Chapter 1
Examples

This chapter illustrates by examples some situations where the use of biset functors is natural. It is meant to be an informal introduction, most of the technical details being omitted, and postponed to subsequent chapters.

1.1. Representation Groups

1.1.1. Let $\mathbb{F}$ be a field. If G is a finite group, denote by $R_{\mathbb{F}}(G)$ the representation group of G over $\mathbb{F}$, i.e. the Grothendieck group of the category of finitely generated $\mathbb{F}G$-modules. It can be computed as the quotient of the free abelian group on the set of isomorphism classes of finitely generated $\mathbb{F}G$-modules, by the subgroup generated by all the elements of the form $[U]-[V]-[W]$ where U, V and W are finitely generated $\mathbb{F}G$-modules appearing in a short exact sequence of $\mathbb{F}G$-modules $0 \to V \to U \to W \to 0$, and $[U]$ denotes the isomorphism class of U.

1.1.2. Operations. There are various natural operations connecting the groups $R_{\mathbb{F}}(G)$ and $R_{\mathbb{F}}(H)$ for finite groups G and H:

- If H is a subgroup of G, then restriction of modules from $\mathbb{F}G$ to $\mathbb{F}H$ induces a *restriction map* $\mathrm{Res}_H^G : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(H)$.
- In the same situation, induction of modules from $\mathbb{F}H$ to $\mathbb{F}G$ yields an *induction map* $\mathrm{Ind}_H^G : R_{\mathbb{F}}(H) \to R_{\mathbb{F}}(G)$.
- If $\varphi : G \to H$ is a *group isomorphism*, there is an obvious associated linear map $\mathrm{Iso}(\varphi) : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(H)$.
- If N is a normal subgroup of G, and $H = G/N$, then inflation of modules from $\mathbb{F}H$ to $\mathbb{F}G$ yields an *inflation map* $\mathrm{Inf}_{G/N}^G : R_{\mathbb{F}}(G/N) \to R_{\mathbb{F}}(G)$.
- Another operation can be defined in the same situation, with the additional hypothesis that the characteristic of $\mathbb{F}$ is coprime to the order of N: starting with an $\mathbb{F}G$-module V, one can consider the module V_N of coinvariants of N on V, i.e. the largest quotient vector space of V on which N acts trivially. Then V_N is a $\mathbb{F}H$-module, and the construction $V \mapsto V_N$ is an exact functor from the category of $\mathbb{F}G$-modules to the category of $\mathbb{F}H$-modules, because of the hypothesis on

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the characteristic of $\mathbb{F}$: indeed $V_N \cong \mathbb{F} \otimes_{\mathbb{F}N} V$, and the $\mathbb{F}N$-module $\mathbb{F}$ is projective, hence flat. So the correspondence $V \mapsto V_N$ induces a *deflation map* $\mathrm{Def}^G_{G/N} : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(G/N)$.

1.1.3. Relations. All these operations are subject to various compatibility conditions (in the following list, all the conditions involving a deflation map hold whenever they are defined, i.e. when the orders of the corresponding normal subgroups are not divisible by the characteristic of $\mathbb{F}$):

1. *Transitivity conditions:*
   a. If K and H are subgroups of G with $K \leq H \leq G$, then
   $$\mathrm{Res}^H_K \circ \mathrm{Res}^G_H = \mathrm{Res}^G_K \ , \qquad \mathrm{Ind}^G_H \circ \mathrm{Ind}^H_K = \mathrm{Ind}^G_K \ .$$
   b. If $\varphi : G \to H$ and $\psi : H \to K$ are group isomorphisms, then
   $$\mathrm{Iso}(\psi) \circ \mathrm{Iso}(\varphi) = \mathrm{Iso}(\psi\varphi) \ .$$
   c. If N and M are normal subgroups of G with $N \leq M$, then
   $$\mathrm{Inf}^G_{G/N} \circ \mathrm{Inf}^{G/N}_{G/M} = \mathrm{Inf}^G_{G/M} \ , \qquad \mathrm{Def}^{G/N}_{G/M} \circ \mathrm{Def}^G_{G/N} = \mathrm{Def}^G_{G/M} \ .$$
2. *Commutation conditions:*
   a. If $\varphi : G \to H$ is a group isomorphism, and K is a subgroup of G, then
   $$\begin{aligned} \mathrm{Iso}(\varphi') \circ \mathrm{Res}^G_K &= \mathrm{Res}^H_{\varphi(K)} \circ \mathrm{Iso}(\varphi) \\ \mathrm{Iso}(\varphi) \circ \mathrm{Ind}^G_K &= \mathrm{Ind}^H_{\varphi(K)} \circ \mathrm{Iso}(\varphi') \ , \end{aligned}$$
   where $\varphi' : K \to \varphi(K)$ is the restriction of φ.
   b. If $\varphi : G \to H$ is a group isomorphism, and N is a normal subgroup of G, then
   $$\begin{aligned} \mathrm{Iso}(\varphi") \circ \mathrm{Def}^G_{G/N} &= \mathrm{Def}^H_{H/\varphi(N)} \circ \mathrm{Iso}(\varphi) \\ \mathrm{Iso}(\varphi) \circ \mathrm{Inf}^G_{G/N} &= \mathrm{Inf}^H_{H/\varphi(N)} \circ \mathrm{Iso}(\varphi") \ , \end{aligned}$$
   where $\varphi" : G/N \to H/\varphi(N)$ is the group isomorphism induced by φ.
   c. *(Mackey formula)* If H and K are subgroups of G, then
   $$\mathrm{Res}^G_H \circ \mathrm{Ind}^G_K = \sum_{x \in [H\backslash G/K]} \mathrm{Ind}^H_{H \cap {}^xK} \circ \mathrm{Iso}(\gamma_x) \circ \mathrm{Res}^K_{H^x \cap K} \ ,$$
   where $[H\backslash G/K]$ is a set of representatives of (H,K)-double cosets in G, and $\gamma_x : H^x \cap K \to H \cap {}^xK$ is the group isomorphism induced by conjugation by x.

d. If N and M are normal subgroups of G, then

$$\mathrm{Def}^G_{G/N} \circ \mathrm{Inf}^G_{G/M} = \mathrm{Inf}^{G/N}_{G/NM} \circ \mathrm{Def}^{G/M}_{G/NM} \ .$$

e. If H is a subgroup of G, and if N is a normal subgroup of G, then

$$\mathrm{Def}^G_{G/N} \circ \mathrm{Ind}^G_H = \mathrm{Ind}^{G/N}_{HN/N} \circ \mathrm{Iso}(\varphi) \circ \mathrm{Def}^H_{H/H\cap N} \ ,$$
$$\mathrm{Res}^G_H \circ \mathrm{Inf}^G_{G/N} = \mathrm{Inf}^H_{H/H\cap N} \circ \mathrm{Iso}(\varphi^{-1}) \circ \mathrm{Res}^{G/N}_{HN/N} \ ,$$

where $\varphi : H/H\cap N \to HN/N$ is the canonical group isomorphism.

f. If H is a subgroup of G, if N is a normal subgroup of G, and if $N \leq H$, then

$$\mathrm{Res}^{G/N}_{H/N} \circ \mathrm{Def}^G_{G/N} = \mathrm{Def}^H_{H/N} \circ \mathrm{Res}^G_H$$
$$\mathrm{Ind}^G_H \circ \mathrm{Inf}^H_{H/N} = \mathrm{Inf}^G_{G/N} \circ \mathrm{Ind}^{G/N}_{H/N} \ .$$

3. *Triviality conditions:* If G is a group, then

$$\mathrm{Res}^G_G = \mathrm{Id} \ , \ \ \mathrm{Ind}^G_G = \mathrm{Id} \ , \ \ \mathrm{Def}^G_{G/\mathbf{1}} = \mathrm{Id} \ , \ \ \mathrm{Inf}^G_{G/\mathbf{1}} = \mathrm{Id} \ ,$$

$$\mathrm{Iso}(\varphi) = \mathrm{Id} \ , \ \ \text{if } \varphi \text{ is an inner automorphism} \ .$$

1.1.4. Simplifications. So at this point, there is a rather complicate formalism involving the natural operations introduced in 1.1.2 and relations between them. The first observation that allows for a simplification, is that for each of the operations of 1.1.2, the map $R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(H)$ is induced by a functor sending an $\mathbb{F}G$-module M to the $\mathbb{F}H$-module $L \otimes_{\mathbb{F}G} M$, where L is some finite dimensional $(\mathbb{F}H, \mathbb{F}G)$-bimodule:

- When H is a subgroup of G, and M is an $\mathbb{F}G$-module, then $\mathrm{Res}^G_H M \cong \mathbb{F}G \otimes_{\mathbb{F}G} M$, so $L = \mathbb{F}G$ in this case, for the $(\mathbb{F}H, \mathbb{F}G)$-bimodule structure given by left multiplication by elements of $\mathbb{F}H$ and right multiplication by elements of $\mathbb{F}G$.
- In the same situation, if N is an $\mathbb{F}H$-module, then $\mathrm{Ind}^G_H N \cong \mathbb{F}G \otimes_{\mathbb{F}H} N$, so $L = \mathbb{F}G$ again, but with its $(\mathbb{F}G, \mathbb{F}H)$-bimodule structure given by left multiplication by $\mathbb{F}G$ and right multiplication by $\mathbb{F}H$.
- If $\varphi : G \to H$ is a group isomorphism, and M is an $\mathbb{F}G$-module, then the image of M by $\mathrm{Iso}(\varphi)$ is the $\mathbb{F}H$-module $\mathrm{Iso}(\varphi)(M) \cong \mathbb{F}H \otimes_{\mathbb{F}G} M$, so $L = \mathbb{F}H$ in this case, for the $(\mathbb{F}H, \mathbb{F}G)$-bimodule structure given by left multiplication by $\mathbb{F}H$, and by first taking images by φ of elements of $\mathbb{F}G$, and then multiplying on the right.
- If N is a normal subgroup of G, and $H = G/N$, then the inflated module from $\mathbb{F}H$ to $\mathbb{F}G$ of the $\mathbb{F}H$-module V is isomorphic to $\mathbb{F}H \otimes_{\mathbb{F}H} V$, so

$L = \mathbb{F}H$ in this case, with the $(\mathbb{F}G, \mathbb{F}H)$-bimodule structure given by multiplication on the right by $\mathbb{F}H$, and projection from $\mathbb{F}G$ onto $\mathbb{F}H$, followed by left multiplication.

- In the same situation, the module of coinvariants by N on the $\mathbb{F}G$-module M is isomorphic to $\mathbb{F}H \otimes_{\mathbb{F}G} M$, so $L = \mathbb{F}H$ in this case, with the $(\mathbb{F}H, \mathbb{F}G)$-bimodule structure obtained by reversing the actions in the previous case.

The second observation is that in each case, the $(\mathbb{F}H, \mathbb{F}G)$-bimodule L is actually a permutation bimodule: there exist an $\mathbb{F}$-basis U of L which is globally invariant under the action of both H and G, i.e. such that $hUg = U$, for any $h \in H$ and $g \in G$. In particular, the set U is endowed with a left H-action, and a right G-action, which commute, i.e. such that $(hu)g = h(ug)$ for any $h \in H$, $u \in U$ and $g \in G$. Such a set is called *an (H,G)-biset*.

Conversely, if U is a finite (H,G)-biset, then the finite dimensional $\mathbb{F}$-vector space $\mathbb{F}U$ with basis U inherits a natural structure of $(\mathbb{F}H, \mathbb{F}G)$-bimodule. If the functor $M \mapsto L \otimes_{\mathbb{F}G} M$ from $\mathbb{F}G$-modules to $\mathbb{F}H$-modules is exact, it induces a group homomorphism $R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(H)$, that will be denoted by $R_{\mathbb{F}}(U)$. The exactness condition means that the module $\mathbb{F}U$ is flat as a right $\mathbb{F}G$-module. Equivalently, since it is finitely generated, it is a projective $\mathbb{F}G$-module. Using Higman's criterion (see [45] III 14.4 Lemme 20), it is easy to see that this is equivalent to say that for each $u \in U$, the order of the stabilizer of u in G is not divisible by the characteristic l of $\mathbb{F}$. In this case, the biset U will be called *right l-free*. Note that if $l = 0$, this conditions is always fulfilled.

1.1.5. Formalism. Now the situation is the following: to each finite group G is associated an abelian group $R_{\mathbb{F}}(G)$. If G and H are finite groups, then to any a finite right l-free (H,G)-biset U corresponds a group homomorphism $R_{\mathbb{F}}(U) : R_{\mathbb{F}}(G) \to R_{\mathbb{F}}(H)$, with the following properties:

1. Let G and H be finite groups, and let U_1 and U_2 be finite right l-free (H,G)-bisets. If U_1 and U_2 are isomorphic as bisets, i.e. if there exists a bijection $f : U_1 \to U_2$ such that $f(hug) = hf(u)g$ for any $h \in H$, $u \in U_1$ and $g \in G$, then $R_{\mathbb{F}}(U_1) = R_{\mathbb{F}}(U_2)$. This is because the $(\mathbb{F}H, \mathbb{F}G)$-bimodules $\mathbb{F}U_1$ and $\mathbb{F}U_2$ are isomorphic in this case. This first property can be summarized as

   (**B1**) $$U_1 \cong U_2 \Rightarrow R_{\mathbb{F}}(U_1) = R_{\mathbb{F}}(U_2) \ .$$

2. If G and H are finite groups, and if U and U' are finite right l-free (H,G)-bisets, then $R_{\mathbb{F}}(U \sqcup U') = R_{\mathbb{F}}(U) + R_{\mathbb{F}}(U')$, where $U \sqcup U'$ is the disjoint union of U and U', endowed with the obvious (H,G)-biset structure. Indeed, the $(\mathbb{F}H, \mathbb{F}G)$-bimodules $\mathbb{F}(U \sqcup U')$ and $\mathbb{F}U \oplus \mathbb{F}U'$ are isomorphic. This property can be recorded as

   (**B2**) $$R_{\mathbb{F}}(U \sqcup U') = R_{\mathbb{F}}(U) + R_{\mathbb{F}}(U') \ .$$

3. Let G, H, and K be finite groups, let U be a finite right l-free (H,G)-biset, and let V be a finite right l-free (K,H)-biset. Then consider the set
$$V \times_H U = (V \times U)/H \ ,$$
where the right action of H on $V \times U$ is given by $(v,u)h = (vh, h^{-1}u)$, for $v \in V$, $u \in U$, and $h \in H$. The set $V \times_H U$ is a (K,G)-biset for the action induced by $k(v,u)g = (kv, ug)$, for $k \in K$, $v \in V$, $u \in U$, and $g \in G$. Moreover, it is right l-free if V and U are: the pair (v,u) of $V \times_H U$ is invariant by $g \in G$ if and only if there exists $h \in H$ such that $vh = v$ and $hu = ug$. Since V is l-free, there is an integer m, not divisible by l, such that $h^m = 1$. But then $h^m u = ug^m = u$, and since U is l-free, there is an integer m', not divisible by l, such that $(g^m)^{m'} = g^{mm'} = 1$. In this situation, one checks easily that

$$\textbf{(B3)} \qquad R_{\mathbb{F}}(V) \circ R_{\mathbb{F}}(U) = R_{\mathbb{F}}(V \times_H U) \ .$$

This is because the $(\mathbb{F}K, \mathbb{F}G)$-bimodules $\mathbb{F}V \otimes_{\mathbb{F}H} \mathbb{F}U$ and $\mathbb{F}(V \times_H U)$ are isomorphic.
4. Finally, if G is a finite group, and if Id_G is the set G, viewed as a (G,G)-biset for left and right multiplication, then Id_G is left and right free, hence right l-free, and $R_{\mathbb{F}}(\mathrm{Id}_G)$ is the identity map: this is because the functor $\mathbb{F}G \otimes_{\mathbb{F}G} -$ is isomorphic to the identity functor on the category of $\mathbb{F}G$-modules. Thus:

$$\textbf{(B4)} \qquad R_{\mathbb{F}}(\mathrm{Id}_G) = \mathrm{Id}_{R_{\mathbb{F}}(G)} \ .$$

This formalism of maps associated to bisets yields a nice way to encode all the relations listed in Sect. 1.1.3: more precisely, the triviality conditions follow from Properties (B4) and (B1). For transitivity and commutation conditions, the left hand side can always be expressed as $R_{\mathbb{F}}(V) \circ R_{\mathbb{F}}(U)$, where V is a (K,H)-biset and U is an (H,G)-biset, for suitable finite groups K, H, and G. By property (B3), this is equal to $R_{\mathbb{F}}(V \times_H U)$.

Now the right hand side of the transitivity conditions is of the form $R_{\mathbb{F}}(W)$, where W is some (K,G)-biset, and one checks easily in this case that the (K,G)-bisets $V \times_H U$ and W are isomorphic. So the transitivity conditions follow from property (B1).

Similarly, the right hand side of the commutation conditions can always be written as a composition of two or three maps of the form $R_{\mathbb{F}}(T_i)$, for suitable bisets T_i, or a sum of such compositions in the case of the Mackey formula. In any case, using properties (B2) and (B3), this right hand side can always we written as $R_{\mathbb{F}}(W)$, for a suitable (K,G)-biset W, and the corresponding relation follows from a biset isomorphism $V \times_H U \cong W$, using property (B1). For example, the Mackey formula

$$\mathrm{Res}_H^G \circ \mathrm{Ind}_K^G = \sum_{x \in [H\backslash G/K]} \mathrm{Ind}_{H\cap {}^xK}^H \circ \mathrm{Iso}(\gamma_x) \circ \mathrm{Res}_{H^x\cap K}^K \ ,$$

when H and K are subgroups of the group G, can be seen as a translation of the isomorphism of (H,K)-bisets

$$G \times_G G \cong \bigsqcup_{x \in [H\backslash G/K]} H \times_{H\cap {}^xK} \mathit{\Gamma}_x \times_{H^x\cap K} K \ ,$$

where $\mathit{\Gamma}_x$ is the $(H \cap {}^xK, H^x \cap K)$-biset associated to the group isomorphism $H^x \cap K \to H \cap {}^xK$ given by conjugation by x on the left. This biset isomorphism is nothing but the decomposition of the (H,K)-biset $G \times_G G \cong G$ as a disjoint union

$$G = \bigsqcup_{x \in [H\backslash G/K]} HxK$$

of its (H,K) double cosets, keeping track of the (H,K)-biset structure of each orbit.

1.2. Other Examples

1.2.1. Groups of Projective Modules. Let $\mathbb{F}$ be a field. If G is a finite group, denote by $P_{\mathbb{F}}(G)$ *the group of finitely generated projective* $\mathbb{F}G$*-modules*. Recall that $P_{\mathbb{F}}(G)$ is the quotient of the free abelian group on the set of isomorphism classes of finitely generated projective $\mathbb{F}G$-modules, by the subgroup generated by the elements of the form $[P \oplus Q] - [P] - [Q]$, where P and Q are two such modules, and $[P]$ denotes the isomorphism class of P.

If H is another finite group, and U is a finite (H,G)-biset, a natural question is to ask if the functor $M \mapsto \mathbb{F}U \otimes_{\mathbb{F}G} M$ maps a projective $\mathbb{F}G$-module M to a projective $\mathbb{F}H$-module. In this case in particular, it maps the module $\mathbb{F}G$ to a projective $\mathbb{F}H$-module, hence $\mathbb{F}U$ is a projective $\mathbb{F}H$-module. Conversely, if $\mathbb{F}U$ is a projective $\mathbb{F}H$-module, and if M is a projective $\mathbb{F}G$-module, then $\mathbb{F}U \otimes_{\mathbb{F}G} M$ is a projective $\mathbb{F}H$-module: indeed M is a direct summand of some free module $(\mathbb{F}G)^{(I)}$, where I is some set. Thus $\mathbb{F}U \otimes_{\mathbb{F}G} M$ is a direct summand of $\mathbb{F}U \otimes_{\mathbb{F}G} (\mathbb{F}G)^{(I)} \cong (\mathbb{F}U)^{(I)}$, which is a projective $\mathbb{F}H$-module.

Using Higman's criterion, it is easy to see that $\mathbb{F}U$ is a projective $\mathbb{F}H$-module if and only if the biset U is *left l-free*, i.e. if for any $u \in U$, the order of the stabilizer of u in H is not divisible by l. So if U is a finite left l-free (H,G)-biset, the functor $P \mapsto \mathbb{F}U \otimes_{\mathbb{F}G} P$ maps a finitely projective $\mathbb{F}G$-module to a finitely generated projective $\mathbb{F}H$-module. Since this functor preserves direct sums, it induces a map $P_{\mathbb{F}}(U) : P_{\mathbb{F}}(G) \to P_{\mathbb{F}}(H)$.

Now the situation is similar to the case of representation groups, except that the condition "right l-free" is replaced by "left l-free": to each finite group G is associated an abelian group $P_{\mathbb{F}}(G)$, and to any finite *left l-free* (H,G)-biset U is associated a group homomorphism $P_{\mathbb{F}}(U)$ from $P_{\mathbb{F}}(G)$ to $P_{\mathbb{F}}(H)$. These operations are easily seen to have properties (B1)–(B4).

Considering the formalization procedure of Sect. 1.1 backwards, it means that for the correspondence $G \mapsto P_{\mathbb{F}}(G)$, there are natural operations of restriction, induction, deflation, and transport by group isomorphism, and also an inflation operation $\mathrm{Inf}_{G/N}^{G} : P_{\mathbb{F}}(G/N) \to P_{\mathbb{F}}(G)$, which is only defined when N is a normal subgroup of G whose order is not divisible by l. All these operations satisfy the relations of Sect. 1.1.3, whenever they are defined.

1.2.2. Burnside Groups. If G is a finite group, denote by $B(G)$ the Burnside group of G, i.e. the Grothendieck group of the category of finite G-sets. Recall that $B(G)$ is the quotient of the free abelian group on the set of isomorphism classes of finite (left) G-sets, by the subgroup generated by the elements of the form $[X \sqcup Y]-[X]-[Y]$, where X and Y are finite G-sets, and $X \sqcup Y$ is their disjoint union, and $[X]$ denotes the isomorphism class of X (see Sect. 2.4 for details).

If G and H are finite groups, and if U is a finite (H,G)-biset, then the correspondence $X \mapsto U \times_G X$ from G-sets to H-sets induces a map $B(U) : B(G) \to B(H)$. One can check easily that these maps have the properties (B1)–(B4), with $R_{\mathbb{F}}$ replaced by B.

1.2.3. Remark : Let $\mathbb{F}$ be a field, of characteristic l. If X is a finite G-set, the $\mathbb{F}$-vector space $\mathbb{F}X$ with basis X has a natural $\mathbb{F}G$-module structure, induced by the action of G on X. The construction $X \mapsto \mathbb{F}X$ maps disjoint unions of G-sets to direct sums of $\mathbb{F}G$-modules, so it induces a map

$$\chi_{\mathbb{F},G} : B(G) \to R_{\mathbb{F}}(G) \ ,$$

called the *linearization morphism* at the group G. These maps are compatible with the maps $B(U)$ and $R_{\mathbb{F}}(U)$ corresponding to bisets, in the following sense: if G and H are finite groups, and if U is a finite right l-free (H,G)-biset, then the diagram

$$\begin{tikzcd} B(G) \arrow[r, "\chi_{\mathbb{F},G}"] \arrow[d, "B(U)"'] & R_{\mathbb{F}}(G) \arrow[d, "R_{\mathbb{F}}(U)"] \\ B(H) \arrow[r, "\chi_{\mathbb{F},H}"'] & R_{\mathbb{F}}(H) \end{tikzcd}$$

is commutative.

1.2.4. Cohomology and Inflation Functors. Let G be a finite group, and R be a commutative ring with identity. When k is a non negative integer, *the k-th cohomology group* $H^k(G,R)$ of G with values in R is defined as the extension group $\mathrm{Ext}_{RG}^{k}(R,R)$. In other words

$$H^k(G,R) = \mathrm{Hom}_{D(RG)}(R, R[k]) \ ,$$

where $D(RG)$ is the derived category of RG-modules.

Now suppose that H is another finite group, and that U is a finite *right free* (H,G)-biset. This hypothesis implies that RU is free as a right RG-module. It follows that the total tensor product $RU\otimes^{\mathbb{L}}_{RG}-$ coincides with the ordinary tensor product $RU\otimes_{RG}-$. This yields a map

$$\Theta_U : \mathrm{Hom}_{D(RG)}(R, R[k]) \to \mathrm{Hom}_{D(RH)}(RU\otimes_{RG} R, RU\otimes_{RG} R[k]) \ .$$

Now $RU\otimes_{RG} R \cong R(U/G)$, and there are two maps of RH-modules

$$\varepsilon_U : R \to R(U/G) \qquad \eta_U : R(U/G) \to R$$

defined by $\varepsilon_U(1) = \sum_{uG\in U/G} uG$, and $\eta_U(uG) = 1$ for any $uG \in U/G$. By composition, this gives a map

$$H^k(U) : H^k(G,R) = \mathrm{Hom}_{D(RG)}(R, R[k]) \to \mathrm{Hom}_{D(RH)}(R, R[k]) = H^k(H,R)$$

defined by $H^k(U)(\varphi) = \eta_U[k]\circ(RU\otimes_{RG}\varphi)\circ\varepsilon_U$.

If K is another finite group, and V is a finite right free (K,H)-biset, one can check that $H^k(V)\circ H^k(U) = H^k(V\times_H U)$: this follows from the fact that since U and V are right free, the set $(V\times_H U)/G$ is in one to one correspondence with $(V/H)\times(U/G)$.

It follows easily that the maps $H^k(U)$ between cohomology groups, for finite right free bisets, have the properties (B1)–(B4), with $R_{\mathbb{F}}$ replaced by H^k.

In this case, the formalism of maps associated to bisets encodes the usual operations of restriction, transfer, transport by isomorphism, and inflation on group cohomology. The condition imposed on bisets to be right free expresses the fact that there is no natural deflation map for group cohomology, that would be compatible with the other operations in the sense of relations of Sect. 1.1.3. Group cohomology is an example of *inflation functor*. These functors have been considered by P. Symonds [47], and also by P. Webb [54], who gave their name. More recently, E. Yaraneri [59] studied the composition factors of the inflation functor $R_{\mathbb{F}}$.

1.2.5. Global Mackey Functors. It may happen that for some construction similar to the previous examples, the only operations that are naturally defined are those of restriction to a subgroup, induction from a subgroup, and transport by group isomorphism. Functors of this type are called *global Mackey functors*, as opposed to the Mackey functors for a fixed finite group G (see [51]). These global Mackey functors have also been considered by P. Webb [54]. They can be included in the general formalism of biset functors, by restricting bisets to be *left and right free*.

1.3. Biset Functors

The above examples lead to the following informal definition: let $\mathcal{D}$ be a class of finite groups, and for each G and H in $\mathcal{D}$, let $\beta(H,G)$ be a class of finite (H,G)-bisets. *A biset functor* F (for $\mathcal{D}$ and β) with values in the category R-Mod of R-modules, where R is a commutative ring with identity, consists of the following data:

1. For each $G \in \mathcal{D}$, an R-module $F(G)$.
2. For each G and H in $\mathcal{D}$, and for each finite (H,G)-biset U in $\beta(H,G)$, a map of R-modules $F(U) : F(G) \to F(H)$.

These data are subject to the following conditions:

(B1) If G and H are in $\mathcal{D}$, then $\beta(H,G)$ is closed by isomorphism of (H,G)-bisets, and

$$\forall U_1, U_2 \in \beta(H,G), \ \ U_1 \cong U_2 \Rightarrow F(U_1) = F(U_2) \ .$$

(B2) If G and H are in $\mathcal{D}$, then $\beta(H,G)$ is closed by disjoint union of (H,G)-bisets, and

$$\forall U, U' \in \beta(H,G), \ \ F(U \sqcup U') = F(U) + F(U') \ .$$

(B3) If G, H, and K are in $\mathcal{D}$, then $\beta(K,H) \times_H \beta(H,G) \subseteq \beta(K,G)$, and

$$\forall V \in \beta(K,H), \ \forall U \in \beta(H,G), \ \ F(V) \circ F(U) = F(V \times_H U) \ .$$

(B4) If $G \in \mathcal{D}$, then the (G,G)-biset Id_G is in $\beta(G,G)$, and

$$F(\mathrm{Id}_G) = \mathrm{Id}_{F(G)} \ .$$

If F and F' are such biset functors, then *a morphism of biset functors* $f : F \to F'$ is a collection of maps $f_G : F(G) \to F'(G)$, for $G \in \mathcal{D}$, such that all the diagrams

$$\begin{tikzcd} F(G) \arrow[r, "f_G"] \arrow[d, "F(U)"'] & F'(G) \arrow[d, "F'(U)"] \\ F(H) \arrow[r, "f_H"'] & F'(H) \end{tikzcd}$$

are commutative, where G and H are in $\mathcal{D}$, and $U \in \beta(H,G)$.

Morphisms of biset functors can be composed in the obvious way, so biset functors (for $\mathcal{D}$ and β) with values in R-Mod form a category.

Equivalently, biset functors (for $\mathcal{D}$ and β) can be seen as additive functors from some additive subcategory of the *biset category* (see Definition 3.1.1), depending on $\mathcal{D}$ and β, to the category of R-modules.

1.4. Historical Notes

The notion of biset functor and related categories has been considered by various authors, under different names: Burnside functors, global Mackey functors, globally defined Mackey functors, functors with a Mackey structure, inflation functors. I wish to thank P. Webb for the major part of the following list of references [56]:

- 1981: Haynes Miller wrote to Frank Adams, describing *the Burnside category*, which is the (non full) subcategory of the biset category in which morphisms are provided by right-free bisets.
- 1985: The Burnside category appears in print: J.F. Adams, J.H. Gunawardena, H. Miller [1].
- 1987: T. tom Dieck [53, page 278] uses the term *global Mackey functor* for functors on this category.
- 1990: I. Hambleton, L. Taylor, B. Williams [37] consider very similar categories and functors (the main difference being that morphisms in their category RG-Morita are *permutation bimodules* instead of bisets).
- 1991: P. Symonds [47] also considers Mackey functors with inflation, that he calls *functors with Mackey structure*.
- 1993: P. Webb [54] considers *global Mackey functors* and *inflation functors*.
- 1996: I consider *foncteurs d'ensembles munis d'une double action* (in french [6]), i.e. functors on specific subcategories of the (yet unnamed) biset category.
- 2000: In Sect. 6 of [8], I use the name *biset-functor* for the Burnside functor.
- 2000: P. Webb [55] uses the name *globally defined Mackey functors* for biset functors.
- 2000: J. Thévenaz and I use *a functorial approach* to study the Dade group of a finite p-group.
- 2005: In Sect. 7 of [14], I define *rational biset functors*, now called rational p-biset functors.
- 2007: E. Yalçın and I [22] use the name of *biset category*.

1.5. About This Book

This book is organized as follows: Part I exposes a few generalities on bisets and biset functors, in a rather general framework. Some details on simple biset functors can be found in Chap. 4. Part II focuses on biset functors defined on *replete subcategories* of the biset category (see Definition 4.1.7), i.e. functors with the above five type of operations, but possibly defined over some particular class of finite groups, closed under taking subquotients (see Definition 4.1.7). The special case of p-biset functors is handled in Part III, and some important applications are detailed in Chaps. 11 and 12.

It follows from this rough summary that many interesting aspects and applications of biset functors, such as connections to homotopy theory, or highest weight categories, are not handled here: the main reason for these omissions is my lack of sufficient knowledge in those fields to treat them properly. I still hope this book will be useful to anyone interested in biset functors and their applications.

1.5.1. Notation and Conventions.

- The trivial group is denoted by $\mathbf{1}$.
- If G is a group, then $H \leq G$ means that H is a subgroup of G. If H and K are subgroups of G, then $H =_G K$ means that H and K are conjugate in G, and $H \leq_G K$ means that H is conjugate in G to some subgroup of K.
- If G is a group, if $g \in G$, and $H \leq G$, then $H^g = g^{-1}Hg$ and ${}^gH = gHg^{-1}$. The normalizer of H in G is denoted by $N_G(H)$, the centralizer of G in H by $C_G(H)$, and the center of G by $Z(G)$.
- If G is a group, then s_G is the set of subgroups of G, and $[s_G]$ is a set of representatives of conjugacy classes of G on s_G.
- The cardinality of a set X is denoted by $|X|$.
- If R is a ring, then $R\text{-}\mathsf{Mod}$ is the category of left R-modules. If B is a $\mathbb{Z}$-module, then RB denotes the R-module $R \otimes_{\mathbb{Z}} B$.
- If $\mathcal{C}$ is a category, then $\mathrm{Ob}(\mathcal{C})$ is its class of objects, and $x \in \mathrm{Ob}(\mathcal{C})$ is often abbreviated to $x \in \mathcal{C}$, to denote that x is an object of $\mathcal{C}$.
- If $\mathcal{C}$ is a category, then $\mathcal{D} \subseteq \mathcal{C}$ means that $\mathcal{D}$ is a subcategory of $\mathcal{C}$. Subcategories need not be full, but are assumed non empty.
- If $\mathcal{C}$ is a category, then $\mathcal{C}^{op}$ denotes the opposite category of $\mathcal{C}$, i.e. the category with the same objects as $\mathcal{C}$, and morphisms reversed.

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