite number of branches.

Leray (1967a) remarked that this result in combination with results of Leray (1962b) is immediately applicable to the investigation of (algebraic) singularities of fundamental solutions of hyperbolic equations.

Among more recent investigations on this subject we note the works of Nilsson (1980), J. Vaillant (1986) and E. Andronikof (1992). For other fundamental applications of the Leray residue theory to analysis and geometry see in D. Zeilberger (1978), L. Aizenberg, A. Yuzhakov (1983), P. Dolbeault (1990), A. Tsikh (1992), C. Berenstein, R. Gay, A. Vidras, and A. Yger (1993).

6. Cauchy-Fantappiè-Leray formulas

There are many different generalizations of the Cauchy formula (1831) for functions of one complex variable to analytic functions of several complex variables (Cauchy (1841), S. Bergman (1922,34), A. Weil (1932, 35), E. Martinelli (1938, 43), S. Bochner (1943, 44), L. Fantappiè (1943), Hua Lo Ken (1958)....)

During a long time it remained unclear which integral representation seemed to be a true multidimensional analog of the classical Cauchy formula.

Leray (1956d, 59b), developing on the one hand the Herglotz-Petrowski-Leray (1953a) formula and on the other hand the theory of the analytic Fantappiè functionals (1943), found the formula (called by him the Cauchy-Fantappiè formula), which was later found to include, as particular cases, all possible integral representations for functions of several complex variables.

Let Ω be a domain in \mathbb{C}_z^n with coordinates $z = (z_1, \dots, z_n)$. Consider the projective space $\mathbb{C}P_\xi^n$ with homogeneous coordinates $\xi = (\xi_0, \xi_1, \dots, \xi_n)$. In $\mathbb{C}P_\xi^n \times \Omega$ we consider the submanifolds

$$\begin{aligned} Q &= \{(\xi, z) \in \mathbb{C}P_\xi^n \times \Omega : \langle \xi \cdot z \rangle = 0\} \\ P_w &= \{(\xi, z) \in \mathbb{C}P_\xi^n \times \Omega : \langle \xi \cdot w \rangle = 0\}, \end{aligned}$$

where w is a point fixed in Ω .

Let $H(Q \setminus P_w)$ denote the homology class on $Q \setminus P_w$ that contains the cycles whose projections onto $\Omega \setminus \{w\}$ are homologous to $\partial\Omega$. Let h_w be a cycle belonging to $H(Q \setminus P_w)$.

First Cauchy-Fantappiè-Leray formula (Leray 1956, 1956d). *For any function $f \in \mathcal{H}(\Omega)$, i.e. holomorphic function in the domain Ω we have*

$$(6.1) \quad f(w) = \frac{(n-1)!}{(2\pi i)^n} \int_{h_w \in H(Q \setminus P_w)} \frac{f(z) \omega'(\xi) \wedge \omega(z)}{\langle \xi \cdot w \rangle^n}.$$

This remarkable formula is both fundamental and simple. What really matters for proving it is to note that the differential form under the integral sign is closed on $Q \setminus P_w$. So, it is sufficient to verify (6.1) for some one cycle $h_w \in H(Q \setminus P_w)$. If as h_w we take the graph of the mapping $z \mapsto \xi(z) = \bar{z} - \bar{w}$, where z belongs to a small sphere with center in w , then the formula (6.1) turns into the classical formula for the average value of a harmonic function.

The residue formula transforms the formula (6.1) into the following.

Second Cauchy-Fantappiè-Leray formula (Leray, 1959b). *If h_w is a cycle from the homology class $H(Q \cap P_w)$, containing the cycle $\{(\xi, z) \in Q \cap P_w : z = w\}$ then*

$$(6.2) \quad f(w) = \frac{(-1)^{n(n-1)/2+1}}{(2\pi i)^{n-1}} \int_{h_w \in H(Q \cap P_w)} \frac{d^{n-1}[f(z) \omega'(\xi) \wedge \omega(z)]}{[d \langle \xi \cdot w \rangle]^n}.$$

The Stokes formula transforms the formula (6.2) into the following.

Third Cauchy-Fantappiè-Leray formula (Leray, 1959b). *If h_w is a cycle from the homology class $H(P_w, Q)$ such that ∂h_w is homologous to the cycle $\{(\xi, z) \in Q \cap P_w : z = w\}$, then*

$$(6.3) \quad f(w) = \frac{(-1)^{n(n-1)/2+1}}{(2\pi i)^{n-1}} \int_{h_w} \frac{d^n[f(z) \omega^*(\xi) \wedge \omega(z)]}{[d \langle \xi \cdot w \rangle]^{n+1}}.$$

Comparing formulas (6.3) and (2.2) leads to the very important observation, that (6.3) is a particular case of (2.2), when $m = 0$ and $a(\xi, x) \equiv 1$.

Leray (1959b) estimates the usefulness of these formulas rather reticently: "La formule de Cauchy-Fantappiè....permet de calculer quelques résidus".

The universal character of the Cauchy-Leray formulas (6.1)-(6.3) is that almost all known (and unknown) integral formulas turn out to be special cases for some choice of the cycle h_w in (6.1)-(6.3) and the representative forms in the residue classes in (6.2),(6.3).

The choice of the cycle h_w in the homology class $H(Q \setminus P_w)$ or $H(Q \cap P_w)$ or $H(P_w, Q)$ or the choice of the residue form in the class

$$\frac{d^{n-1}[f(z) \omega'(\xi) \wedge \omega(z)]}{[d \langle \xi \cdot w \rangle]^n} \quad \text{or} \quad \frac{d^n[f(z) \omega^*(\xi) \wedge \omega(z)]}{[d \langle \xi \cdot w \rangle]^{n+1}}$$

is a problem, which requires special consideration.

What really mattered, for most problems connected with the Cauchy problem and considered by J.Leray himself, was only the choice of the the homology classes $H(Q \setminus P_w)$, $H(Q \cap P_w)$, or $H(P_w, Q)$, and not the choice of cycles in them. Nevertheless, for the case of convex domains with smooth boundary $D = \{z \in \mathbb{C}^n : \rho(z) < 0\}$, Leray (1956d) himself found a rather successful realization of the formula (6.1). Namely, for this case, as a cycle h_w in (6.1) one can take the graph of the mapping

$z \mapsto \xi(z) = \{\xi_0(z), \xi_1(z), \dots, \xi_n\}$; where $\xi'(z) = \{\xi_1(z), \dots, \xi_n(z)\}$,

$$\xi_j(z) = \frac{\partial \rho}{\partial w_j}(z), \quad j = 1, 2, \dots, n,$$

$$\xi_0(z) = - \sum_{j=1}^n z_j \frac{\partial \rho}{\partial w_j}(z).$$

Then (6.1) takes the form

$$(6.4) \quad f(w) = \frac{(n-1)!(-1)^n}{(2\pi i)^n} \int_{z \in bD} \frac{f(z) \omega'(\xi(z)) \wedge \omega(z)}{\langle \xi(z) \cdot w \rangle^n},$$

where $f \in \mathcal{H}(D) \cap C(\bar{D})$, $w \in D$.

Having obtained formulas (6.1)–(6.4), J.Leray (1959b) remarks

“L. Fantappiè a plus généralement exprimé $f(w)$ comme de puissances p -ièmes de fonctions linéaires de w (p : entier négatif); d’où l’un des résultats essentiels de sa théorie des “fonctionnelles linéaires analytiques”: une telle fonctionnelle $\mathcal{F}[f]$ est connue quand on connaît les valeurs qu’elle prend lorsque f est la puissance p -ième d’une fonction linéaire. Quand $p = -n$ ce théorème de L.Fantappiè s’explicite de façon particulièrement simple: c’est la formule...., que j’ai énoncée dans la Note (1956d), ...”

We formulate here still two more significant applications of the Leray formula (6.4): to the theory of the analytic Fantappiè-Martineau functionals and to the “fundamental principle” of Euler-Ehrenpreis-Palamodov.

Let D be a linearly concave domain in $\mathbb{C}P^n$ in the sense that for every $z \in D$ there exists a projective hyperplane $\mathbb{C}P_{\xi(z)}^{n-1} = \{w \in \mathbb{C}P^n : \xi(z) \cdot w = 0\}$ depending continuously on z , passing through the point z and contained in D . The compact set $K = \mathbb{C}P^n \setminus D$ is called linearly convex. Suppose, $\{w_0 = 0\} \subset D$. The set of projective hyperplanes, contained in D , forms in the dual space $(\mathbb{C}P^n)^*$ an open set D^* said to be dual to D .

Let $M = \{z \in \mathbb{C}P^n : \tilde{P}_1(z) = \dots = \tilde{P}_r(z) = 0\}$ be k -dimensional algebraic subset of $\mathbb{C}P^n$ of the dimension k , where the homogeneous polynomials $\tilde{P}_1, \tilde{P}_2, \dots, \tilde{P}_r$ are such that $\text{rank}[\text{grad } \tilde{P}_1, \dots, \text{grad } \tilde{P}_r] = n - k$ almost everywhere on M .

Let $\mathcal{H}^*(K)$ denote the space of linear functionals on the space $\mathcal{H}(K)$ of holomorphic functions on K . For the functional $\mu \in \mathcal{H}^*(K)$ we define the Fantappiè indicatrix as the function

$$f(\xi) = \mathcal{F} \mu(\xi) = \langle \mu, \frac{z_0}{\xi \cdot z} \rangle, \quad \xi \in D^*.$$

We have $f \in \mathcal{H}(D^*, \mathcal{O}(-1))$, where $\mathcal{O}(l)$ denotes the line bundle over $(\mathbb{C}P^n)^*$, the sections of which are functions of $(\xi_0, \xi_1, \dots, \xi_n)$ homogeneous of degree l .

The main result of the theory of analytic functionals of A. Martineau (1962, 67) can be formulated as follows:

i) the mapping $\mu \mapsto \mathcal{F} \mu$ realizes an isomorphism between the space $\mathcal{H}^*(K)$ and the space $\mathcal{H}(D^*, \mathcal{O}(-1))$.

In the original work Fantappiè (1943), this result was, mainly, proved for the case when K is a polydisc.

The main application of analytic functionals according to Fantappiè (1943, 56) consists of different methods of integration of partial differential equations with constant coefficients, including an explicit solution of the Cauchy problem.

This application in its generalized form can be summarized as the following result (Henkin (1995)):

ii) The functional $\mu \in H^*(K)$ has support on $K \cap M$ iff its indicatrix of Fantappiè $f = \mathcal{F} \mu$ satisfies the system of differential equations

$$(6.5) \quad \tilde{P}_j \left(\frac{d}{d\xi} \right) f(\xi) = 0, \quad j = 1, 2, \dots, r.$$

Suppose, further, M be a complete intersection (i.e. $r = n - k$) and $D = \cup D_\varepsilon$, where D_ε are linearly concave domains with smooth boundaries, compactly supported in D and $D_{\varepsilon_1} \supset D_{\varepsilon_2}$ for $\varepsilon_1 < \varepsilon_2$.

iii) In this case the solution $f(\eta)$, $\eta \in D^*$ of the system (6.5) from the space $\mathcal{H}(D^*, \mathcal{O}(-1))$ can be represented by the following "sum" of elementary solutions

$$(6.6) \quad \frac{1}{\eta_0 + \eta' \cdot z}, \quad z \in (bD_\varepsilon) \cap M, \quad \varepsilon < \varepsilon_0,$$

$$f(\eta) = \frac{(-1)}{(2\pi i)^k (k-1)!} \int_{z \in (bD_\varepsilon) \cap M} \frac{\det [Q(z, D), \xi'(z), \bar{\partial} \xi'(z)] \frac{\partial^{k-1} f(\xi(z))}{\partial \xi_0^{k-1}} \wedge dP]}{\eta_0 + \eta' \cdot z} dz,$$

where $z \mapsto \xi(z)$ is the mapping as in the Leray formula (6.4),

$$D = \left\{ \left(\frac{\partial}{\partial \xi_0} \right)^{-1} \frac{\partial}{\partial \xi_1}, \dots, \left(\frac{\partial}{\partial \xi_0} \right)^{-1} \frac{\partial}{\partial \xi_n} \right\},$$

$$P_j(z) = \tilde{P}_j(1, z_1, \dots, z_n), \quad j = 1, 2, \dots, n - k,$$

$$dP] dz = \text{res} \frac{dz_1 \wedge \dots \wedge dz_n}{P_1 \dots P_{n-k}},$$

$$\det [Q(z, w), \xi'(z), \bar{\partial} \xi'(z)] \stackrel{\text{def}}{=} \det [Q^{(1)}(z, w), \dots, Q^{(n-k)}(z, w), \xi'(z), \bar{\partial} \xi'(z), \dots, \bar{\partial} \xi'(z)],$$

$$Q^{(j)}(z, w) = \{Q_1^{(j)}(z, w), \dots, Q_n^{(j)}(z, w)\}, \quad z, w \in \mathbb{C}^n,$$

are polynomials such that

$$P_j(z) - P_j(w) = \langle (z - w) \cdot Q^{(j)}(z, w) \rangle, \quad j = 1, 2, \dots, n - k.$$

The result (ii) can be interpreted as a variant of the Euler (1743), Ehrenpreis (1956-1960, 70) Palamodov (1961, 67) and also R.Gay (1980) "fundamental principle" for the Fantappiè transforms instead of the classical Fourier-Laplace transform.

The result (iii) is an explicit version of (ii) in the spirit of the explicit version of the “fundamental principle” in the Berndtsson-Passare (1989) paper.

The formula (6.6) can be considered on the one hand as a generalized Leray formula (6.4) and on the other hand as an analog of the Leray formula (2.2).

For other fundamental applications of the Cauchy-Fantappiè-Leray formula to functional analysis, to partial differential equations, to complex analysis and to complex integral geometry see L. Waelbroeck (1960), J. Leray (1961), W. Koppelman (1967), N. Bourbaki (1967), J. Bony (1976), B. Berndtsson (1983, 91), L. Aizenberg and A. Yuzhakov (1983), G. Henkin and J. Leiterer (1984), M. Range (1986), I. Lieb and M. Range (1987), D. Schiltz (1988), G. Henkin (1990, 95), B. Sternin and V. Shatalov (1994), A. D’Agnolo and P. Schapira (1996), M. Anderson (1996) and S. Rigat (1997).

7. The strengthened Cauchy-Kowalewska theorem and the ramified Cauchy problem

The theory of the ramified Cauchy problem is based on the quantitative refinement of the Cauchy-Kowalewska theorem obtained by Leray as a lemma in the proof of his uniformization theorem (section 3). The following statement, which is only the first lemma in the first article of the series (Problème de Cauchy I-VI), is, strange though it may seem, one of the most frequently cited statements among the numerous results of Leray on the Cauchy problem.

Lemma (Leray, 1957b). *Let $a(x, \frac{\partial}{\partial x})$ be a differential operator of degree m with holomorphic coefficients in the ball $B_n(0, R) = \{x \in \mathbb{C}^n : \|x\| < R\}$. Let $g(x, \frac{\partial}{\partial x})$ be the principal part of $a(x, \frac{\partial}{\partial x})$. Let $S = \{x \in B_n(0, R) : s(x) = 0\}$, where $s(x) = x_1$. Then, the solution $u(x)$ of the Cauchy problem (3.1) is holomorphic in the ball $B_n(0, \theta R)$, where*

$$\theta = \frac{1}{12mn} \frac{|g(0, \frac{\partial s}{\partial x})|}{\sup_{\|x\|=R, \|p\|=1} |g(x, p)|}.$$

On the one hand this lemma refines versions of the Cauchy-Kowalewska theorem due to J. Schauder (1935) and I. Petrowski (1937) and on the other hand this theorem was repeatedly refined in works of P. Rosenbloom (1961), L. Hörmander (1963), L. Gårding, T. Kotake, J. Leray (1964) and so on.

Before passing to applications of this Lemma to the ramified Cauchy problem we give three further fundamental generalizations of this Lemma.

We shall start with a result about the possibility of solving a nonhomogeneous linear differential equation with holomorphic coefficients in the neighbourhood of a non-characteristic point of the boundary.

Theorem (Bony-Schapira, 1972). *Consider a linear differential equation with holomorphic coefficients $a(x, \frac{\partial}{\partial x}) f = \varphi$ in a domain $\Omega = \{x \in \mathbb{C}^n : \rho(x) < 0\}$ with a smooth boundary $b\Omega = \{x \in \mathbb{C}^n : \rho(x) = 0\}$.*

Let the real hypersurface $b\Omega$ be non-characteristic with respect to the operator $a(x, \frac{\partial}{\partial x})$ at the point $x_0 \in b\Omega$, i.e. let $g(x_0, \frac{\partial \rho}{\partial x}(x_0)) \neq 0$. Then, there exists a

neighbourhood U_{x_0} of the point x_0 and a holomorphic function $f_0 \in \mathcal{O}(\Omega \cap U_{x_0})$ such that $a(x, \frac{\partial}{\partial x}) f_0 = \varphi$.

The following result is the most complete and exact generalization of the above Leray lemma both regarding the statement as well as the method of proving.

Let Ω be a connected complex n - dimensional manifold, $x' = (x_1, \dots, x_n)$ be local coordinates of the point $x' \in \Omega$. Let Ω_0 be a 1-connected domain in \mathbb{C} . Let $T(\Omega)$ and $T^*(\Omega)$ be the tangent and cotangent bundles to Ω . We provide Ω , $T(\Omega)$ and $T^*(\Omega)$ respectively with Riemannian metrics.

Let $X = \Omega_0 \times \Omega$ and $x = (x_0, x')$, $x_0 \in \Omega_0$, $x' \in \Omega$. Let $a(x, \frac{\partial}{\partial x})$ be a differential operator of order m with holomorphic coefficients on X and with principal part $g(x, \frac{\partial}{\partial x})$. Suppose that for any $\alpha \in \Omega_0$ the complex hypersurface $\alpha \times \Omega = \{(x_0, x') \in X : x_0 = \alpha\}$ is non- characteristic for $a(x, \frac{\partial}{\partial x})$. We represent the characteristic polynomial $g(x, \xi)$ in the form

$$g(x, \xi) = \sum_{r=0}^m g_r(x_0, x', \xi') \xi_0^r, \quad \text{where } \xi' \in \Gamma_{x'}^*(\Omega).$$

Let $\rho(x_0, x', \xi')$ be a non-negative root of the equation:

$$\sum_{r=0}^{m-1} |g_r(x_0, x', \xi')| \rho^r = |g_m(x_0, x')| \rho^m.$$

Set

$$\rho_{x_0} = \sup_{\{(x', \xi') \in T^*(\Omega) : |\xi'|=1\}} \rho(x_0, x', \xi').$$

Theorem (Hamada, Leray, Takeuchi, 1985). *Suppose, that the Riemannian metric on Ω is such that the function $x_0 \mapsto \rho_{x_0}$ is locally bounded (this condition certainly holds if we replace the given metric by a conformally equivalent metric increasing at infinity (i.e. at $b\Omega$) rather rapidly) . Then the Cauchy problem*

$$(7.1) \quad a(x, \frac{\partial}{\partial x}) u(x) = v(x); \quad u(x) = O(|x_0 - \alpha|^m), \quad \alpha \in \Omega_0$$

has a unique solution holomorphic in the domain

$$\Delta = \{(x_0, x') \in X : \text{dist}(x_0, \alpha) \subset \text{dist}(x', b\Omega)\},$$

if the given a and v are holomorphic on Δ .

This beautiful statement is the best possible in these terms. In Leray's recent work (1992) the generalization of this theorem to a large class of nonlinear holomorphic systems of equations was obtained.

Another interesting variant in the development of the above mentioned Leray lemma was realized in D. Schiltz's work (1988) (as a thesis under Leray's and J. Vaillant's direction).

Consider the homogeneous Cauchy problem

$$(7.2) \quad a\left(x, \frac{\partial}{\partial x}\right) u = 0, \quad \left(\frac{\partial}{\partial x_0}\right)^k u|_S = w_k(x'), \quad k = 0, 1, \dots, m-1,$$

where $x = (x_0, x') = (x_0, x_1, \dots, x_n) \in \mathbb{C}^{n+1}$; S is the hyperplane $x_0 = 0$ in \mathbb{C}^{n+1} ; the functions w_k are holomorphic in a pseudoconvex domain Ω^n with a smooth boundary $\partial\Omega^n$ in $S \simeq \mathbb{C}^n$; $a\left(x, \frac{\partial}{\partial x}\right)$ is an operator of order m with holomorphic coefficients in a domain $\Omega^{n+1} \subset \mathbb{C}^{n+1}$, containing $\overline{\Omega}^n$; $g(x, \xi)$ is the characteristic polynomial for $a\left(x, \frac{\partial}{\partial x}\right)$.

The point $y \in \partial\Omega^n$ is by definition non-characteristic for $a\left(x, \frac{\partial}{\partial x}\right)$, if the m complex analytic bicharacteristics of the operator a , passing through y are mutually different.

Theorem (Schiltz, 1988). *The complex analytic bicharacteristics of the operator $a\left(x, \frac{\partial}{\partial x}\right)$ issuing from $\partial\Omega^n \cap G$ in a neighbourhood $G \subset \mathbb{C}^{n+1}$ of the non-characteristic point $\partial\Omega^n$ generate m real characteristic hypersurfaces Γ_j , $j = 1, 2, \dots, m$. Any holomorphic solution of the Cauchy problem (7.2) can be expressed as a sum of functions which are holomorphic in subdomains G_j in G with boundaries $\partial G_j \supset \Gamma_j \cap G$, $j = 1, 2, \dots, m$. This means, in particular, that a solution of the Cauchy problem u extends analytically to the domain $\cap_{j=1}^m G_j$ and, in general, does not extend to a larger domain inside G .*

The proof of this result uses the Cauchy-Fantappiè-Leray formula in the following interesting way. Due to well-known theorems of approximation one can suppose, without loss of generality, that the domain Ω^n is strictly pseudoconvex and the functions $\{w_k\}$ are continuous on $\overline{\Omega}^n$. Using the generalization of the Cauchy-Fantappiè-Leray formula for strictly pseudoconvex domains (see Henkin-Leiterer (1984)), the functions $\{w_k\}$ can be represented as sums of meromorphic functions with poles outside Ω^n and tangent to $\partial\Omega^n$.

Due to the below mentioned Hamada-Leray-Wagschal theorem (1976a) for data $\{w_k\}$ meromorphic in the neighbourhood $\overline{\Omega}^n$, a holomorphic solution of the Cauchy problem (7.2) exists in the form of functions ramified along analytic bicharacteristics issuing from the singularities of $\{w_k\}$. Consequently, for any $\{w_k\}$ holomorphic on Ω^n in a neighbourhood G of any non-characteristic point on $\partial\Omega$, a solution $u(x)$ of the problem (7.2) is certainly holomorphic in the connected component, containing $\Omega^n \cap G$, of the complement in G to the union of the characteristics issuing from $\partial\Omega^n \cap G$.

Remark. Another proof of this result (and its generalization to general systems of linear differential equations with holomorphic coefficients) based on the microlocal version of the Leray sheaf theory was obtained by D'Agnolo and Schapira (1991).

Now we consider the main results concerning the ramified Cauchy problem aimed at proving the following part of the Leray conjecture (1956b): the singularities of the solution of the non-characteristic Cauchy problem belong to the union of the characteristics issuing from the singularities of the initial data.

Consider again the Cauchy problem (7.2), supposing that the hyperplane $\Omega^n = \{x \in \Omega^{n+1} : x_0 = 0\}$ is non-characteristic for $a\left(x, \frac{\partial}{\partial x}\right)$ and the $\{w_k(x')\}$ are (multivalued) holomorphic functions on $\Omega^n \setminus \Omega^{n-1}$ ramified along the plane

$\Omega^{n-1} = \{x \in \Omega^{n+1} : x_0 = x_1 = 0\}$ or having singularities (polar or essential) on Ω^{n-1} .

After the first results of Y. Hamada (1969, 70) and C. Wagschal (1972) on the propagation of polar singularities in the problem (7.2) Hamada, Leray, and Wagschal obtained the following general result.

Theorem (Hamada, Leray, Wagschal, 1976a). *Suppose the operator $a(x, \frac{\partial}{\partial x})$ has characteristics polynomial of constant multiplicity, in the neighbourhood G of zero in \mathbb{C}^{n+1} i.e. suppose the characteristic polynomial of the operator a have the form*

$$(7.3) \quad g(x, \xi) = \Pi_s g_s(x, \xi)^{d_s},$$

where $g_s(x, \xi)$ are polynomials with respect to ξ with coefficients holomorphic with respect to x and all the roots ξ_0 of the equation $\Pi_s g_s(0, \xi_0, 1, 0, \dots, 0) = 0$ are different.

Let d be the number of these roots. Let K_j be characteristic hypersurfaces in G issuing from Ω^{n-1} , i.e. $K_j = \{x \in G : k_j(x) = 0\}$, $j = 1, 2, \dots, d$, where $k_j(x)$ are solutions of the equations

$$\Pi_s g_s(x, \frac{\partial k_j(x)}{\partial x}) = 0, \quad k_j(x)|_{\Omega^n \cap G} = x_1.$$

Then, the Cauchy problem (7.2) has in G a unique solution of the form $u(x) = \sum_{j=1}^d u_j(x)$, where $u_j(x)$ are ramified on G along K_j . More precisely $u(x)$ admits in G a representation

$$(7.4) \quad u(x) = \sum_{j=1}^d U_j(k_j(x), x),$$

where the functions $U_j(t, x)$ are holomorphic on $\tilde{\Omega}^1 \times (\Omega^n \cap G)$, $\tilde{\Omega}^1$ is the universal covering of some neighbourhood of zero in \mathbb{C} with zero deleted.

Moreover, M. Kashiwara and P. Schapira (1978) proposed a new proof and an extension of this result to general systems (D -modules) and to the case when the initial data and solution of the problem (7.2) are ramified functions of logarithmic type.

More recently, an even more general result was obtained by E. Leichtnam. Consider, under conditions of the previous theorem, the following nonhomogeneous ramified Cauchy problem

$$(7.5) \quad \begin{aligned} a(x, \frac{\partial}{\partial x}) u(x) &= v(x), \\ (\frac{\partial}{\partial x_0})^k u|_S &= w_k(x'), \quad k = 0, 1, \dots, m-1, \end{aligned}$$

where $v(x)$ and $\{w_k(x')\}$ are ramified holomorphic functions respectively on $G \setminus (\cup_{j=1}^d K_j)$ and $G \cap (\Omega^n \setminus \Omega^{n-1})$, $S = \Omega^n \cap G$.

Leichtnam (1990) proved that the Cauchy problem (7.5) has a unique solution holomorphic on the universal covering of $G \setminus (\cup_{j=1}^d K_j)$.

Examples show that in the above conditions a tempered growth of data $v(x)$ and $\{w_k(x')\}$ near $\cup_{j=1}^d K_j$ does not lead, in general, to a tempered growth of the solution $u(x)$ of the problems (7.2) or (7.5) near $\cup_{j=1}^d K_j$. However, for operators $a(x, \frac{\partial}{\partial x})$ with simple characteristics (i.e. when in the representation (7.3) all $d_s = 1$) it is proved (Leichtnam (1990)) that a tempered growth in the sense of Nilsson for the Cauchy data near $\cup_{j=1}^d K_j$ implies a tempered growth for the solution $u(x)$.

The systematic investigation of the ramified Cauchy problem for operators with characteristics of non constant multiplicity was carried on after 1977 by Y. Hamada, G. Nakamura, J. Vaillant, and C. Wagschal. In the the works of Wagschal (1983) and D. Schiltz, J. Vaillant, and C. Wagschal (1982), they obtained significant generalizations of the result (7.4) of Hamada, Leray, and Wagschal (1976) which lead, in particular, to a more precise form of the original Leray conjecture about the propagation of singularities in the ramified Cauchy problem.

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