Lecture Notes in Mathematics 1997

Intersection Spaces, Spatial Homology Truncation, and String Theory

Bearbeitet von
Markus Banagl

1st Edition. 2010. Taschenbuch. xvi, 224 S. Paperback
ISBN 978 3 642 12588 1
Format (B x L): 15,5 x 23,5 cm
Gewicht: 760 g

2.2 The Intersection Space in the Isolated Singularities Case

Let $\bar{p}$ be a perversity. The intersection space of a stratified pseudomanifold M with one stratum is by definition $I^{\bar{p}}M = M$. (Such a space is a manifold, but a manifold is not necessarily a one-stratum space.) Let X be an n-dimensional compact oriented CAT pseudomanifold with isolated singularities $x_1, \ldots, x_w$, $w \geq 1$, and simply connected links $L_i = \mathrm{Link}(x_i)$, where CAT is PL or DIFF or TOP. (Pseudomanifolds whose links are all simply connected are sometimes called *supernormal* in the literature, see [CW91].) Thus X has two strata: the bottom pure stratum is $\{x_1, \ldots, x_w\}$ and the top stratum is the complement. By a DIFF pseudomanifold we mean a Whitney stratified pseudomanifold. By a TOP pseudomanifold we mean a topological stratified pseudomanifold as defined in [GM83]. In the present isolated singularities situation, this means that the L_i are closed topological manifolds and a small neighborhood of x_i is homeomorphic to the open cone on L_i. If CAT=TOP, assume for the moment $n \neq 5$. We shall define the *perversity $\bar{p}$ intersection space $I^{\bar{p}}X$* for X.

Lemma 2.8. *Every link L_i, $i = 1, \ldots, w$, can be given the structure of a CW-complex.*

Proof. We begin with the case CAT=PL. Every link is then a closed PL manifold, which can be triangulated. The triangulation defines the CW-structure. For the case CAT=DIFF, i.e. the Whitney stratified case, we observe that links in Whitney stratified sets are again canonically Whitney stratified by intersecting with the strata of X. Since the links are contained in the top stratum, they are thus smooth manifolds. By the triangulation theorem of J. H. C. Whitehead, the link can then be smoothly triangulated. Again, the triangulation defines the desired CW-structure. Lastly, suppose CAT=TOP. If $n \leq 1$, then X has no singularities. If $n = 2$, the links are finite disjoint unions of circles. By the simple connectivity assumption, such unions must be empty. If $n = 3$, then by simple connectivity every link is a 2-sphere, so again X would be nonsingular. (Simple connectivity is of course not essential here, as circles and surfaces are certainly CW-complexes.) If $n = 4$, then the links are closed topological 3-manifolds. Since they are simply connected, the links must be 3-spheres according to the Poincaré conjecture, proved by Perelman. The space X would be nonsingular. (Simply connectivity is once more not essential for the existence of a CW-structure on the links because we could appeal to Moise's theorem [Moi52], asserting that every compact 3-manifold can be triangulated.) If $n \geq 6$, the links are closed topological manifolds of dimension at least 5. In this dimension range, topological manifolds have CW-structures by [KS77, FQ90]. □

Remark 2.9. The preceding lemma makes a statement that is more refined than necessary for constructing the intersection space. CW-structures arising from triangulations for example, while having the virtue of being regular, typically are very large and have lots of cells that are not closely tied to the global topology of the space. To form the intersection space, it is enough to know that every link is homotopy equivalent to a CW-complex. Using such an equivalence, one is free to choose smaller CW-structures, indeed minimal cell structures consistent with the homology, or to obtain a CW-structure when it is not known to exist on the given link per se. This latter situation arises in the case TOP and $n = 5$, not covered by the lemma.

In this case, the links L_i are simply connected closed topological 4-manifolds. It is at present not known whether such a manifold possesses a CW-structure. It is not possible to obtain such a structure from a handlebody because a closed topological 4-manifold admits a topological handle decomposition if and only if it is smoothable, since the attaching maps can always the smoothed by an isotopy. For example, Freedman's closed simply connected 4-manifold with intersection form E_8 does not admit a handle decomposition. However, such links L_i are homotopy equivalent to a cell complex with one 0-cell, a finite number of 2-cells and one 4-cell. In the case TOP and $n = 5$, after having removed small open cone neighborhoods of the singularities, we glue in the mapping cylinders of these homotopy equivalences and now have CW-complexes sitting on the "boundary." The intersection space can then be defined, following the recipe below, in all dimensions, even when CAT=TOP.

We shall now invoke the spatial homology truncation machine of Section 1.1. If $k = n-1-\bar{p}(n) \geq 3$, we can and do fix completions (L_i, Y_i) of L_i so that every (L_i, Y_i) is an object in $\mathbf{CW}_{k\supset\partial}$. If $k \leq 2$, no groups Y_i have to be chosen and we simply apply the low-degree truncation of Section 1.1.5. Applying the truncation $t_{<k} : \mathbf{CW}_{k\supset\partial} \to \mathbf{HoCW}_{k-1}$ as defined on page 50, we obtain a CW-complex $t_{<k}(L_i, Y_i) \in Ob\,\mathbf{HoCW}_{k-1}$. The natural transformation $\mathrm{emb}_k : t_{<k} \to t_{<\infty}$ of Theorem 1.41 gives homotopy classes of maps

$$f_i = \mathrm{emb}_k(L_i, Y_i) : t_{<k}(L_i, Y_i) \longrightarrow L_i$$

such that for $r < k$,

$$f_{i*} : H_r(t_{<k}(L_i, Y_i)) \cong H_r(L_i),$$

while $H_r(t_{<k}(L_i, Y_i)) = 0$ for $r \geq k$. Let M be the compact manifold with boundary obtained by removing from X open cone neighborhoods of the singularities $x_1, \ldots, x_w$. The boundary is the disjoint union of the links,

$$\partial M = \bigsqcup_{i=1}^{w} L_i.$$

Let

$$L_{<k} = \bigsqcup_{i=1}^{w} t_{<k}(L_i, Y_i)$$

and define a homotopy class $g : L_{<k} \longrightarrow M$ by composing

$$L_{<k} \xrightarrow{f} \partial M \longrightarrow M,$$

where $f = \bigsqcup_i f_i$. The intersection space will be the homotopy cofiber of g:

Definition 2.10. The *perversity $\bar{p}$ intersection space* $I^{\bar{p}}X$ of X is defined to be

$$\boxed{I^{\bar{p}}X = \mathrm{cone}(g) = M \cup_g \mathrm{cone}(L_{<k}).}$$

More precisely, $I^{\bar{p}}X$ is a homotopy type of a space. If g_1 and g_2 are both representatives of the class g, then $\mathrm{cone}(g_1) \simeq \mathrm{cone}(g_2)$ by the following proposition.

Proposition 2.11. *If*

$$\begin{array}{ccc} Y & \xleftarrow{\;f\;} & A \\ {\scriptstyle \phi_Y}\downarrow \simeq & & {\scriptstyle \phi_A}\downarrow \simeq \\ Y' & \xleftarrow{\;f'\;} & A' \end{array}$$

is a homotopy commutative diagram of continuous maps such that ϕ_Y and ϕ_A are homotopy equivalences, then there is a homotopy equivalence

$$Y \cup_f \mathrm{cone}A \longrightarrow Y' \cup_{f'} \mathrm{cone}A'$$

extending ϕ_Y.

This is Theorem 6.6 in [Hil65], where a proof can be found. The preceding construction of the intersection space $I^{\bar{p}}X$ depends on choices of cellular subgroups Y_i. If a link L_i is an object of the interleaf category **ICW**, then we may replace $t_{<k}(L_i, Y_i)$ in the construction by $t_{<k}L_i$, where $t_{<k}$: **ICW** $\to$ **HoCW** is the truncation functor of Section 1.9. The corresponding homotopy class f_i is to be replaced by the homotopy class $\mathrm{emb}_k(L_i) : t_{<k}L_i \to L_i$ given by the natural transformation

$$\mathrm{emb}_k : t_{<k} \longrightarrow t_{<\infty}$$

from Section 1.9. The construction of the intersection space thus becomes technically much simpler. The following theorem establishes generalized Poincaré duality for the rational reduced homology of intersection spaces and describes the relation to the intersection homology of Goresky and MacPherson.

Theorem 2.12. *Let X be an n-dimensional compact oriented supernormal singular CAT pseudomanifold with only isolated singularities. Let $\bar{p}$ and $\bar{q}$ be complementary perversities. Then:*

(1) The pair $(\widetilde{H}_*(I^{\bar{p}}X), IH_*^{\bar{p}}(X))$ is $(n-1-\bar{p}(n))$-reflective across the homology of the links, and

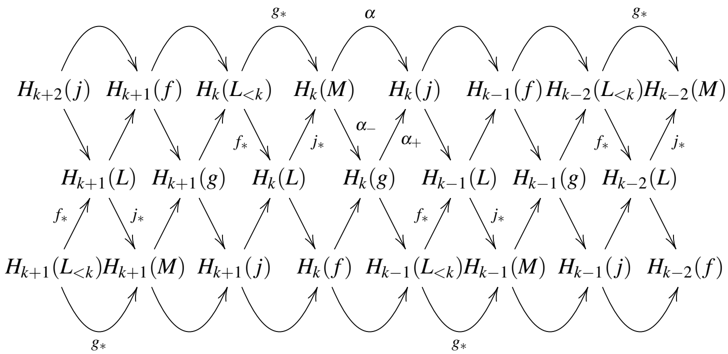
(2) $(\widetilde{H}_*(I^{\bar{p}}X;\mathbb{Q}), IH_*^{\bar{p}}(X;\mathbb{Q}))$ and $(\widetilde{H}_*(I^{\bar{q}}X;\mathbb{Q}), IH_*^{\bar{q}}(X;\mathbb{Q}))$ are n-dual reflective pairs.

Remark 2.13. Note that, as stated in the hypotheses, the theorem cannot formally be applied to a nonsingular X that is stratified with one stratum. The reason is simply that the reduced homology of a manifold $X = M$ does not possess Poincaré duality. If M is connected, then $\widetilde{H}_0(M) = 0$ but $\widetilde{H}_n(M) \cong \mathbb{Z}$ generated by the fundamental class.

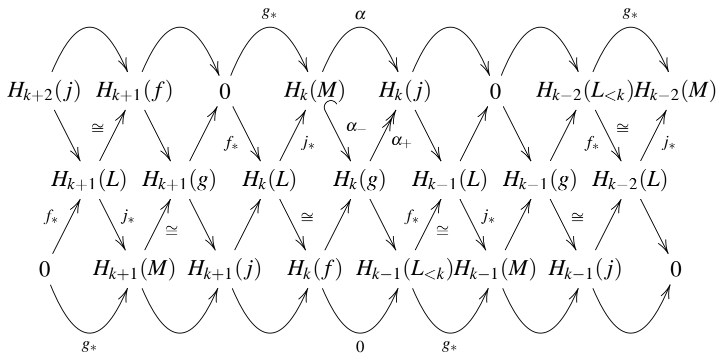
We begin the proof of Theorem 2.12:

Proof. We prove statement (1) first. Put $L = \partial M$ and let $j : L \hookrightarrow M$ be the inclusion of the boundary. We will study the braid of the triple

$$\begin{array}{ccc} L_{<k} & \xrightarrow{f} & L \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle j} \\ & & M, \end{array}$$



Using the fact that f_* is an isomorphism in degrees less than k, as well as $H_r(L_{<k}){=}0$ for $r \geq k$, the braid becomes



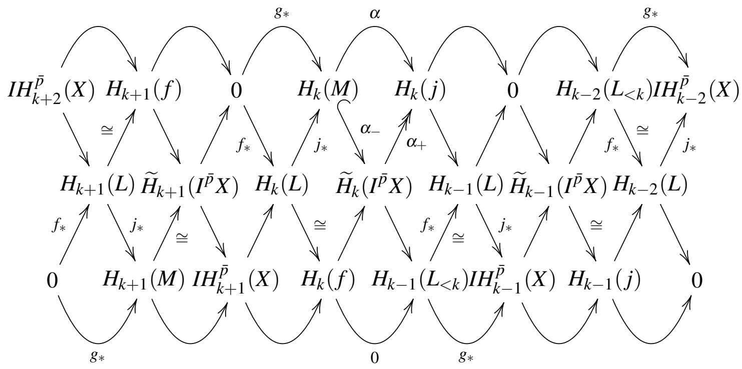
Since

$$H_*(g) = \widetilde{H}_*(\mathrm{cone}(g)) = \widetilde{H}_*(I^{\bar{p}}X)$$

and

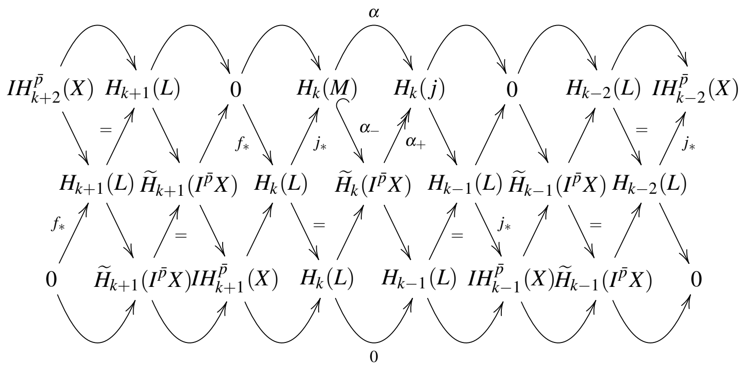
$$IH_r^{\bar{p}}(X) = \begin{cases} H_r(M,L) = H_r(j), & r > k \\ H_r(M), & r < k, \end{cases}$$

this can be rewritten as

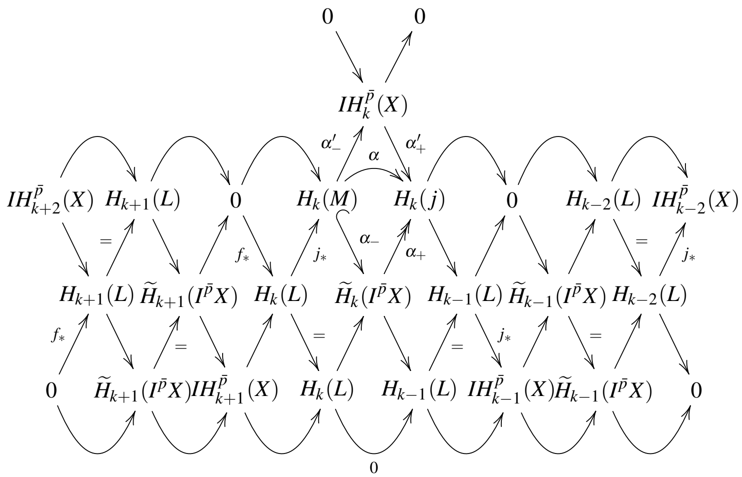


(2.3)

By composing with the indicated isomorphisms and their inverses, we may replace $H_r(f)$ by $H_r(L)$ for $r \geq k$, $H_r(L_{<k})$ by $H_r(L)$ for $r < k$, $H_r(M)$ by $\widetilde{H}_r(I^{\bar{p}}X)$ for $r > k$, and $H_r(j)$ by $\widetilde{H}_r(I^{\bar{p}}X)$ for $r < k$ to obtain



Finally, $IH_k^{\bar{p}}(X) = \operatorname{im}\alpha$, and we arrive at



where α'_- is given by regarding α as a map onto its image and α'_+ is the inclusion of $\operatorname{im}\alpha$ into $H_k(j)$. This braid contains the desired k-reflective diagram and all the required exact sequences.

For the remainder of the proof we will work with rational coefficients. To prove statement (2), we shall first construct duality isomorphisms

$$d : \widetilde{H}_r(I^{\bar{p}}X)^* \xrightarrow{\cong} \widetilde{H}_{n-r}(I^{\bar{q}}X).$$

There are three cases to consider: $r > k$, $r = k$, and $r < k$. For $r > k$, braid (2.3) contains the isomorphisms

$$H_r(M) \xrightarrow{\cong} \widetilde{H}_r(I^{\bar{p}}X).$$

For $I^{\bar{q}}X$, the cut-off degree k' is given by $k' = n-1-\bar{q}(n) = n-k$. Since $n-r < k'$, we have isomorphisms

$$\widetilde{H}_{n-r}(I^{\bar{q}}X) \xrightarrow{\cong} H_{n-r}(j)$$

by the braid of the $(n-k)$-reflective pair $(\widetilde{H}_*(I^{\bar{q}}X), IH^{\bar{q}}_*(X))$ analogous to braid (2.3). Using the Poincaré duality isomorphism $H_r(M)^* \cong H_{n-r}(j)$, we define d to be the unique isomorphism such that

$$\begin{array}{ccc} \widetilde{H}_r(I^{\bar{p}}X)^* & \xrightarrow{\cong} & H_r(M)^* \\ {\scriptstyle d}\downarrow{\scriptstyle \cong} & & {\scriptstyle PD}\downarrow{\scriptstyle \cong} \\ \widetilde{H}_{n-r}(I^{\bar{q}}X) & \xrightarrow{\cong} & H_{n-r}(j) \end{array}$$

commutes. Then

$$\begin{array}{ccccc} IH^{\bar{p}}_r(X)^* & \longrightarrow & \widetilde{H}_r(I^{\bar{p}}X)^* & \longrightarrow & H_r(L)^* \\ {\scriptstyle GMD}\downarrow{\scriptstyle \cong} & & {\scriptstyle d}\downarrow{\scriptstyle \cong} & & {\scriptstyle PD}\downarrow{\scriptstyle \cong} \\ IH^{\bar{q}}_{n-r}(X) & \longrightarrow & \widetilde{H}_{n-r}(I^{\bar{q}}X) & \longrightarrow & H_{n-r-1}(L) \end{array}$$

commutes, where *GMD* denotes Goresky–MacPherson duality on intersection homology. Indeed, via the universal coefficient isomorphism (which is natural), this diagram is isomorphic to

$$\begin{array}{ccccc} H^r(M,\partial M) & \longrightarrow & H^r(M) & \xrightarrow{j^*} & H^r(\partial M) \\ {\scriptstyle -\cap[M,\partial M]}\downarrow{\scriptstyle \cong} & & {\scriptstyle -\cap[M,\partial M]}\downarrow{\scriptstyle \cong} & & {\scriptstyle -\cap[\partial M]}\downarrow{\scriptstyle \cong} \\ H_{n-r}(M) & \longrightarrow & H_{n-r}(M,\partial M) & \xrightarrow{\partial_*} & H_{n-r-1}(\partial M). \end{array}$$

It commutes on the nose, not only up to sign, because

$$\partial_*(\xi \cap [M,\partial M]) = j^*\xi \cap \partial_*[M,\partial M] = j^*\xi \cap [\partial M],$$

see [Spa66], Chapter 5, Section 6, 20, page 255. (Recall that we are using Spanier's sign conventions.) For $r < k$, we proceed by "reflecting the construction of the previous case." That is, using the isomorphisms

$$\widetilde{H}_r(I^{\bar p}X) \xrightarrow{\cong} H_r(j),\ H_{n-r}(M) \xrightarrow{\cong} \widetilde{H}_{n-r}(I^{\bar q}X),\ PD: H_r(j)^* \cong H_{n-r}(M),$$

we define d to be the unique isomorphism such that

$$\begin{array}{ccc} H_r(j)^* & \xrightarrow{\cong} & \widetilde{H}_r(I^{\bar p}X)^* \\ {\scriptstyle PD}\downarrow{\scriptstyle\cong} & & {\scriptstyle d}\downarrow{\scriptstyle\cong} \\ H_{n-r}(M) & \xrightarrow{\cong} & \widetilde{H}_{n-r}(I^{\bar q}X) \end{array}$$

commutes. It follows that

$$\begin{array}{ccccc} H_{r-1}(L)^* & \longrightarrow & \widetilde{H}_r(I^{\bar p}X)^* & \longrightarrow & IH^{\bar p}_r(X)^* \\ {\scriptstyle PD}\downarrow{\scriptstyle\cong} & & {\scriptstyle d}\downarrow{\scriptstyle\cong} & & {\scriptstyle GMD}\downarrow{\scriptstyle\cong} \\ H_{n-r}(L) & \longrightarrow & \widetilde{H}_{n-r}(I^{\bar q}X) & \longrightarrow & IH^{\bar q}_{n-r}(X) \end{array}$$

commutes as well. The remaining case $r = k$ is perhaps the most interesting one. Let

$$\begin{array}{ccccccccccc} & & & & & & & & IH^{\bar q}_{n-k}(X) & & \\ & & & & & & & {\scriptstyle\gamma'_-}\nearrow & & \searrow{\scriptstyle\gamma'_+} & \\ \widetilde{H}_{n-k+1}(I^{\bar q}X) & \longrightarrow & IH^{\bar q}_{n-k+1}(X) & \longrightarrow & H_{n-k}(L) & \xrightarrow{\delta_-} & H_{n-k}(M) & & \xrightarrow{\gamma} & & H_{n-k}(j) \xrightarrow{\delta_+} \\ & & & & & & & {\scriptstyle\gamma_-}\searrow & & \nearrow{\scriptstyle\gamma_+} & \\ & & & & & & & & \widetilde{H}_{n-k}(I^{\bar q}X) & & \end{array}$$

$$H_{n-k-1}(L) \longrightarrow IH^{\bar q}_{n-k-1}(X) \longrightarrow \widetilde{H}_{n-k-1}(I^{\bar q}X) \longrightarrow \cdots$$

be the $(n-k)$-reflective diagram for the pair $(\widetilde{H}_*(I^{\bar q}X), IH^{\bar q}_*(X))$. The dual of the k-reflective diagram for $(\widetilde{H}_*(I^{\bar p}X), IH^{\bar p}_*(X))$ near k is

$$\begin{array}{ccccccc} & & & IH^{\bar p}_k(X)^* & & & \\ & & {\scriptstyle\alpha'^*_+}\nearrow & & \searrow{\scriptstyle\alpha'^*_-} & & \\ H_{k-1}(L)^* \xrightarrow{\beta^*_+} H_k(j)^* & & & \xrightarrow{\alpha^*} & & & H_k(M)^* \xrightarrow{\beta^*_-} H_k(L)^*. \\ & & {\scriptstyle\alpha^*_+}\searrow & & \nearrow{\scriptstyle\alpha^*_-} & & \\ & & & \widetilde{H}_k(I^{\bar p}X)^* & & & \end{array} \quad (2.4)$$

The following Poincaré duality isomorphisms will play a role in the construction of d:

$$d_M : H_k(M)^* \xrightarrow{\cong} H_{n-k}(j),$$
$$d'_M : H_k(j)^* \xrightarrow{\cong} H_{n-k}(M),$$
$$d_L : H_k(L)^* \xrightarrow{\cong} H_{n-k-1}(L).$$

Since the square

$$\begin{array}{ccc} H_k(M)^* & \xrightarrow{\beta_-^*} & H_k(L)^* \\ {\scriptstyle d_M}\downarrow \cong & & {\scriptstyle d_L}\downarrow \cong \\ H_{n-k}(j) & \xrightarrow{\delta_+} & H_{n-k-1}(L) \end{array}$$

commutes, d_L restricts to an isomorphism

$$d_L : \operatorname{im} \beta_-^* \xrightarrow{\cong} \operatorname{im} \delta_+.$$

Pick any splitting

$$s_{p\beta} : \operatorname{im} \beta_-^* \longrightarrow H_k(M)^*$$

for the surjection $\beta_-^* : H_k(M)^* \twoheadrightarrow \operatorname{im} \beta_-^*$. Set

$$s_{q\delta} = d_M s_{p\beta} d_L^{-1} : \operatorname{im} \delta_+ \longrightarrow H_{n-k}(j).$$

Then $s_{q\delta}$ splits $\delta_+ : H_{n-k}(j) \twoheadrightarrow \operatorname{im} \delta_+$ because

$$\delta_+ s_{q\delta} = \delta_+ d_M s_{p\beta} d_L^{-1} = d_L \beta_-^* s_{p\beta} d_L^{-1} = \operatorname{id}.$$

Pick any splitting

$$s_{p\alpha} : H_k(M)^* \longrightarrow \widetilde{H}_k(I^{\bar{p}}X)^*$$

for the surjection $\alpha_-^* : \widetilde{H}_k(I^{\bar{p}}X)^* \twoheadrightarrow H_k(M)^*$ and any splitting

$$s_{q\gamma} : H_{n-k}(j) \longrightarrow \widetilde{H}_{n-k}(I^{\bar{q}}X)$$

for the surjection $\gamma_+ : \widetilde{H}_{n-k}(I^{\bar{q}}X) \twoheadrightarrow H_{n-k}(j)$. The composition

$$s_p = s_{p\alpha} s_{p\beta} : \operatorname{im} \beta_-^* \longrightarrow \widetilde{H}_k(I^{\bar{p}}X)^*$$

is a splitting for $\beta_-^* \alpha_-^* : \widetilde{H}_k(I^{\bar{p}}X)^* \twoheadrightarrow \operatorname{im} \beta_-^*$. Similarly, the composition

$$s_q = s_{q\gamma} s_{q\delta} : \operatorname{im} \delta_+ \longrightarrow \widetilde{H}_{n-k}(I^{\bar{q}}X)$$

is a splitting for $\delta_+ \gamma_+ : \widetilde{H}_{n-k}(I^{\bar{q}}X) \twoheadrightarrow \operatorname{im} \delta_+$. Next, choose a splitting

$$t_p : IH_k^{\bar{p}}(X)^* \longrightarrow H_k(j)^*$$

for $\alpha_+'^* : H_k(j)^* \twoheadrightarrow IH_k^{\bar{p}}(X)^*$. Since duals of reflective diagrams are again reflective, diagram (2.4) has an associated T-diagram of type (2.2):

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
0 & \longrightarrow & \operatorname{im}\beta_+^* & \longrightarrow & H_k(j)^* & \xrightarrow{\alpha_+'^*} & IH_k^{\bar{p}}(X)^* & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle \alpha_+^*} & & & & \\
 & & & & \widetilde{H}_k(I^{\bar{p}}X)^* & & & & \\
 & & & & \downarrow{\scriptstyle \beta_-^*\alpha_-^*} & & & & \\
 & & & & \operatorname{im}\beta_-^* & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Thus we obtain a decomposition

$$\widetilde{H}_k(I^{\bar{p}}X)^* = \alpha_+^*(\operatorname{im}\beta_+^*) \oplus \alpha_+^* t_p IH_k^{\bar{p}}(X)^* \oplus s_p(\operatorname{im}\beta_-^*)$$

and every $v \in \widetilde{H}_k(I^{\bar{p}}X)^*$ can be written uniquely as

$$v = \alpha_+^*(b_+ + t_p(h)) + s_p(b_-)$$

with $b_+ \in \operatorname{im}\beta_+^*$, $h \in IH_k^{\bar{p}}(X)^*$ and $b_- \in \operatorname{im}\beta_-^*$. Write $x = b_+ + t_p(h)$. Setting

$$d(v) = \gamma_- d_M'(x) + s_q d_L(b_-)$$

defines a map

$$d : \widetilde{H}_k(I^{\bar{p}}X)^* \longrightarrow \widetilde{H}_{n-k}(I^{\bar{q}}X).$$

We claim that d is an isomorphism: By construction, the square

$$\begin{array}{ccc}
H_k(j)^* & \xrightarrow{\alpha_+^*} & \widetilde{H}_k(I^{\bar{p}}X)^* \\
{\scriptstyle d_M'}\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle d} \\
H_{n-k}(M) & \xrightarrow{\gamma_-} & \widetilde{H}_{n-k}(I^{\bar{q}}X)
\end{array}$$

commutes. The square

$$\begin{array}{ccc}
\widetilde{H}_k(I^{\bar{p}}X)^* & \xrightarrow{\beta_-^*\alpha_-^*} & \operatorname{im}\beta_-^* \\
{\scriptstyle d}\downarrow & & {\scriptstyle \cong}\downarrow{\scriptstyle d_L} \\
\widetilde{H}_{n-k}(I^{\bar{q}}X) & \xrightarrow{\delta_+\gamma_+} & \operatorname{im}\delta_+
\end{array}$$

commutes also, since

$$\begin{aligned} d_L\beta_-^*\alpha_-^*(v) &= d_L\beta_-^*\alpha^*(x) + d_L\beta_-^*\alpha_-^* s_p(b_-) \\ &= d_L(b_-) \\ &= \delta_+\gamma_+\gamma_- d'_M(x) + \delta_+\gamma_+ s_q d_L(b_-) \\ &= \delta_+\gamma_+ d(v). \end{aligned}$$

Hence we have a morphism of short exact sequences

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H_k(j)^* & \xrightarrow{\alpha_+^*} & \widetilde{H}_k(I^{\bar{p}}X)^* & \xrightarrow{\beta_-^*\alpha_-^*} & \operatorname{im}\beta_-^* & \longrightarrow & 0 \\ & & \cong\downarrow d'_M & & \downarrow d & & \cong\downarrow d_L & & \\ 0 & \longrightarrow & H_{n-k}(M) & \xrightarrow{\gamma_-} & \widetilde{H}_{n-k}(I^{\bar{q}}X) & \xrightarrow{\delta_+\gamma_+} & \operatorname{im}\delta_+ & \longrightarrow & 0 \end{array}$$

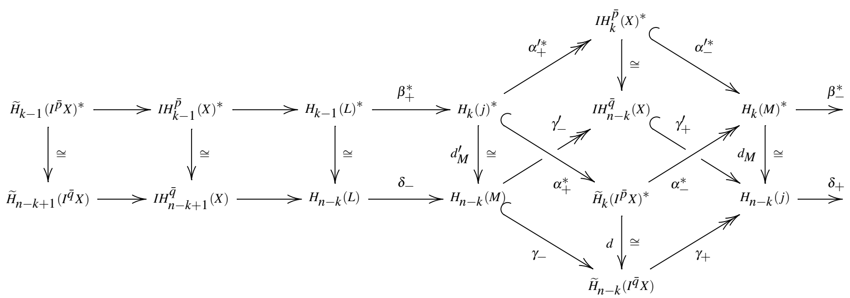
By the five-lemma, d is an isomorphism. It remains to be shown that the square

$$\begin{array}{ccc} \widetilde{H}_k(I^{\bar{p}}X)^* & \xrightarrow{\alpha_-^*} & H_k(M)^* \\ d\downarrow\cong & & \cong\downarrow d_M \\ \widetilde{H}_{n-k}(I^{\bar{q}}X) & \xrightarrow{\gamma_+} & H_{n-k}(j) \end{array}$$

commutes. This is established by the calculation

$$\begin{aligned} \gamma_+ d(v) &= \gamma d'_M(x) + \gamma_+ s_q d_L(b_-) \\ &= d_M\alpha^*(x) + \gamma_+ s_{q\gamma} s_{q\delta} d_L(b_-) \\ &= d_M\alpha_-^*(\alpha_+^*(x)) + s_{q\delta} d_L(b_-) \\ &= d_M\alpha_-^*(\alpha_+^*(x)) + d_M s_{p\beta} d_L^{-1}\circ d_L(b_-) \\ &= d_M\alpha_-^*(\alpha_+^*(x)) + d_M s_{p\beta}(b_-) \\ &= d_M\alpha_-^*(\alpha_+^*(x)) + d_M(\alpha_-^* s_{p\alpha}) s_{p\beta}(b_-) \\ &= d_M\alpha_-^*(\alpha_+^*(x)) + d_M\alpha_-^* s_p(b_-) \\ &= d_M\alpha_-^*(v). \end{aligned}$$

In summary, we have constructed the duality isomorphism



$$\begin{array}{ccccccc} H_k(L)^* & \longrightarrow & IH^{\bar{p}}_{k+1}(X)^* & \longrightarrow & \widetilde{H}_{k+1}(I^{\bar{p}}X)^* & \longrightarrow & \cdots \\ d_L \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \\ H_{n-k-1}(L) & \longrightarrow & IH^{\bar{q}}_{n-k-1}(X) & \longrightarrow & \widetilde{H}_{n-k-1}(I^{\bar{q}}X) & \longrightarrow & \cdots \end{array}$$

between the dual of the k-reflective diagram of the pair $(\widetilde{H}_*(I^{\bar{p}}X), IH^{\bar{p}}_*(X))$ and the $(n-k)$-reflective diagram of the pair $(\widetilde{H}_*(I^{\bar{q}}X), IH^{\bar{q}}_*(X))$. □

Corollary 2.14. *If $n = \dim X$ is even, then the difference between the Euler characteristics of $\widetilde{H}_*(I^{\bar{p}}X)$ and $IH^{\bar{p}}_*(X)$ is given by*

$$\chi(\widetilde{H}_*(I^{\bar{p}}X)) - \chi(IH^{\bar{p}}_*(X)) = -2\chi_{<n-1-\bar{p}(n)}(L),$$

where L is the disjoint union of the links of all the isolated singularities of X.
If $n = \dim X$ is odd, then

$$\chi(\widetilde{H}_*(I^{\bar{n}}X)) - \chi(IH^{\bar{n}}_*(X)) = (-1)^{\frac{n-1}{2}} b_{(n-1)/2}(L),$$

where $b_{(n-1)/2}(L)$ is the middle dimensional Betti number of L and $\bar{n}$ is the upper middle perversity. Regardless of the parity of n, the identity

$$\operatorname{rk}\widetilde{H}_k(I^{\bar{p}}X) + \operatorname{rk} IH^{\bar{p}}_k(X) = \operatorname{rk} H_k(M) + \operatorname{rk} H_k(M,L) \tag{2.5}$$

always holds in degree $k = n-1-\bar{p}(n)$, where M is the exterior of the singular set of X.

Proof. By Theorem 2.12, the pair $(H_*, H'_*) = (\widetilde{H}_*(I^{\bar{p}}X), IH^{\bar{p}}_*(X))$ is $(n-1-\bar{p}(n))$-reflective across the homology of L. Therefore, Proposition 2.5 applies and we obtain

$$\chi(\widetilde{H}_*(I^{\bar{p}}X)) - \chi(IH^{\bar{p}}_*(X)) = \chi(L) - 2\chi_{<n-1-\bar{p}(n)}(L).$$

If n is even, then L is an odd-dimensional closed oriented manifold and thus $\chi(L) = 0$ by Poincaré duality. If n is odd, then the cut-off value k for the upper middle perversity is $k = n-1-\bar{n}(n) = (n-1)/2$, the middle dimension of L. We have

$$\chi(L) = \chi_{<k}(L) + (-1)^k b_k(L) + \chi_{>k}(L) = 2\chi_{<k}(L) + (-1)^k b_k(L),$$

by Poincaré duality for L. Finally, as $A_- = H_k(M)$ and $A_+ = H_k(M,L)$, identity (2.5) follows from the equation

$$\operatorname{rk} H_k + \operatorname{rk} H'_k = \operatorname{rk} A_- + \operatorname{rk} A_+$$

of Proposition 2.5. □

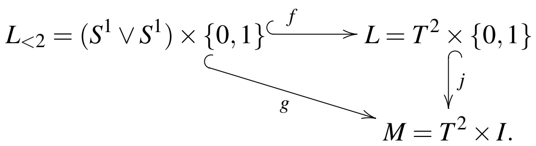
If a link of some singularity is not simply connected, so that the general construction of the intersection space as described above does not strictly apply, then one can in practice still often construct the intersection space provided one can find an ad hoc spatial homology truncation for this specific link. One then uses this truncation in place of the $t_{<k}L_i$ applied above; the rest of the construction remains the same. The simple connectivity assumption was adopted because our truncation machine required it (which in turn is due to the employment of the Hurewicz theorem). Inspection of the above proof on the other hand reveals that simple connectivity is nowhere necessary, only the existence of a spatial homology truncation of the link in the required dimension, dictated by the dimension of the pseudomanifold and the perversity. The following example illustrates this.

Example 2.15. Let us study Poincaré's own example of a three-dimensional space whose ordinary homology does not possess the duality that bears his name: $X^3 = \Sigma T^2$, the unreduced suspension of the 2-torus. This pseudomanifold has two singularities x_1, x_2, whose links are $L_1 = L_2 = T^2$, not simply connected. There are only two possible perversity functions to consider: $\bar{p}(3) = 0$ and $\bar{q}(3) = 1$. These two functions are complementary to each other.

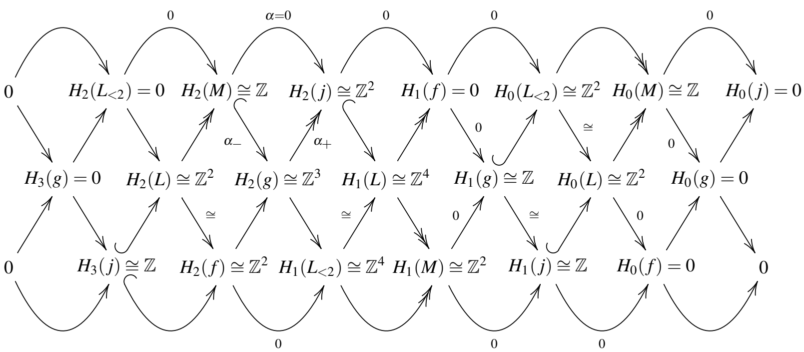
Let us build the intersection space $I^{\bar{p}}X$ first. The cut-off value k is $k = n-1-\bar{p}(n) = 2$. We have spatial homology truncations

$$t_{<2}(L_1) = t_{<2}(L_2) = S^1 \vee S^1,$$

the 1-skeleton of T^2. The $\bar{p}$-intersection space is $I^{\bar{p}}X = \operatorname{cone}(g)$, where g is the composition



We shall proceed to work out its reduced homology. The braid utilized in the proof of Theorem 2.12 looks like this:



Therefore, the reduced homology of $I^{\bar{p}}X$,

$$\widetilde{H}_*(I^{\bar{p}}X) = H_*(g) = H_*(T^2 \times I, (S^1 \vee S^1) \times \{0,1\}),$$

is

$$\begin{aligned}
\widetilde{H}_0(I^{\bar{p}}X) &= 0, \\
\widetilde{H}_1(I^{\bar{p}}X) &= \mathbb{Z}\langle \mathrm{pt} \times I\rangle, \\
\widetilde{H}_2(I^{\bar{p}}X) &= \mathbb{Z}\langle T^2 \times \{\tfrac{1}{2}\}\rangle \oplus \mathbb{Z}\langle S^1 \times \mathrm{pt} \times I\rangle \oplus \mathbb{Z}\langle \mathrm{pt} \times S^1 \times I\rangle, \\
\widetilde{H}_3(I^{\bar{p}}X) &= 0.
\end{aligned}$$

Let us now build the intersection space $I^{\bar{q}}X$. The cut-off value k is $k = n-1-\bar{q}(n) = 1$. The spatial homology truncations are

$$t_{<1}(L_1) = t_{<1}(L_2) = \mathrm{pt},$$

the 0-skeleton of T^2. The $\bar{q}$-intersection space is $I^{\bar{q}}X = \mathrm{cone}(g)$, where g is the composition

$$\begin{array}{ccc}
L_{<1} = \mathrm{pt} \times \{0,1\} & \xhookrightarrow{f} & L = T^2 \times \{0,1\} \\
 & \searrow^{g} & \downarrow j \\
 & & M = T^2 \times I.
\end{array}$$

Thus $I^{\bar{q}}X$ is obtained from a cylinder on the 2-torus by picking two points on it, one on each of the two boundary components, and then joining the two points by an arc outside of the cylinder. Its reduced homology

$$\widetilde{H}_*(I^{\bar{q}}X) = H_*(g) = H_*(T^2 \times I, \mathrm{pt} \times \{0,1\}),$$

can be determined from the long exact sequence of the pair and is given by

$$\begin{aligned}
\widetilde{H}_0(I^{\bar{q}}X) &= 0, \\
\widetilde{H}_1(I^{\bar{q}}X) &= \mathbb{Z}\langle \mathrm{pt} \times I\rangle \oplus \mathbb{Z}\langle S^1 \times \mathrm{pt} \times \{\tfrac{1}{2}\}\rangle \oplus \mathbb{Z}\langle \mathrm{pt} \times S^1 \times \{\tfrac{1}{2}\}\rangle, \\
\widetilde{H}_2(I^{\bar{q}}X) &= \mathbb{Z}\langle T^2 \times \{\tfrac{1}{2}\}\rangle, \\
\widetilde{H}_3(I^{\bar{q}}X) &= 0.
\end{aligned}$$

The table below contrasts the intersection space homology with the intersection homology of X, listing the generators in each dimension.

r	$IH_r^{\bar{p}}(X)$	$IH_r^{\bar{q}}(X)$	$\widetilde{H}_r(I^{\bar{p}}X)$	$\widetilde{H}_r(I^{\bar{q}}X)$
0	pt	pt	0	0
1	$S^1 \times \mathrm{pt}$ $\mathrm{pt} \times S^1$	0	$\mathrm{pt} \times I$	$\mathrm{pt} \times I$ $S^1 \times \mathrm{pt}$ $\mathrm{pt} \times S^1$
2	0	$\Sigma(S^1 \times \mathrm{pt})$ $\Sigma(\mathrm{pt} \times S^1)$	$T^2 \times \{\frac{1}{2}\}$ $S^1 \times \mathrm{pt} \times I$ $\mathrm{pt} \times S^1 \times I$	$T^2 \times \{\frac{1}{2}\}$
3	$\Sigma(S^1 \times S^1)$	$\Sigma(S^1 \times S^1)$	0	0

The relative 2-cycle $S^1 \times \mathrm{pt} \times I$ in the $\bar{p}$-intersection space homology corresponds to the suspension $\Sigma(S^1 \times \mathrm{pt})$ in the $\bar{q}$-intersection homology, similarly $\mathrm{pt} \times S^1 \times I$ corresponds to $\Sigma(\mathrm{pt} \times S^1)$. In dimension 1, we have an analogous correspondence between the cycles $S^1 \times \mathrm{pt}$, $\mathrm{pt} \times S^1$. The fundamental class $\Sigma(S^1 \times S^1)$ is present in intersection homology but is not seen in the homology of the intersection spaces. This is a general phenomenon and explains why the duality holds for the reduced, not the absolute, homology. Except for this phenomenon, the homology of the intersection spaces sees more cycles than the intersection homology. The 2-cycle $T^2 \times \{\frac{1}{2}\}$, geometrically present in X, is recorded by both the homology of $I^{\bar{p}}X$ and $I^{\bar{q}}X$, but remains invisible to intersection homology, though an echo of it is the 3-cycle ΣT^2 in intersection homology. By the duality theorem, the 2-cycle $T^2 \times \{\frac{1}{2}\}$ must have a dual partner. Indeed, the intersection space homology automatically finds the geometrically dual partner as well: It is the suspension of a point, the relative cycle $\mathrm{pt} \times I$. The relative $\bar{p}$-cycle $S^1 \times \mathrm{pt} \times I$ is dual to the $\bar{q}$-cycle $\mathrm{pt} \times S^1$ and the relative $\bar{p}$-cycle $\mathrm{pt} \times S^1 \times I$ is dual to the $\bar{q}$-cycle $S^1 \times \mathrm{pt}$. In the table, one can also observe the reflective nature of the relationship between intersection homology and the homology of the intersection spaces. The example shows that in degrees other than $k = n - 1 - \bar{p}(n)$, the homology of $I^{\bar{p}}X$ need not contain a copy of intersection homology. (We shall return to this point in Section 3.7.) In degree k it always does, as the proof of the theorem shows.

Let us also illustrate Corollary 2.14, relating the Euler characteristics of $\widetilde{H}_*(I^{\bar{p}}X)$ and $IH_*^{\bar{p}}(X)$, in the context of this example. In general, see also Proposition 2.5,

$$\chi(\widetilde{H}_*(I^{\bar{p}}X)) - \chi(IH_*^{\bar{p}}(X)) = \chi(L) - 2\chi_{<n-1-\bar{p}(n)}(L).$$

We have $\chi(L) = \chi(T^2 \times \{0,1\}) = 0$ and, since $k = 2$ for perversity $\bar{p}$, $\chi_{<2}(L) = 2 - 4 = -2$, whence

$$\chi(\widetilde{H}_*(I^{\bar{p}}X)) - \chi(IH_*^{\bar{p}}(X)) = 4.$$

Indeed, $\chi(\widetilde{H}_*(I^{\bar{p}}X)) = 0 - 1 + 3 - 0 = 2$ and $\chi(IH_*^{\bar{p}}(X)) = 1 - 2 + 0 - 1 = -2$. Furthermore, since $a_- = \mathrm{rk}\, H_2(T^2 \times I) = 1$ and $a_+ = \mathrm{rk}\, H_2(T^2 \times I, \partial) = 2$, we have according to equation (2.5),

$$\mathrm{rk}\, \widetilde{H}_2(I^{\bar{p}}X) + \mathrm{rk}\, IH_2^{\bar{p}}(X) = \mathrm{rk}\, H_2(T^2 \times I) + \mathrm{rk}\, H_2(T^2 \times I, \partial) = 1 + 2 = 3,$$

in concurrence with the ranks listed in the table. Since $\bar{q} = \bar{n}$ and the dimension $n = 3$ is odd, we have for $I^{\bar{q}}X$:

$$\chi(\widetilde{H}_*(I^{\bar{q}}X)) - \chi(IH_*^{\bar{q}}(X)) = -\mathrm{rk}\, H_1(T^2 \times \{0,1\}) = -4,$$

consistent with $\chi(\widetilde{H}_*(I^{\bar{q}}X)) = 0 - 3 + 1 - 0 = -2$ and $\chi(IH_*^{\bar{q}}(X)) = 1 - 0 + 2 - 1 = 2$. Formula (2.5) states that

$$\mathrm{rk}\, \widetilde{H}_1(I^{\bar{q}}X) + \mathrm{rk}\, IH_1^{\bar{q}}(X) = \mathrm{rk}\, H_1(T^2 \times I) + \mathrm{rk}\, H_1(T^2 \times I, \partial) = 2 + 1 = 3,$$

again in agreement with the ranks listed in the table.

Example 2.16. (The intersection space construction applied to a manifold point.) The intersection space construction may in principle also be applied to a nonsingular, two-strata pseudomanifold. What happens when the construction is applied to a manifold point x? One must remove a small open neighborhood of x and gets a compact oriented manifold M with boundary $\partial M = S^{n-1}$. The open neighborhood of x is an open n-ball, that is, the open cone on the link S^{n-1}. For a perversity $\bar{p}$, the cut-off degree $k = n-1-\bar{p}(n)$ is at most equal to $n-1$. Thus the spatial homology truncation is $t_{<k}S^{n-1} = \mathrm{pt}$. The fundamental class of the sphere is lost, no matter which $\bar{p}$ one takes. Thus $I^{\bar{p}}N$ is M together with a whisker attached to the 0-cell of the boundary sphere of M. This space is homotopy equivalent to M and to $N - \{x\}$. The reduced homology of M satisfies Poincaré duality since $\widetilde{H}_n(M)$ is dual to $\widetilde{H}_0(M)$ and $H_r(M) \to H_r(M,\partial M) = H_r(M,S^{n-1})$ is an isomorphism for $0 < r < n$.

Remark 2.17. There are two ways to truncate a chain complex C_* algebraically. The "good" truncation $\tau_{<k}C_*$ truncates the homology cleanly and corresponds to the spatial homology truncation as introduced in Chapter 1. The so-called "stupid" truncation $\sigma_{<k}C_*$, defined by $(\sigma_{<k}C_*)_i = C_i$ for $i < k$ and $(\sigma_{<k}C_*)_i = 0$ for $i \geq k$, does not truncate the homology cleanly. On spaces, the stupid truncation $\sigma_{<k}L$ of a CW-complex L would be $\sigma_{<k}L = L^{k-1}$, the $(k-1)$-skeleton of L, and is thus much easier to define and to handle than the good spatial truncation. In light of these advantages, one may wonder whether in the construction of the intersection space, one could replace the good spatial truncation $t_{<k}(L,Y)$ of the link L by the above stupid truncation $\sigma_{<k}L$ and still get a space that possesses generalized Poincaré duality. The following example will show that this is in fact not possible. Let X^n be the 4-sphere, thought of as a stratified space

$$X = S^4 = D^4 \cup_{S^3} D^4 = M^4 \cup_{L^3} \mathrm{cone}(L^3),$$

where $M^4 = D^4$ and $L^3 = S^3$ is the link of the cone point, thought of as the bottom stratum. Suppose L is equipped with the CW-structure

$$L = e_1^0 \cup e_2^0 \cup e_1^1 \cup e_2^1 \cup e_1^2 \cup e_2^2 \cup e_1^3 \cup e_2^3,$$

so that the equatorial spheres $S^0 \subset S^1 \subset S^2 \subset L$ are all subcomplexes. Is $\mathrm{cone}(g)$, where g is the composition

$$\begin{array}{ccc} \sigma_{<k}L = L^{k-1} & \stackrel{f}{\hookrightarrow} & L = \partial M \\ & {\scriptstyle g}\searrow & \downarrow{\scriptstyle j} \\ & & M, \end{array}$$

a viable candidate for an intersection space of X? Since $\widetilde{H}_*(M) = \widetilde{H}_*(D^4) = 0$, the exact sequence of the pair (M, L^{k-1}) shows

$$\widetilde{H}_*(\mathrm{cone}(g)) \cong \widetilde{H}_{*-1}(L^{k-1}).$$

For the middle perversity, one would take $k = n/2 = 2$. Thus $\sigma_{<2}L = L^1 = S^1$ and the middle homology of cone(g),

$$\widetilde{H}_2(\mathrm{cone}(g)) \cong \widetilde{H}_1(S^1),$$

has rank one. If cone(g) had Poincaré duality, then the signature of the nondegenerate, symmetric intersection form on $\widetilde{H}_2(\mathrm{cone}(g))$ would have to be nonzero. (Zero signature would imply even rank.) But the signature of $X = S^4$ is zero. Thus $\widetilde{H}_*(\mathrm{cone}(g))$ is a meaningless theory, unrelated to the geometry of X. It is therefore necessary to choose a subgroup $Y \subset C_2(L) = \mathbb{Z}e_1^2 \oplus \mathbb{Z}e_2^2$ such that $(L,Y) \in Ob\mathbf{CW}_{2\supset\partial}$ and apply the good spatial truncation $t_{<2}(L,Y)$, not the stupid truncation $\sigma_{<2}L$. (Using $\sigma_{<1}L$ or $\sigma_{<3}L$ does not yield self-dual homology groups either.) Any such Y arises as the image of a splitting $s : \mathrm{im}\,\partial_2 \to C_2(L)$ for $\partial_2 : C_2(L) \twoheadrightarrow \mathrm{im}\,\partial_2 = \ker\partial_1 = \mathbb{Z}\langle e_1^1 - e_2^1\rangle$. So we could for instance take $Y = \mathbb{Z}e_1^2$ or $Y = \mathbb{Z}e_2^2$ because $\partial_2(e_1^2) = e_1^1 - e_2^1 = \partial_2(e_2^2)$.

2.3 Independence of Choices of the Intersection Space Homology

The construction of the intersection spaces $I^{\bar{p}}X$ involves choices of subgroups $Y_i \subset C_k(L_i)$, where the L_i are the links of the singularities, such that (L_i,Y_i) is an object in $\mathbf{CW}_{k\supset\partial}$ with $k = n-1-\bar{p}(n)$, $n = \dim X$. Moreover, the chain complexes $C_*(L_i)$ depend on the CW-structures on the links and these structures are another element of choice. In this section we collect some results on the independence of these choices of the intersection space homology $\widetilde{H}_*(I^{\bar{p}}X)$.

Theorem 2.18. *Let X^n be a compact oriented pseudomanifold with isolated singularities and fixed, simply connected links L_i that can be equipped with CW-structures. Then*
(1) $\widetilde{H}_*(I^{\bar{p}}X;\mathbb{Q})$ is independent of the choices involved in the construction of the intersection space $I^{\bar{p}}X$,
(2) $\widetilde{H}_r(I^{\bar{p}}X;\mathbb{Z})$ is independent of choices for $r \neq n-1-\bar{p}(n)$, and
(3) $\widetilde{H}_k(I^{\bar{p}}X;\mathbb{Z})$, $k = n-1-\bar{p}(n)$, is independent of choices if either

$$\mathrm{Ext}(\mathrm{im}(H_k(M,L) \to H_{k-1}(L)), H_k(M)) = 0,$$

or

$$\mathrm{Ext}(H_k(M,L), \mathrm{im}(H_k(L) \to H_k(M))) = 0.$$

Proof. We shall first look at the integral homology groups. For $r > k$, the proof of Theorem 2.12 exhibits isomorphisms

$$H_r(M) \xrightarrow{\cong} \widetilde{H}_r(I^{\bar{p}}X).$$