Chapter 2
The Special Theory of Relativity

In this chapter we shall give a short introduction to the fundamental principles of the special theory of relativity and deduce some of the consequences of the theory.

The special theory of relativity was presented by Albert Einstein in 1905. It was founded on two postulates:

1. The laws of physics are the same in all Galilean frames.
2. The velocity of light in empty space is the same in all Galilean frames and independent of the motion of the light source.

Einstein pointed out that these postulates are in conflict with Galilean kinematics, in particular with the Galilean law for addition of velocities. According to Galilean kinematics two observers moving relative to each other cannot measure the same velocity for a certain light signal. Einstein solved this problem by a thorough discussion of how two distant clocks should be synchronized.

2.1 Coordinate Systems and Minkowski Diagrams

The most simple physical phenomenon that we can describe is called an event. This is an incident that takes place at a certain point in space and at a certain point in time. A typical example is the flash from a flashbulb.

A complete description of an event is obtained by giving the position of the event in space and time. Assume that our observations are made with reference to a reference frame. We introduce a coordinate system into our reference frame. Usually it is advantageous to employ a Cartesian coordinate system. This may be thought of as a cubic lattice constructed by measuring rods. If one lattice point is chosen as origin, with all coordinates equal to zero, then any other lattice point has three spatial coordinates equal to the distances of that point along the coordinate axes that pass through the origin. The spatial coordinates of an event are the three coordinates of the lattice point at which the event happens.

It is somewhat more difficult to determine the point of time of an event. If an observer is sitting at the origin with a clock, then the point of time when he catches sight of an event is not the point of time when the event happened. This is because

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the light takes time to pass from the position of the event to the observer at the origin. Since observers at different positions have to make different such corrections, it would be simpler to have (imaginary) observers at each point of the reference frame such that the point of time of an arbitrary event can be measured locally.

But then a new problem appears. One has to synchronize the clocks, so that they show the same time and go at the same rate. This may be performed by letting the observer at the origin send out light signals so that all the other clocks can be adjusted (with correction for light travel time) to show the same time as the clock at the origin. These clocks show the *coordinate time* of the coordinate system, and they are called *coordinate clocks*.

By means of the lattice of measuring rods and coordinate clocks, it is now easy to determine four coordinates ($x^0 = ct, x, y, z$) for every event. (We have multiplied the time coordinate t by the velocity of light c in order that all four coordinates shall have the same dimension.)

This coordinatization makes it possible to describe an event as a point P in a so-called *Minkowski diagram*. In this diagram we plot ct along the vertical axis and one of the spatial coordinates along the horizontal axis.

In order to observe particles in motion, we may imagine that each particle is equipped with a flash-light and that they flash at a constant frequency. The flashes from a particle represent a succession of events. If they are plotted into a Minkowski diagram, we get a series of points that describe a curve in the continuous limit. Such a curve is called a *world line* of the particle. The world line of a free particle is a straight line, as shown to left of the time axis in Fig. 2.1.

A particle acted upon by a net force has a curved world line as the velocity of the particle changes with time. Since the velocity of every material particle is less than the velocity of light, the tangent of a world line in a Minkowski diagram will always make an angle less than 45° with the time axis.

A flash of light gives rise to a light front moving onwards with the velocity of light. If this is plotted in a Minkowski diagram, the result is a light cone. In Fig. 2.1 we have drawn a light cone for a flash at the origin. It is obvious that we could have

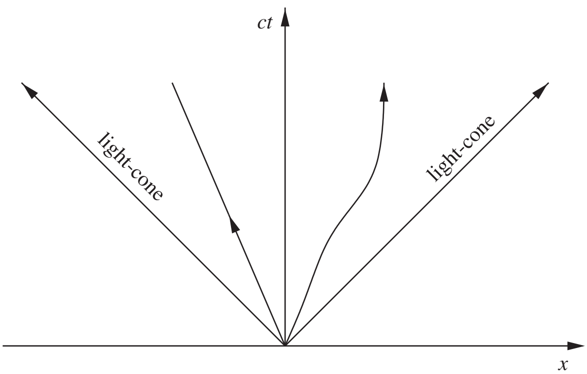

Fig. 2.1 World lines

drawn light cones at all points in the diagram. An important result is that *the world line of any particle at a point is inside the light cone of a flash from that point*. This is an immediate consequence of the special principle of relativity, and is also valid locally in the presence of a gravitational field.

2.2 Synchronization of Clocks

There are several equivalent methods that can be used to synchronize clocks. We shall here consider the radar method.

We place a mirror on the x-axis and emit a light signal from the origin at time t_A. This signal is reflected by the mirror at t_B, and received again by the observer at the origin at time t_C. According to the second postulate of the special theory of relativity, the light moves with the same velocity in both directions, giving

$$t_B = \frac{1}{2}(t_A + t_C) . \tag{2.1}$$

When this relationship holds we say that the clocks at the origin and at the mirror are *Einstein synchronized*. Such synchronization is presupposed in the special theory of relativity. The situation corresponding to synchronization by the radar method is illustrated in Fig. 2.2.

The radar method can also be used to measure distances. The distance L from the origin to the mirror is given by

$$L = \frac{c}{2}(t_C - t_A) . \tag{2.2}$$

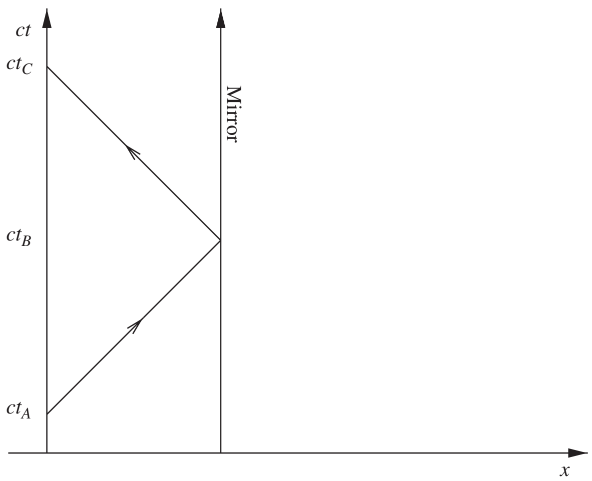

Fig. 2.2 Clock synchronization by the radar method

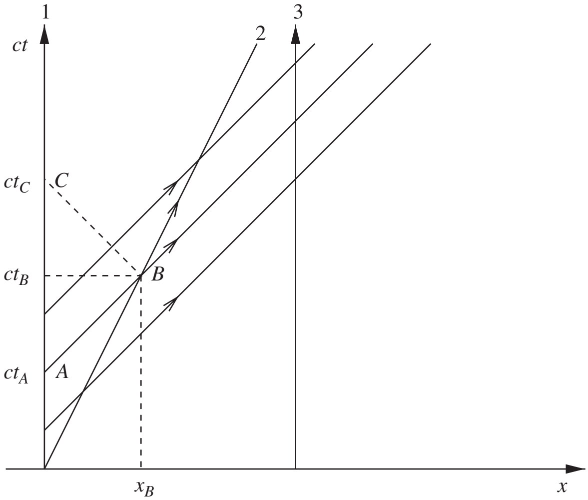

Fig. 2.3 The Doppler effect

2.3 The Doppler Effect

Consider three observers (1, 2 and 3) in an inertial frame. Observers 1 and 3 are at rest, while 2 moves with constant velocity along the *x*-axis. The situation is illustrated in Fig. 2.3.

Each observer is equipped with a clock. If observer 1 emits light pulses with a constant period τ_1, then observer 2 receives them with a longer period τ_2 according to his or her[1] clock. The fact that these two periods are different is a well-known phenomenon, called the *Doppler effect*. The same effect is observed with sound; the tone of a receding vehicle is lower than that of an approaching one.

We are now going to deduce a relativistic expression for the Doppler effect. First, we see from Fig. 2.3 that the two periods τ_1 and τ_2 are proportional to each other,

$$\tau_2 = K\tau_1 \ . \tag{2.3}$$

The constant $K(v)$ is called Bondi's *K*-factor. Since observer 3 is at rest, the period τ_3 is equal to τ_1 so that

$$\tau_3 = \frac{1}{K}\tau_2 \ . \tag{2.4}$$

These two equations imply that if 2 moves away from 1, so that $\tau_2 > \tau_1$, then $\tau_3 < \tau_2$. This is because 2 moves towards 3.

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[1] For simplicity we shall – without any sexist implications – follow the grammatical convention of using masculine pronouns, instead of the more cumbersome "his or her".

The K-factor is most simply determined by placing observer 1 at the origin, while letting the clocks show $t_1 = t_2 = 0$ at the moment when 2 passes the origin. This is done in Fig. 2.3. The light pulse emitted at the point of time t_A, is received by 2 when his clock shows $\tau_2 = Kt_A$. If 2 is equipped with a mirror, the reflected light pulse is received by 1 at a point of time $t_C = K\tau_2 = K^2 t_A$. According to Eq. (2.1) the reflection event then happens at a point of time

$$t_B = \frac{1}{2}(t_C + t_A) = \frac{1}{2}(K^2 + 1)t_A \,. \tag{2.5}$$

The mirror has then arrived at a distance x_B from the origin, given by Eq. (2.2),

$$x_B = \frac{c}{2}(t_C - t_A) = \frac{c}{2}(K^2 - 1)t_A \,. \tag{2.6}$$

Thus, the velocity of observer 2 is

$$v = \frac{x_B}{t_B} = c\frac{K^2 - 1}{K^2 + 1} \,. \tag{2.7}$$

Solving this equation with respect to the K-factor we get

$$K = \left(\frac{c + v}{c - v}\right)^{1/2} \,. \tag{2.8}$$

This result is relativistically correct. The special theory of relativity was included through the tacit assumption that the velocity of the reflected light is c. This is a consequence of the second postulate of special relativity; the velocity of light is isotropic and independent of the velocity of the light source.

Since the wavelength λ of the light is proportional to the period τ, Eq. (2.3) gives the observed wavelength λ' for the case when the observer moves away from the source,

$$\boxed{\lambda' = K\lambda = \left(\frac{c + v}{c - v}\right)^{1/2} \lambda.} \tag{2.9}$$

This Doppler effect represents a redshift of the light. If the light source moves towards the observer, there is a corresponding blueshift given by K^{-1}.

It is common to express this effect in terms of the relative change of wavelength,

$$z = \frac{\lambda' - \lambda}{\lambda} = K - 1 \tag{2.10}$$

which is positive for redshift. If $v \ll c$, Eq. (2.9) gives

$$\frac{\lambda'}{\lambda} = K \approx 1 + \frac{v}{c} \tag{2.11}$$

to lowest order in v/c. The redshift is then

$$z = \frac{v}{c} \,. \tag{2.12}$$

This result is well known from non-relativistic physics.

2.4 Relativistic Time Dilation

Every periodic motion can be used as a clock. A particularly simple clock is called the light clock. This is illustrated in Fig. 2.4.

The clock consists of two parallel mirrors that reflect a light pulse back and forth. If the period of the clock is defined as the time interval between each time the light pulse hits the lower mirror, then $\Delta t' = 2L_0/c$.

Assume that the clock is at rest in an inertial reference frame Σ' where it is placed along the y-axis, as shown in Fig. 2.4. If this system moves along the ct-axis with a velocity v relative to another inertial reference frame Σ, the light pulse of the clock will follow a zigzag path as shown in Fig. 2.5.

The light signal follows a different path in Σ than in Σ'. The period Δt of the clock as observed in Σ is different from the period $\Delta t'$ which is observed in the rest frame. The period Δt is easily found from Fig. 2.5. Since the pulse takes the time $(1/2)\Delta t$ from the lower to the upper mirror and since the light velocity is always the same, we find

$$\left(\frac{1}{2}c\Delta t\right)^2 = L_0^2 + \left(\frac{1}{2}v\Delta t\right)^2 \,, \tag{2.13}$$

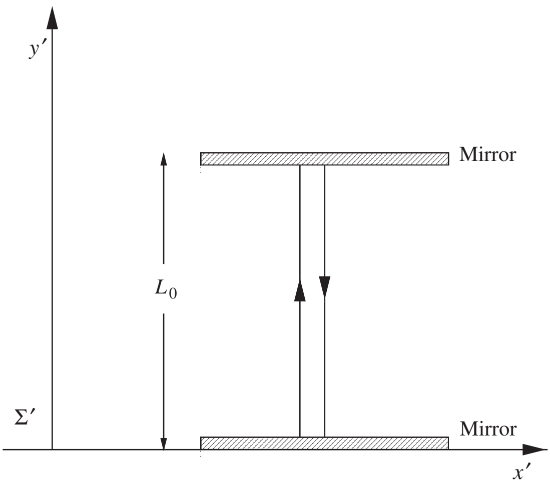

Fig. 2.4 Light clock

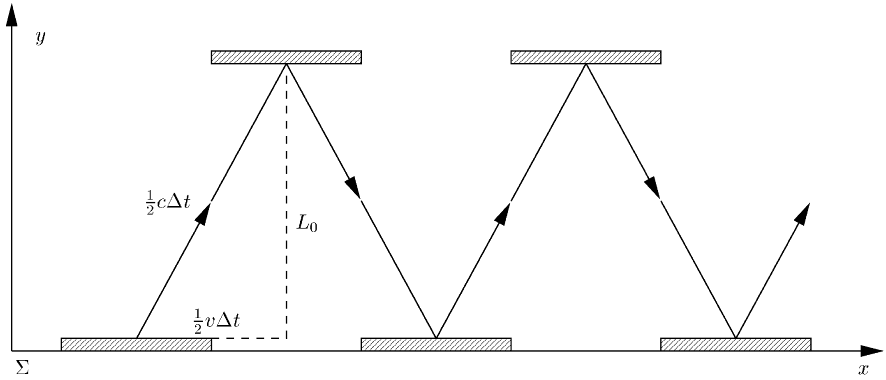

Fig. 2.5 Moving light clock

i.e.

$$\Delta t = \gamma \frac{2L_0}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}}\,. \tag{2.14}$$

The γ factor is a useful short-hand notation for a term which is often used in relativity theory. It is commonly known as the Lorentz factor.

Since the period of the clock in its rest frame is $\Delta t' = 2L_0/c$, we get

$$\boxed{\Delta t = \gamma \Delta t'.} \tag{2.15}$$

Thus, we have to conclude that the period of the clock when it is observed to move (Δt) is greater that its rest period ($\Delta t'$). In other words: *a moving clock goes slower than a clock at rest*. This is called *the relativistic time dilation*. The period $\Delta t'$ of the clock as observed in its rest frame is called the proper period of the clock. The corresponding time t' is called the proper time of the clock.

One might be tempted to believe that this surprising consequence of the special theory of relativity has something to do with the special type of clock that we have employed. This is not the case. If there had existed a mechanical clock in Σ that did not show the time dilation, then an observer at rest in Σ might measure his velocity by observing the different rates of his light clock and this mechanical clock. In this way he could measure the absolute velocity of Σ. This would be in conflict with the special principle of relativity.

2.5 The Relativity of Simultaneity

Events that happen at the same point of time are said to be *simultaneous events*. We shall now show that according to the special theory of relativity, events that are simultaneous in one reference frame are not simultaneous in another reference

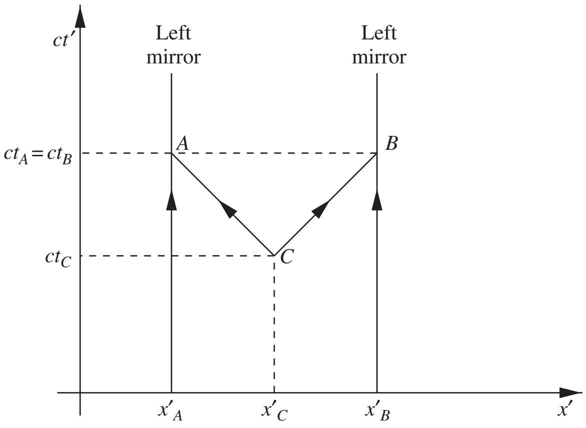

Fig. 2.6 Simultaneous events A and B

frame moving with respect to the first. This is what is meant by the expression "the relativity of simultaneity".

Consider again two mirrors connected by a line along the x'-axis, as shown in Fig. 2.6. Halfway between the mirrors there is a flash-lamp emitting a spherical wave front at a point of time t_C.

The points at which the light front reaches the left-hand and the right-hand mirrors are denoted by A and B, respectively. In the reference frame Σ' of Fig. 2.6 the events A and B are simultaneous.

If we describe the same course of events from another reference frame (Σ), where the mirror moves with constant velocity v in the positive x-direction, we find the Minkowski diagram shown in Fig. 2.7. Note that the light follows world lines making an angle of 45° with the axes. This is the case in every inertial frame.

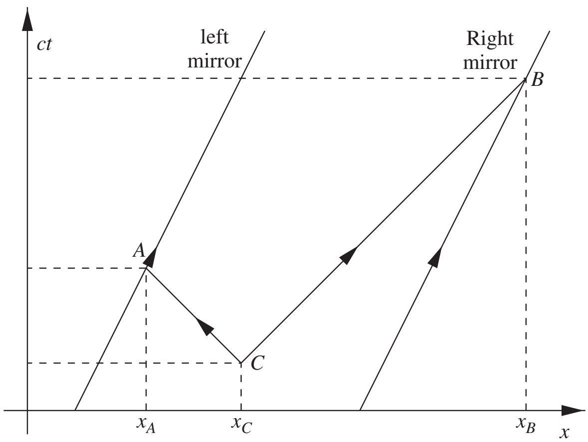

Fig. 2.7 The simultaneous events of Fig. 2.6 in another frame

In Σ the light pulse reaches the left mirror, which moves towards the light, before it reaches the right mirror, which moves in the same direction as the light. In this reference frame the events when the light pulses hit the mirrors are not simultaneous.

As an example illustrating the relativity of simultaneity, Einstein imagined that the events A, B and C happen in a train which moves past the platform with a velocity v. The event C represents the flash of a lamp at the mid-point of a wagon. A and B are the events when the light is received at the back end and at the front end of the wagon, respectively. This situation is illustrated in Fig. 2.8.

As observed in the wagon, A and B happen simultaneously. As observed from the platform the rear end of the wagon moves towards the light which moves backwards, while the light moving forwards has to catch up with the front end. Thus, as observed from the platform A will happen before B.

The time difference between A and B as observed from the platform will now be calculated. The length of the wagon, as observed from the platform, will be denoted by L. The time coordinate is chosen such that $t_C = 0$. The light moving backwards hits the rear wall at a point of time t_A. During the time t_A the wall has moved a distance vt_A forwards, and the light has moved a distance ct_A backwards. Since the distance between the wall and the emitter is $L/2$, we get

$$\frac{L}{2} = vt_A + ct_A \ . \tag{2.16}$$

Thus

$$t_A = \frac{L}{2(c+v)} \ . \tag{2.17}$$

In the same manner one finds

$$t_B = \frac{L}{2(c-v)} \ . \tag{2.18}$$

It follows that the time difference between A and B as observed from the platform is

$$\Delta t = t_B - t_A = \frac{\gamma^2 vL}{c^2} \ . \tag{2.19}$$

As observed from the wagon A and B are simultaneous. As observed from the platform the rear event A happens at a time interval Δt before the event B. This is the relativity of simultaneity.

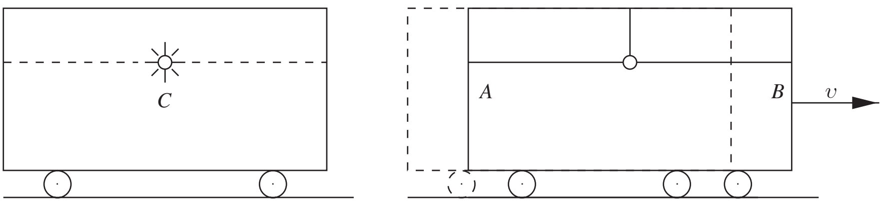

Fig. 2.8 Light flash in a moving train

2.6 The Lorentz Contraction

During the first part of the nineteenth century the so-called luminiferous ether was introduced into physics to account for the propagation and properties of light. After J.C. Maxwell showed that light is electromagnetic waves the ether was still needed as a medium in which electromagnetic waves propagated.

It was shown that Maxwell's equations do not obey the principle of relativity, when coordinates are changed using the Galilean transformations. If it is assumed that the Galilean transformations are correct, then Maxwell's equations can only be valid in one coordinate system. This coordinate system was the one in which the ether was at rest. Hence, Maxwell's equations in combination with the Galilean transformations implied the concept of "absolute rest". This made the measurement of the velocity of the Earth relative to the ether of great importance.

An experiment sufficiently accurate to measure this velocity to order v^2/c^2 was carried out by Michelson and Morley in 1887. A simple illustration of the experiment is shown in Fig. 2.9.

Our earlier photon clock is supplied by a mirror at a distance L along the x-axis from the emitter. The apparatus moves in the x-direction with a velocity v. In the rest frame (Σ') of the apparatus, the distance between A and B is equal to the distance between A and C. This distance is denoted by L_0 and is called the *rest length* between A and B.

Light is emitted from A. Since the velocity of light is isotropic and the distances to B and C are equal in Σ', the light reflected from B and that reflected from C have the same travelling time. This was the result of the Michelson–Morley experiment, and it seems that we need no special effects such as the Lorentz contraction to explain the experiment.

However, before 1905 people believed in the physical reality of absolute velocity. The Earth was considered to move through an "ether" with a velocity that changed with the seasons. The experiment should therefore be described under the assumption that the apparatus is moving.

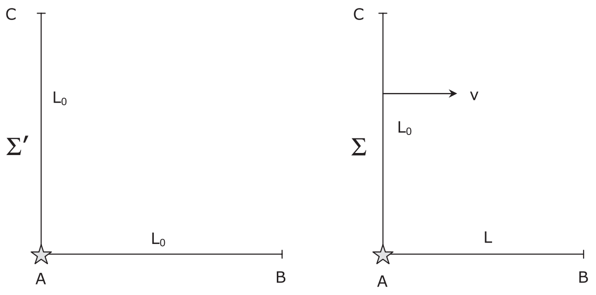

Fig. 2.9 Length contraction

Let us therefore describe an experiment from our reference frame Σ, which may be thought of as at rest in the "ether". Then according to Eq. (2.14) the travel time of the light being reflected at C is

$$\Delta t_C = \gamma \frac{2L_0}{c} \,. \tag{2.20}$$

For the light moving from A to B we may use Eq. (2.18), and for the light from B to A Eq. (2.17). This gives

$$\Delta t_B = \frac{L}{c-v} + \frac{L}{c+v} = \gamma^2 \frac{2L}{c} \,. \tag{2.21}$$

If length is independent of velocity, then $L = L_0$. In this case the travelling times of the light signals will be different. The travelling time difference is

$$\Delta t_B - \Delta t_C = \gamma(\gamma - 1)\frac{2L_0}{c} \,. \tag{2.22}$$

To the lowest order in v/c, $\gamma \approx 1 + \frac{1}{2}(v/c)^2$, so that

$$\Delta t_B - \Delta t_C \approx \frac{1}{2}\left(\frac{v}{c}\right)^2 , \tag{2.23}$$

which depends upon the velocity of the apparatus.

According to the ideas involving an absolute velocity of the Earth through the ether, if one lets the light reflected at B interfere with the light reflected at C (at the position A) then the interference pattern should vary with the season. This was not observed. On the contrary, observations showed that $\Delta t_B = \Delta t_C$.

Assuming that length varies with velocity, Eqs. (2.20) and (2.21), together with this observation, gives

$$\boxed{L = \gamma^{-1} L_0.} \tag{2.24}$$

The result that $L < L_0$ (i.e. the length of a rod is less when it moves than when it is at rest) is called the *Lorentz contraction*.

2.7 The Lorentz Transformation

An event P has coordinates $(t', x', 0, 0)$ in a Cartesian coordinate system associated with a reference frame Σ'. Thus the distance from the origin of Σ' to P measured with a measuring rod at rest in Σ' is x'. If the distance between the origin of Σ' and the position at the x-axis where P took place is measured with measuring rods at rest in a reference frame moving with velocity v in the x-direction relative to Σ', one finds the length $\gamma^{-1}x$ due to the Lorentz contraction. Assuming that the origin of Σ and Σ' coincided at the point of time $t = 0$, the origin of Σ' has an x-coordinate vt at a point of time t. The event P thus has an x-coordinate

$$x = vt + \gamma^{-1} x' \tag{2.25}$$

or

$$x' = \gamma(x - vt) . \tag{2.26}$$

The x-coordinate may be expressed in terms of t' and x' by letting $v \to -v$,

$$\boxed{x = \gamma(x' + vt').} \tag{2.27}$$

The y and z coordinates are associated with axes directed perpendicular to the direction of motion. Therefore, they are the same in the two-coordinate systems

$$y = y' \text{ and } z = z' . \tag{2.28}$$

Substituting x' from Eq. (2.26) in to Eq. (2.27) reveals the connection between the time coordinates of the two-coordinate systems,

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{2.29}$$

and

$$\boxed{t = \gamma\left(t' + \frac{vx'}{c^2}\right).} \tag{2.30}$$

The latter term in this equation is nothing but the deviation from simultaneity in Σ for two events that are simultaneous in Σ'.

The relations (2.27)–(2.30) between the coordinates of Σ and Σ' represent a special case of the *Lorentz transformations*. The above relations are special since the two-coordinate systems have the same spatial orientation, and the x and x'-axes are aligned along the relative velocity vector of the associated frames. Such transformations are called *boosts*.

For non-relativistic velocities $v \ll c$, the Lorentz transformations (2.27)–(2.30) pass over into the corresponding Galilei transformations.

The Lorentz transformation gives a connection between the relativity of simultaneity and the Lorentz contraction. The *length* of a body is defined as the difference between the coordinates of its end points, *as measured by simultaneity in the rest frame of the observer*.

Consider the wagon of Sect. 2.5. Its rest length is $L_0 = x'_B - x'_A$. The difference between the coordinates of the wagon's end points, $x_A - x_B$ as measured in Σ, is given implicitly by the Lorentz transformation

$$x'_B - x'_A = \gamma[x_B - x_A - v(t_B - t_A)] . \tag{2.31}$$

According to the above definition the length (L) of the moving wagon is given by $L = x_B - x_A$ with $t_B = t_A$.

From Eq. (2.31) we then get

$$L_0 = \gamma L \tag{2.32}$$

which is equivalent to Eq. (2.24).

The Lorentz transformation will now be used to deduce the relativistic formulae for velocity addition. Consider a particle moving with velocity u along the x'-axis of Σ'. If the particle was at the origin at $t' = 0$, its position at t' is $x' = u't'$. Using this relation together with Eqs. (2.27) and (2.28) we find the velocity of the particle as observed in Σ

$$\boxed{u = \frac{x}{t} = \frac{u' + v}{1 + \frac{u'v}{c^2}}.} \tag{2.33}$$

A remarkable property of this expression is that by adding velocities less than c one cannot obtain a velocity greater than c. For example, if a particle moves with a velocity c in Σ' then its velocity in Σ is also c regardless of Σ's velocity relative to Σ'.

Equation (2.33) may be written in a geometrical form by introducing the so-called *rapidity* η defined by

$$\boxed{\tanh\eta = \frac{u}{c}} \tag{2.34}$$

for a particle with velocity u. Similarly the rapidity of Σ' relative to Σ is

$$\tanh\theta = \frac{v}{c}\ . \tag{2.35}$$

Since

$$\tanh(\eta' + \theta) = \frac{\tanh\eta' + \tanh\theta}{1 + \tanh\eta' \tanh\theta}\ , \tag{2.36}$$

the relativistic velocity addition formula, Eq. (2.33), may be written as

$$\eta = \eta' + \theta\ . \tag{2.37}$$

Since rapidities are additive, their introduction simplifies some calculations and they have often been used as variables in elementary particle physics.

With these new hyperbolic variables we can write the Lorentz transformation in a particularly simple way. Using Eq. (2.35) in Eqs. (2.27) and (2.30) we find

$$\boxed{x = x'\cosh\theta + ct'\sinh\theta, \quad ct = x'\sinh\theta + ct'\cosh\theta.} \tag{2.38}$$

2.8 Lorentz-Invariant Interval

Let two events be given. The coordinates of the events, as referred to two different reference frames Σ and Σ', are connected by a Lorentz transformation. The coordinate differences are therefore connected by

$$\begin{aligned} \Delta t &= \gamma(\Delta t' + \tfrac{v}{c^2}\Delta x'), \; \Delta x = \gamma(\Delta x' + v\Delta t'), \\ \Delta y &= \Delta y', \qquad\qquad\quad \Delta z = \Delta z'. \end{aligned} \tag{2.39}$$

Just like $(\Delta y)^2 + (\Delta z)^2$ is invariant under a rotation about the x-axis, $-(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$ is invariant under a Lorentz transformation, i.e.

$$\begin{aligned} (\Delta s)^2 &= -(c\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2 \\ &= -(c\Delta t')^2 + (\Delta x')^2 + (\Delta y')^2 + (\Delta z')^2 \,. \end{aligned} \tag{2.40}$$

This combination of squared coordinate intervals is called the spacetime interval, or the *interval*. It is invariant under both rotations and Lorentz transformations.

Due to the minus sign in Eq. (2.40), the interval between two events may be positive, zero or negative. These three types of intervals are called

$$\begin{aligned} &(\Delta s)^2 > 0 \text{ space-like} \\ &(\Delta s)^2 = 0 \text{ light-like} \\ &(\Delta s)^2 < 0 \text{ time-like.} \end{aligned} \tag{2.41}$$

The reasons for these names are the following. Given two events with a space-like interval (A and B in Fig. 2.10), there exists a Lorentz transformation to a new reference frame where A and B happen simultaneously. In this frame the distance between the events is purely spatial. Two events with a light-like interval (C and D in Fig. 2.10) can be connected by a light signal, i.e. one can send a photon from C to D. The events E and F have a time-like interval between them, and can be observed from a reference frame in which they have the same spatial position, but occur at different points of time.

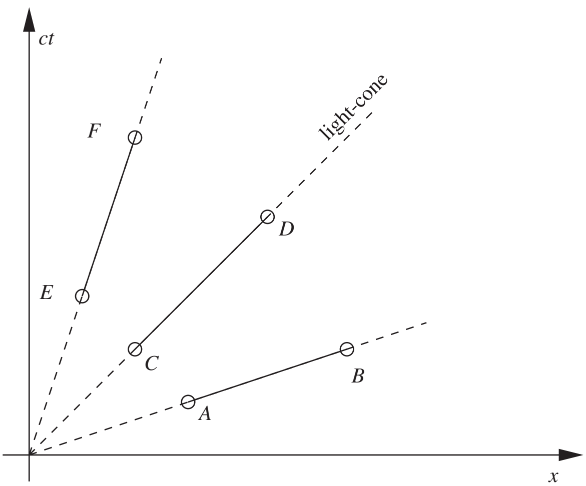

Fig. 2.10 The interval between A and B is space-like, between C and D light-like and between E and F time-like

Since all material particles move with a velocity less than that of light, the points on the world line of a particle are separated by time-like intervals. The curve is then said to be time-like. All time-like curves through a point pass inside the light cone from that point.

If the velocity of a particle is $u = \Delta x/\Delta t$ along the x-axis, Eq. (2.40) gives

$$(\Delta s)^2 = -\left(1 - \frac{u^2}{c^2}\right)(c\Delta t)^2 \,. \tag{2.42}$$

In the rest frame Σ' of the particle, $\Delta x' = 0$, giving

$$(\Delta s)^2 = -(c\Delta t')^2 \,. \tag{2.43}$$

The time t' in the rest frame of the particle is the same as the time measured on a clock carried by the particle. It is called the *proper time* of the particle, and denoted by τ. From Eqs. (2.42) and (2.43) it follows that

$$\Delta\tau = \sqrt{1 - \frac{u^2}{c^2}}\Delta t = \gamma^{-1}\Delta t \tag{2.44}$$

which is an expression of the relativistic time dilation.

Equation (2.43) is important. It gives the physical interpretation of a time-like interval between two events. The interval is a measure of the proper time interval between the events. This time is measured on a clock that moves such that it is present at both events. In the limit $u \to c$ (the limit of a light signal), $\Delta\tau = 0$. This shows that $(\Delta s)^2 = 0$ for a light-like interval.

Consider a particle with a variable velocity, $u(t)$, as indicated in Fig. 2.11. In this situation we can specify the velocity at an arbitrary point of the world line. Equation (2.44) can be used with this velocity, in an infinitesimal interval around this point,

$$d\tau = \sqrt{1 - \frac{u^2(t)}{c^2}}dt \,. \tag{2.45}$$

This equation means that the acceleration has no local effect upon the proper time of the clock. Here the word "local" means as measured by an observer at the position of the clock. Such clocks are called *standard clocks*.

If a particle moves from A to B in Fig. 2.11, the proper time as measured on a standard clock following the particle is found by integrating Eq. (2.45)

$$\tau_B - \tau_A = \int_A^B \sqrt{1 - \frac{u^2(t)}{c^2}}dt \,. \tag{2.46}$$

The relativistic time dilation has been verified with great accuracy by observations of unstable elementary particles with short lifetimes [1].

An infinitesimal spacetime interval

$$ds^2 = -c^2dt^2 + dx^2 + dy^2 + dz^2 \tag{2.47}$$

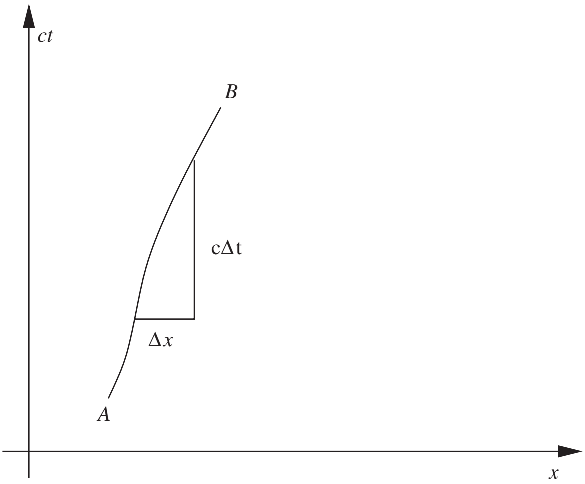

Fig. 2.11 World line of an accelerating particle

is called a *line element*. The physical interpretation of the line element between two infinitesimally close events on a time-like curve is

$$\boxed{ds^2 = -c^2 d\tau^2,} \tag{2.48}$$

where $d\tau$ is the proper time interval between the events, measured with a clock following the curve. The spacetime interval between two events is given by the integral (2.46). It follows that *the proper time interval between two events is path dependent*. This leads to the following surprising result: A time-like interval between two events is *greatest* along the straightest possible curve between them.

2.9 The Twin Paradox

Rather than discussing the life time of elementary particles, we may as well apply Eq. (2.46) to a person. Let her name be Eva. Assume that Eva is rapidly accelerating from rest at the point of time $t = 0$ at origin to a velocity v along the x-axis of a (ct, x) coordinate system in an inertial reference frame Σ. (See Fig. 2.12.)

At a point of time t_P she has come to a position x_P. She then rapidly decelerates until reaching a velocity v in the negative x-direction. At a point of time t_Q, as measured on clocks at rest in Σ, she has returned to her starting location. If we neglect the brief periods of acceleration, Eva's travelling time as measured on a clock which she carries with her is

$$t_{\text{Eva}} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} t_Q \, . \tag{2.49}$$

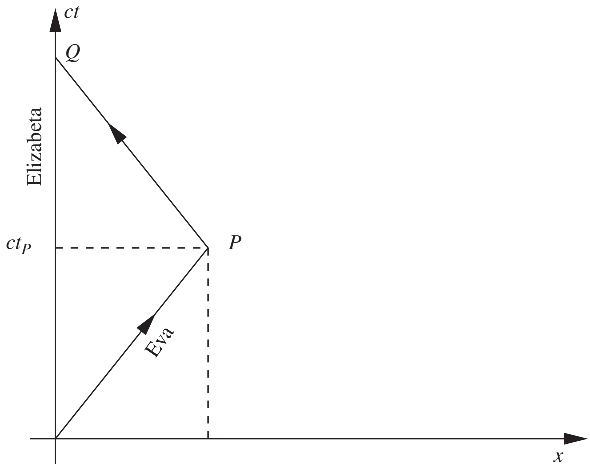

Fig. 2.12 World lines of the twin sisters Eva and Elizabeth

Now assume that Eva has a twin sister named Elizabeth who remains at rest at the origin of Σ.

Elizabeth has become older by $\tau_{\text{Elizabeth}} = t_Q$ during Eva's travel, so that

$$\tau_{\text{Eva}} = \left(1 - \frac{v^2}{c^2}\right)^{1/2} \tau_{\text{Elizabeth}} \,. \tag{2.50}$$

For example, if Eva travelled to Alpha Centauri (the Sun's nearest neighbour at four light years) with a velocity $v = 0.8c$, she would be gone for 10 years as measured by Elizabeth. Therefore Elizabeth has aged 10 years during Eva's travel. According to Eq. (2.50), Eva has only aged 6 years. According to Elizabeth, Eva has aged less than herself during her travels.

The principle of relativity, however, tells that Eva can consider herself as at rest and Elizabeth as the traveller. According to Eva it is Elizabeth who has only aged by 6 years, while Eva has aged by 10 years during the time they are apart.

What happens? How can the twin sisters arrive at the same prediction as to how much each of them age during the travel? In order to arrive at a clear answer to these questions, we shall have to use a result from the general theory of relativity. The twin paradox will be taken up again in Chap. 5.

2.10 Hyperbolic Motion

With reference to an inertial reference frame it is easy to describe relativistic accelerated motion. The special theory of relativity is in no way limited to describe motion with constant velocity.

Let a particle move with a variable velocity $u(t) = dx/dt$ along the x-axis in Σ. The frame Σ' moves with velocity v in the same direction relative to Σ. In this frame the particle velocity is $u'(t') = dx'/dt'$. At every moment the velocities u and u' are connected by the relativistic formula for velocity addition, Eq. (2.33). Thus, a velocity change du' in Σ' and the corresponding velocity change du in Σ are related – using Eq. (2.30) by

$$dt = \frac{dt' + \frac{v}{c^2}dx'}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{1 + \frac{u'v}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}}dt' \,. \tag{2.51}$$

Combining these expressions we obtain the relationship between the acceleration of the particle as measured in Σ and in Σ'

$$a = \frac{du}{dt} = \frac{\left(1 - \frac{v^2}{c^2}\right)^{3/2}}{\left(1 + \frac{u'v}{c^2}\right)^3}a' \,. \tag{2.52}$$

Until now the reference frame Σ' has had an arbitrary velocity. Now we choose $v = u(t)$ so that Σ' is the instantaneous rest frame of the particle at a point of time t. At this moment $u' = 0$. Then Eq. (2.52) reduces to

$$a = \left(1 - \frac{u^2}{c^2}\right)^{3/2} a' \,. \tag{2.53}$$

Here a' is the acceleration of the particle as measured in its instantaneous rest frame. It is called *the rest acceleration* of the particle. Equation (2.53) can be integrated if we know how the rest acceleration of the particle varies with time.

We shall now focus on the case where the particle has uniformly accelerated motion and moves along a straight path in space. The rest acceleration of the particle is constant, say $a' = g$. Integration of Eq. (2.53) with $u(0) = 0$ then gives

$$u = \left[1 + \frac{g^2}{c^2}t^2\right]^{-1/2} gt \,. \tag{2.54}$$

Integrating once more gives

$$x = \frac{c^2}{g}\left[1 + \frac{g^2}{c^2}t^2\right]^{1/2} + x_0 - \frac{c^2}{g} \,, \tag{2.55}$$

where x_0 is a constant of integration corresponding to the position at $t = 0$.

Equation (2.55) can be given the form

$$\boxed{\left(x - x_0 + \frac{c^2}{g}\right)^2 - c^2t^2 = \frac{c^4}{g^2}.} \tag{2.56}$$

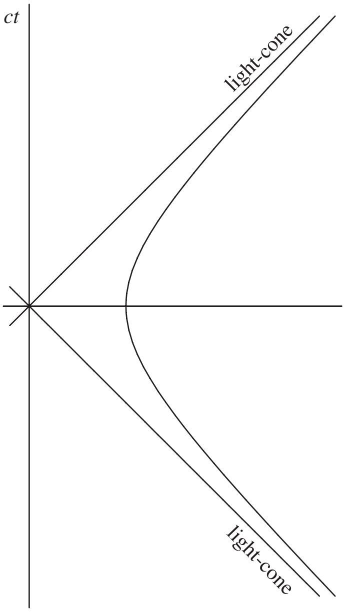

Fig. 2.13 World line of particle with constant rest acceleration

As shown in Fig. 2.13, this is the equation of a hyperbola in the Minkowski-diagram.

Since the world line of a particle with uniformly accelerated, rectilinear motion has the shape of a hyperbola, this type of motion is called *hyperbolic motion*.

Using the proper time τ of the particle as a parameter, we may obtain a simple parametric representation of its world line. Substituting Eq. (2.54) into Eq. (2.45) we get

$$d\tau = \frac{dt}{\sqrt{1+\frac{g^2}{c^2}t^2}} . \tag{2.57}$$

Integration with $\tau(0) = 0$ gives

$$\tau = \frac{c}{g}\operatorname{arcsinh}\left(\frac{gt}{c}\right) \tag{2.58}$$

or

$$\boxed{t = \frac{c}{g}\sinh\left(\frac{g\tau}{c}\right) .} \tag{2.59}$$

Inserting this expression into Eq. (2.55), we get

$$\boxed{x = \frac{c^2}{g}\cosh\left(\frac{g\tau}{c}\right) + x_0 - \frac{c^2}{g} .} \tag{2.60}$$

These expressions shall be used later when describing uniformly accelerated reference frames.

Note that *hyperbolic motion* results when the particle moves with *constant rest acceleration*. Such motion is usually called *uniformly accelerated motion*. Motion with constant acceleration as measured in the "laboratory frame" Σ gives rise to the usual parabolic motion.

2.11 Energy and Mass

The existence of an electromagnetic radiation pressure was well known before Einstein formulated the special theory of relativity. In black body radiation with energy density ρ there is an isotropic pressure $p = (1/3)\rho c^2$. If the radiation moves in a certain direction (laser), then the pressure in this direction is $p = \rho c^2$.

Einstein gave several deductions of the famous equation connecting the inertial mass of a body with its energy content. A deduction he presented in 1906 is as follows.

Consider a box with a light source at one end. A light pulse with radiation energy E is emitted to the other end where it is absorbed. (See Fig. 2.14.)

The box has a mass M and a length L. Due to the radiation pressure of the shooting light pulse the box receives a recoil. The pulse is emitted during a time interval Δt. During this time the radiation pressure is

$$p = \rho c^2 = \frac{E}{V} = \frac{E}{Ac\Delta t}\,, \tag{2.61}$$

where V is the volume of the radiation pulse and A the area of a cross-section of the box. The recoil velocity of the box is

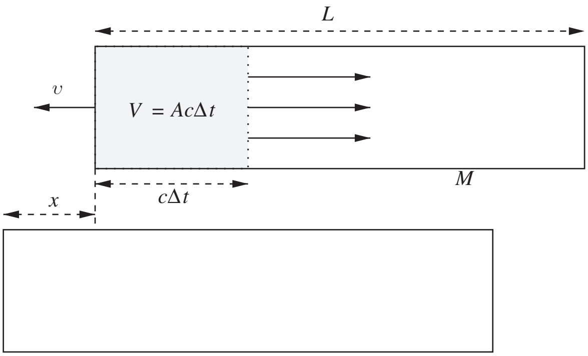

Fig. 2.14 Light pulse in a box

$$\begin{aligned}\Delta v &= -a\Delta t = -\frac{F}{M}\Delta t = -\frac{pA}{M}\Delta t \\ &= -\left(\frac{E}{Ac\Delta t}\right)\left(\frac{A\Delta t}{M}\right) = -\frac{E}{Mc} .\end{aligned} \tag{2.62}$$

The pulse takes the time L/c to move to the other side of the box. During this time the box moves a distance

$$\Delta x = \Delta v \frac{L}{c} = -\frac{EL}{Mc^2} . \tag{2.63}$$

Then the box is stopped by the radiation pressure caused by the light pulse hitting the wall at the other end of the box.

Let m be the mass of the radiation. Before Einstein one would put $m = 0$. Einstein, however, reasoned as follows. Since the box and its contents represents an isolated system, the mass centre has not moved. The mass centre of the box with mass M has moved a distance Δx to the left, the radiation with mass m has moved a distance L to the right. Thus

$$mL + M\Delta x = 0 \tag{2.64}$$

which gives

$$m = -\frac{M}{L}\Delta x = -\left(\frac{M}{L}\right)\left(-\frac{EL}{Mc^2}\right) = \frac{E}{c^2} \tag{2.65}$$

or

$$\boxed{E = mc^2.} \tag{2.66}$$

Here we have shown that radiation energy has an innate mass given by Eq. (2.65). Einstein derived Eq. (2.66) using several different methods showing that it is valid in general for all types of systems.

The energy content of even small bodies is enormous. For example, by transforming 1 g of matter to heat, one may heat 300,000 metric tons of water from room temperature to the boiling point. (The energy corresponding to a mass m is enough to change the temperature by ΔT of an object of mass M and specific heat capacity c_V: $mc^2 = Mc_V\Delta T$.)

2.12 Relativistic Increase of Mass

In the special theory of relativity, force is defined as rate of change of momentum. We consider a body that gets a change of energy dE due to the work performed on it by a force F. According to Eq. (2.66) and the definition of work (force times distance) the body gets a change of mass dm, given by

$$c^2 dm = dE = F ds = F v dt = v d(mv) = mv dv + v^2 dm , \tag{2.67}$$

which gives

$$\int_{m_0}^{m} \frac{dm}{m} = \int_0^v \frac{v dv}{c^2 - v^2} , \tag{2.68}$$

where m_0 is the rest mass of the body – i.e. its mass as measured by an observer comoving with the body – and m its mass when its velocity is equal to v. Integration gives

$$m = \frac{m_0}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m_0 \,. \tag{2.69}$$

In the case of small velocities compared to the velocity of light we may use the approximation

$$\sqrt{1-\frac{v^2}{c^2}} \approx 1 + \frac{1}{2}\frac{v^2}{c^2} \,. \tag{2.70}$$

With this approximation Eqs. (2.66) and (2.69) give

$$E \approx m_0 c^2 + \frac{1}{2} m_0 v^2 \,. \tag{2.71}$$

This equation shows that the total energy of a body encompasses its rest energy m_0 and its kinetic energy. In the non-relativistic limit the kinetic energy is $m_0 v^2/2$. The relativistic expression for the kinetic energy is

$$E_K = E - m_0 c^2 = (\gamma - 1) m_0 c^2 \,. \tag{2.72}$$

Note that $E_K \to \infty$ when $v \to c$.

According to Eq. (2.33), it is not possible to obtain a velocity greater than that of light by adding velocities. Equation (2.72) gives a dynamical reason that material particles cannot be accelerated up to and above the velocity of light.

2.13 Tachyons

Particles cannot pass the velocity-barrier represented by the velocity of light. However, the special theory of relativity permits the existence of particles that have *always* moved with a velocity $v > c$. Such particles are called *tachyons*.

Tachyons have special properties that have been used in the experimental searches for them. There is currently no observational evidence for the physical existence of tachyons.

There are also certain theoretical difficulties with the existence of tachyons. The special theory of relativity applied to tachyons leads to the following paradox. Using a tachyon telephone a person, A, emits a tachyon to B at a point of time t_1. B moves away from A. The tachyon is reflected by B and reach A before it was emitted, see Fig. 2.15. If the tachyon could carry information it might bring an order to destroy the tachyon emitter when it arrives back at A.

To avoid similar problems in regards to the energy-exchange between tachyons and ordinary matter, a reinterpretation principle is introduced for tachyons. For certain observers a tachyon will move backwards in time, i.e. the observer finds that

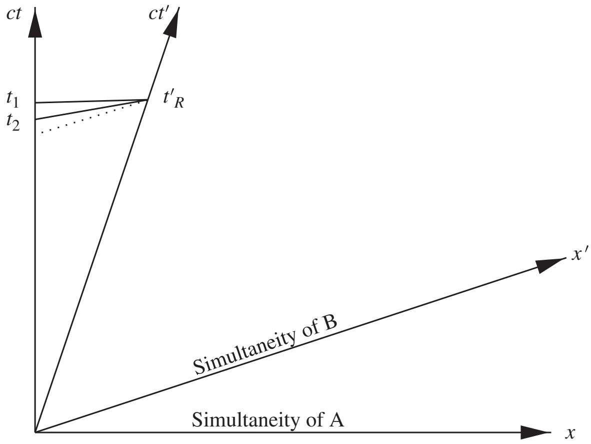

Fig. 2.15 *A* emits a tachyon at the point of time t_1. It is reflected by *B* and arrives at *A* at a point of time t_2 before t_1. Note that the arrival event at *A* is later than the reflection event as measured by *B*

the tachyon is received before it was emitted. Special relativity tells us that such a tachyon is always observed to have negative energy.

According to the reinterpretation principle, the observer will interpret his observations to mean that a tachyon with positive energy moves forward in time. In this way, one finds that the energy-exchange between tachyons and ordinary matter proceeds in accordance with the principle of causality.

However, the reinterpretation principle cannot be used to remove the problems associated with exchange of information between tachyons and ordinary matter. The tachyon telephone paradox cannot be resolved by means of the reinterpretation principle. The conclusion is that if tachyons exist, they cannot be carriers of information in our slowly moving world.

2.14 Magnetism as a Relativistic Second-Order Effect

Electricity and magnetism are described completely by Maxwell's equations of the electromagnetic field,

$$\nabla \cdot \mathbf{E} = \frac{1}{\varepsilon_0}\rho_q \tag{2.73}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{2.74}$$

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{2.75}$$

$$\nabla \times \mathbf{B} = \mu_0 \mathbf{j} + \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} \tag{2.76}$$

together with Lorentz's force law

$$\mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \; . \tag{2.77}$$

However, the relation between the magnetic and the electric force was not fully understood until Einstein had constructed the special theory of relativity. Only then could one clearly see the relationship between the magnetic force on a charge moving near a current carrying wire and the electric force between charges.

We shall consider a simple model of a current carrying wire in which we assume that the positive ions are at rest while the conducting electrons move with the velocity v. The charge per unit length for each type of charged particle is $\hat{\lambda} = Sne$ where S is the cross-sectional area of the wire, n the number of particles of one type per unit length and e the charge of one particle. The current in the wire is

$$J = Snev = \hat{\lambda} v \; . \tag{2.78}$$

The wire is at rest in an inertial frame $\hat{\Sigma}$. As observed in $\hat{\Sigma}$ it is electrically neutral. Let a charge q move with a velocity u along the wire in the opposite direction of the electrons. The rest frame of q is Σ. The wire will now be described from Σ (see Figs. 2.16 and 2.17).

Note that the charge per unit length of the particles as measured in their own rest frames, Σ_0, is

$$\lambda_{0-} = \hat{\lambda}\left(1 - \frac{v^2}{c^2}\right)^{1/2} , \quad \lambda_{0+} = \hat{\lambda} \tag{2.79}$$

since the distance between the electrons is Lorentz contracted in $\hat{\Sigma}$ compared to their distances in Σ_0.

The velocities of the particles as measured in Σ are

$$v_- = -\frac{v+u}{1+\frac{uv}{c^2}} \quad \text{and} \quad v_+ = -u \; . \tag{2.80}$$

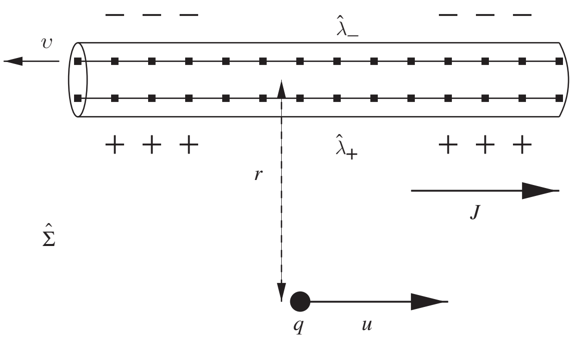

Fig. 2.16 Wire seen from its own rest frame

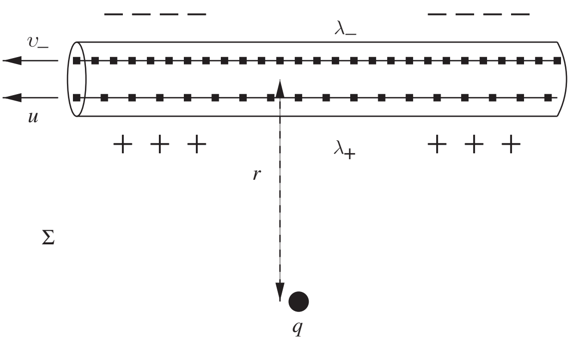

Fig. 2.17 Wire seen from rest frame of moving charge

The charge per unit length of the negative particles as measured in Σ is

$$\lambda_- = \left(1 - \frac{v_-^2}{c^2}\right)^{-1/2} \lambda_0 \,. \tag{2.81}$$

Substitution from Eqs. (2.79) and (2.80) gives

$$\lambda_- = \gamma\left(1 + \frac{uv}{c^2}\right)\hat{\lambda} \,, \tag{2.82}$$

where $\gamma = \left(1 - u^2/c^2\right)^{-1/2}$. In a similar manner, the charge per unit length of the positive particles as measured in Σ is found to be

$$\lambda_+ = \gamma\hat{\lambda} \,. \tag{2.83}$$

Thus, as observed in the rest frame of q the wire has a net charge per unit length

$$\lambda = \lambda_- - \lambda_+ = \frac{\gamma uv}{c^2}\hat{\lambda} \,. \tag{2.84}$$

As a result of the different Lorentz contractions of the positive and negative ions when we transform from their respective rest frames to Σ, a current carrying wire which is electrically neutral in the laboratory frame is observed to be electrically charged in the rest frame of the charge q.

As observed in this frame there is a radial electrical field with field strength

$$E = \frac{\lambda}{2\pi\epsilon_0 r} \,. \tag{2.85}$$

Then a force F acts on q, this is given by

$$F = qE = \frac{q\lambda}{2\pi\epsilon_0 r} = \frac{\hat{\lambda} v}{2\pi\epsilon_0 c^2 r}\gamma q u \,. \tag{2.86}$$

If a force acts upon q as observed in $\hat{\Sigma}$ then a force also acts on q as observed in Σ. According to the relativistic transformation of a force component in the same direction as the relative velocity between $\hat{\Sigma}$ and Σ, this force is

$$\hat{F} = \gamma^{-1} F = \frac{\hat{\lambda} v}{2\pi\epsilon_0 c^2 r} qu \, . \tag{2.87}$$

Inserting $J = \hat{\lambda} v$ from Eq. (2.78) and using $c^2 = (\epsilon_0 \mu_0)^{-1}$ (where μ_0 is the permeability of a vacuum) we obtain

$$\hat{F} = \frac{\mu_0 J}{2\pi r} qu \, . \tag{2.88}$$

This is exactly the expression obtained if we calculate the magnetic flux-density $\hat{B}$ around the current carrying wire using Ampere's circuit law

$$\hat{B} = \mu_0 \frac{J}{2\pi r} \tag{2.89}$$

and use the force law (Eq. (2.77)) for a charge moving in a magnetic field

$$\hat{F} = qu\hat{B} \, . \tag{2.90}$$

We have seen here how a magnetic force appears as a result of an electrostatic force and the special theory of relativity. The considerations above have also demonstrated that a force which is identified as electrostatic in one frame of reference is observed as a magnetic force in another frame. In other words, the electric and the magnetic force are really the same. What an observer names it depends upon his state of motion.

Problems

2.1. *The twin paradox*
On New Years day 2004, an astronaut (A) leaves Earth on an interstellar journey. He is travelling in a spacecraft at the speed of $v = 4/5c$ heading towards Alpha Centauri. This star is at a distance of 4 ly(ly =light years) measured from the reference frame of the Earth. As A reaches the star, he immediately turns around and heads home. He reaches the Earth New Years day 2016 (in Earth's time frame).
The astronaut has a brother (B), who remains on Earth during the entire journey. The brothers have agreed to send each other a greeting every new years day with the aid of radio-telescope.

(a) Show that A only sends 6 greetings (including the last day of travel), while B sends 10.
(b) Draw a Minkowski diagram where A's journey is depicted with respect to the Earth's reference frame. Include all the greetings that B is sending. Show with

the aid of the diagram that while A is outbound, he only receives one greeting, while on his way home he receives nine.

(c) Draw a new diagram, still with respect to Earth's reference frame, where A's journey is depicted. Include the greetings that A is sending to B. Show that B is receiving one greeting every third year the first 9 years after A has left, while the last year before his return he receives three.

(d) Show how the results from (b) and (c) can be deduced from the Doppler-effect.

2.2. *Faster than the speed of light?*
The quasar 3C273 emits a jet of matter that moves with the speed v_0 towards Earth making an angle ϕ to the line of sight (see Fig. 2.18).

(a) Assume that two signals are sent towards the Earth simultaneously, one from A and one from B. How much earlier will the signal from B reach the Earth compared to that from A?

(b) Find an expression of the transverse distance that the emitted part has moved when it reaches B. How much time (relative to the Earth) has this part been travelling?

(c) Let v be the transverse velocity and relate it to v_0. The observed (the transverse) speed of the light source is $v = 10c$. Find v_0 when we assume that $\phi = 10°$. What is the largest possible ϕ?

2.3. *Two successive boosts in different directions*
Let us consider Lorentz transformations without rotation ("boosts"). A boost in the x-direction is given by

$$x = g(x' + bct'), \quad y = y', \quad z = z' \quad t = g(t' + bx'/c)$$
$$\gamma = \frac{1}{\sqrt{1-b^2}} \quad b = \frac{v}{c}\,. \tag{2.91}$$

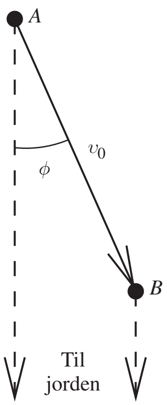

Fig. 2.18 A Quasar emitting a jet of matter

This can be written as

$$x^{\mu} = \Lambda^{\mu}{}_{\mu'} x^{\mu'} , \tag{2.92}$$

where $\Lambda^{\mu}{}_{\mu'}$ is the matrix

$$\Lambda^{\mu}{}_{\mu'} = \begin{bmatrix} g & gb & 0 & 0 \\ gb & g & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} . \tag{2.93}$$

(a) Show that Eqs. (2.92) and (2.93) yield Eq. (2.91). Find the transformation matrix, $\bar{\Lambda}^{\mu}{}_{\mu'}$, for a boost in the negative y-direction.

(b) Two successive Lorentz transformations are given by the matrix product of each matrix. Find $\Lambda^{\mu}_{a}\Lambda^{a}{}_{\mu'}$ and $\Lambda^{\mu}_{a}\bar{\Lambda}^{a}{}_{\mu'}$. Are the product of two boosts a boost? The matrix for a general boost in arbitrary direction is given by

$$\begin{aligned} \Lambda^{0}{}_{0} &= g, \\ \Lambda^{0}{}_{m} &= \Lambda^{m}{}_{0} = gb_m, \\ \Lambda^{m}{}_{m'} &= \delta^{m}_{m'} + \frac{b_m b_{m'}}{b^2}(g-1), \\ g &= \frac{1}{\sqrt{1-b^2}}, \quad b^2 = b^m b_m, \quad m, m' = 1, 2, 3 . \end{aligned} \tag{2.94}$$

Does the set of all possible boosts form a group?

2.4. *Length contraction and time dilation*

(a) A rod with length ℓ is moving with constant velocity $\mathbf{v}$ with respect to the inertial frame Σ. The length of the rod is parallel to $\mathbf{v}$, which we will for the sake of simplicity assume is parallel to the x-axis. At time $t = 0$, the rear end of the rod is in the origin of Σ. What do we mean by the length of such a moving rod? Describe how an observer can find this length. Draw the rod in a Minkowski diagram and explain how the length of the rod can be read from the diagram. Using the Lorentz transformations, calculate the position of the end points of the rod as a function of time t. Show that the length of the rod, as measured in Σ, is shorter than its rest length ℓ.

(b) The rod has the same velocity as before, but now the rod makes an angle with $\mathbf{v}$. In an inertial frame which follows the movement of the rod (Σ'), this angle is $a' = 45°$ (with the x-axis in Σ). What is the angle between the velocity $\mathbf{v}$ and the rod when measured in Σ? What is the length of the rod as a function of a', as measured from Σ?

(c) We again assume that $a' = 0$. At the centre of the rod there is a flash that sends light signals with a time interval τ_0 between every flash. In the frame Σ', the light signals will reach the two ends simultaneously. Show that these two events are not simultaneous in Σ. Find the time difference between these two events. Show that the time-interval τ measured from Σ between each flash is larger than the interval τ_0 measured in Σ'.

An observer in Σ is located at the origin. He measures the time-interval Δt between every time he receives a light signal. Find Δt in terms of the speed v, and check whether Δt is greater or less than τ.

(d) The length of the rod is now considered to be $ell = 1\,\text{m}$ and its speed, as measured in Σ, is $v = \frac{3}{5}c$. As before, we assume that the rod is moving parallel to the x-axis, but this time at a distance of $y = 10\,\text{m}$ from the axis. A measuring ribbon is stretched out along the trajectory of the rod. This ribbon is at rest in Σ. An observer at the origin sees the rod move along the background ribbon. The ribbon has tick-marks along it which correspond to the x coordinates. The rod length can be measured by taking a photograph of the rod and the ribbon. Is the length that is directly measured from the photograph identical to the length of the rod in Σ?
In one of the photographs the rod is symmetrically centred with respect to $x = 0$. What is the length of the rod as measured using this photograph? Another photograph shows the rod with its trailing edge at $x = 10$ m. At what point will the leading edge of the rod be on this photograph? Compare with the length of the rod in the Σ frame.

(e) At one point along the trajectory the rod passes through a box which is open at both ends and stationary in Σ. This box is shorter than the rest length of the rod, but longer than the length of the rod as measured in Σ. At a certain time in Σ, the entire rod is therefore inside the box. At this time the box is closed at both ends, trapping the rod inside. The rod is also brought to rest. It is assumed that the box is strong enough to withstand the impact with the rod.
What happens to the rod? Describe what happens as observed from Σ and Σ'. Draw a Minkowski diagram. This is an example of why the theory of relativity has difficulty with the concept of absolute rigid bodies. What is the reason for this difficulty?

2.5. *Reflection angles off moving mirrors*

(a) The reflection angle of light equals the incidence angle of the light. Show that this is also the case for mirrors that are moving parallel to the reflection surface.

(b) A mirror is moving with a speed v in a direction orthogonal to the reflection surface. Light is sent towards the mirror with an angle ϕ. Find the angle of the reflected light as a function of v and ϕ. What is the frequency of the reflected light expressed in terms of its original frequency f?

2.6. *Minkowski diagram*

The reference frame Σ' is moving relative to the frame Σ at a speed of $v = 0.6c$. The movement is parallel to the x-axes of the two frames.
Draw the x' and the ct'-axis in the Minkowski diagram of Σ. Points separated by 1 m are marked along both axes. Draw these points in the Minkowski diagram as for both frames.
Show where the lines of simultaneity for Σ' are in the diagram. Also show where the $x' =$ constant line is.

Assume that the frames are equipped with measuring rods and clocks that are at rest in their respective frames. How can we use the Minkowski diagram to measure the length contraction of the rod that is in rest at Σ'? Similarly, how can we measure the length contraction of the rod in Σ when measured from Σ? Show how the time dilation of the clocks can be measured from the diagram.

2.7. *Robb's Lorentz-invariant spacetime interval formula* (A.A. Robb, 1936)
Show that the spacetime interval between the origin event and the reflection event in Fig. 2.2 is $s = c\sqrt{t_A t_B}$.

2.8. *The Doppler effect*
A radar antenna emits radio pulses with a wavelength of $\lambda = 1.0$ cm, at a time-interval $\tau = 1.0$ s. An approaching spacecraft is being registered by the radar.
Draw a Minkowski diagram for the reference frame Σ. The antenna is at rest in this frame. In this diagram, indicate the position of

1. the antenna,
2. the spacecraft and
3. the outgoing and reflected radar pulses.

Calculate the time difference Δt_1 between two subsequent pulses as measured in the spacecraft. What is the wavelength of these signals?
Calculate the time difference Δt_2 between two reflected signals, as it is measured from the antenna's receiver? At what wavelength will these signals be?

2.9. *Aberration and Doppler effect*
We shall describe light emitted from a spherical surface that expands with ultra-relativistic velocity. Consider a surface element dA with velocity $v = \beta c$ in the laboratory frame F (i.e. the rest frame of the observer), as shown in Fig. 2.19 .

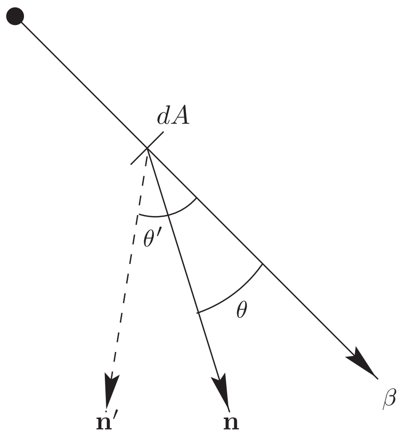

Fig. 2.19 Light is emitted in the direction $\mathbf{n}'$ as measured in the rest frame F' of the emitting surface element. The light is measured to propagate in the $\mathbf{n}$-direction in the rest frame F of the observer

(a) Show by means of the relativistic formula for velocity addition that the relationship between the directions of propagation measured in F and F' is

$$\cos\theta = \frac{\cos\theta' + b}{1 + b\cos\theta'} \,. \tag{2.95}$$

This is the aberration formula.

(b) Show that an observer far away from the surface will only observe light from a spherical cap with opening angle (see Fig. 2.20)

$$\theta_0 = \arccos\beta = \arcsin\frac{1}{\gamma} \approx \frac{1}{\gamma} \text{ for } \gamma \gg 1 \,. \tag{2.96}$$

(c) Assume that the expanding shell emits mononromatic light with frequency ν' in F'. Show that the observer in F will measure an angle-dependent frequency

$$\nu = \frac{\nu'}{\gamma(1 - b\cos\theta)} = \gamma(1 + b\cos\theta')\nu' \,. \tag{2.97}$$

(d) Let the measured frequency of light from M and Q be ν_M and ν_Q, respectively. This is the maximal and minimal frequency. Show that the expansion velocity can be found from these measurements, as

$$v = \frac{\nu_M - \nu_Q}{\nu_Q} c \,. \tag{2.98}$$

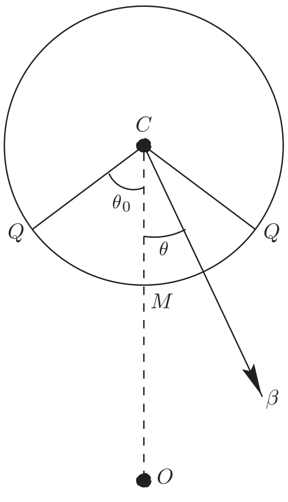

Fig. 2.20 The faraway observer, O, can only see light from the spherical cap with opening angle θ_0

2.10. *A traffic problem*
A driver is in court for driving through a red light. In his defence, the driver claims that the traffic signal appeared green as he was approaching the junction. The judge says that this does not strengthen his case stronger as he would have been travelling at the speed of ...
At what speed would the driver have to travel for the red traffic signal ($\lambda = 6000$ Å) to Doppler shift to a green signal ($\lambda = 5000$ Å)?

2.11. *Work and rotation*
A circular ring is initially at rest. It has radius r, rest mass m and a constant of elasticity k. Find the work that has to be done to give the ring an angular velocity ω. We assume that the ring is accelerated in such a way that its radius is constant. Compare with the non-relativistic case. How can we understand that in the relativistic case we also have to do elastic work?

2.12. *Muon experiment*
How many of the 10 million muons created 10 km above sea level will reach the Earth? If there are initially n_0 muons, $n = n_0 2^{-t/T}$ will survive for a time t (T is the half-lifetime).

(a) Compute the non-relativistic result.
(b) What is the result of a relativistic calculation by an Earth observer?
(c) Make a corresponding calculation from the point of view of an observer co-moving with the muon. The muon has a rest half-lifetime $T = 1.56 \cdot 10^{-6} s$ and moves with a velocity $v = 0.98c$.

2.13. *Cerenkov radiation*
When a particle moves through a medium with a velocity greater than the velocity of light in the medium, it emits a cone of radiation with a half-angle θ given by $\cos\theta = c/nv$ (see Fig. 2.21).

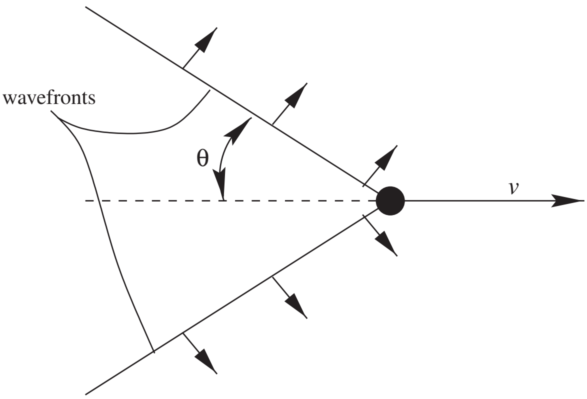

Fig. 2.21 Cerenkov radiation from a particle

(a) What is the threshold kinetic energy (in MeV) of an electron moving through water in order that it shall emit Cerenkov radiation? The index of refraction of water is $n = 1.3$. The rest energy of an electron is $m_e = 0.511$ MeV.
(b) What is the limiting half-angle of the cone for high-speed particles moving through water?

Reference

1. Frisch, D. H. and Smith, J. H. 1963. Measurement of relativistic time-dilation using mu-mesons, *Am. J. Phys.* **31**, 342.