Chapter 2
Determinants, and Linear Independence

2.1 Introduction to Determinants and Systems of Equations

Determinants can be defined and studied independently of matrices, though when square matrices occur they play a fundamental role in the study of linear systems of algebraic equations, in the formal definition of an inverse matrix, and in the study of the eigenvalues of a matrix. So, in anticipation of what is to follow in later chapters, and before developing the properties of determinants in general, we will introduce and motivate their study by examining the solution a very simple system of equations.

The theory of determinants predates the theory of matrices, their having been introduced by Leibniz (1646–1716) independently of his work on the calculus, and subsequently their theory was developed as part of algebra, until Cayley (1821–1895) first introduced matrices and established the connection between determinants and matrices. Determinants are associated with square matrices and they arise in many contexts, with two of the most important being their connection with systems of linear algebraic equations, and systems of linear differential equations like those in (1.12).

To see how determinants arise from the study of linear systems of equations we will consider the simplest linear nonhomogeneous system of algebraic equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1, \\ a_{21}x_1 + a_{22}x_2 &= b_2. \end{aligned} \tag{2.1}$$

These equations can be solved by elimination as follows. Multiply the first equation by a_{22}, the second by a_{12}, and subtract the results to obtain an equation for x_1 from which the variable x_2 has been *eliminated* . Next, multiply the first equation by a_{21}, the second by a_{11}, and subtract the results to obtain an equation for x_2, where this time the variable x_1 has been *eliminated*. The result is the solution set $\{x_1, x_2\}$ with its elements given by given by

$$x_1 = \frac{b_1a_{22} - b_2a_{12}}{a_{11}a_{22} - a_{12}a_{21}}, \quad x_2 = \frac{b_2a_{11} - b_1a_{21}}{a_{11}a_{22} - a_{12}a_{21}}. \tag{2.2}$$

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For this solution set to exist it is necessary that the denominator $a_{11}a_{22} - a_{12}a_{21}$ in the expressions for x_1 and x_2 does not vanish. So setting $\Delta = a_{11}a_{22} - a_{12}a_{21}$, the condition for the existence of the solution set $\{x_1, x_2\}$ becomes $\Delta \neq 0$.

In terms of a square matrix of coefficients whose elements are the *coefficients* associated with (2.1), namely

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}, \tag{2.3}$$

the *second-order* determinant associated with $\mathbf{A}$, written either as det $\mathbf{A}$ or as $|\mathbf{A}|$, is defined as the *number*

$$\det \mathbf{A} = |\mathbf{A}| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21}, \tag{2.4}$$

so the denominator in (2.2) is $\Delta = \det \mathbf{A}$.

Notice how the *value* of the determinant in (2.4) is obtained from the elements of $\mathbf{A}$. The expression on the right of (2.4), called the *expansion* of the determinant, is the product of elements on the leading diagonal of $\mathbf{A}$, from which is subtracted the product of the elements on the cross-diagonal that runs from the bottom left to the top right of the array $\mathbf{A}$. The classification of the type of determinant involved is described by specifying its *order*, which is the number of rows (equivalently columns) in the square matrix $\mathbf{A}$ from which the determinant is derived. Thus the determinant in (2.4) is a *second-order* determinant. Specifying the *order* of a determinant gives some indication of the magnitude of the calculation involved when expanding it, while giving *no* indication of the value of the determinant. If the elements of $\mathbf{A}$ are numbers, det $\mathbf{A}$ is seen to be a number, but if the elements are functions of a variable, say t, then det $\mathbf{A}$ becomes a function of t. In general determinants whose elements are functions, often of several variables, are called *functional determinants*. Two important examples of these determinants called *Jacobian determinants*, or simply *Jacobians*, will be found in Exercises 14 and 15 at the end of this chapter.

Notice that in the conventions used in this book, when a matrix is written out in full, the elements of the matrix are enclosed within square brackets, thus [...], whereas the notation for its determinant, which is only associated with a square matrix, encloses its elements between vertical rules, thus |...|, and these notations should not be confused

Example 2.1. Given (a) $\mathbf{A} = \begin{bmatrix} 1 & 3 \\ -4 & 6 \end{bmatrix}$ and (b) $\mathbf{B} = \begin{bmatrix} e^t & e^t \\ \cos t & \sin t \end{bmatrix}$, find det $\mathbf{A}$ and det $\mathbf{B}$.

Solution. By definition (a)det$\mathbf{A} = \begin{vmatrix} 1 & 3 \\ -4 & 6 \end{vmatrix} = (1 \times 6) - (3) \times (-4) = 18$.
(b) $\det \mathbf{B} = \begin{vmatrix} e^t & e^t \\ \cos t & \sin t \end{vmatrix} = (e^t) \times (\sin t) - (e^t) \times (\cos t) = e^t(\sin t - \cos t)$.

It is possible to express the solution set $\{x_1, x_2\}$ in (2.2) entirely in terms of determinants by defining the three second-order determinants

$$\Delta = \det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}, \quad \Delta_1 = \begin{vmatrix} b_1 & a_{12} \\ b_2 & a_{22} \end{vmatrix}, \quad \Delta_2 = \begin{vmatrix} a_{11} & b_1 \\ a_{21} & b_2 \end{vmatrix}, \tag{2.5}$$

because then the solutions in (2.2) become

$$x_1 = \frac{\Delta_1}{\Delta}, \quad x_2 = \frac{\Delta_2}{\Delta} \ . \tag{2.6}$$

Here Δ is the determinant of the coefficient matrix in system (2.1), while the determinant Δ_1 in the numerator of the expression for x_1 is obtained from Δ by replacing its *first column* by the nonhomogeneous terms b_1 and b_2 in the system, and the determinant Δ_2 in the numerator of the expression for x_2 is obtained from Δ by replacing its *second column* by the nonhomogeneous terms b_1 and b_2. This is the simplest form of a result known as *Cramer's rule* for solving the two simultaneous first-order algebraic equations in (2.1), in terms of determinants, and its generalization to n nonhomogeneous equations in n unknowns will be given later, along with its proof.

2.2 A First Look at Linear Dependence and Independence

Before developing the general properties of determinants, the simple system (2.1) will be used introduce the important concepts of the *linear dependence* and *independence* of equations. Suppose the second equation in (2.1) is proportional to the first equation, then for some constant of proportionality $\lambda \neq 0$ it will follow that $a_{21} = \lambda a_{11}$, $a_{22} = \lambda a_{12}$ and $b_2 = \lambda b_1$. If this happens the equations are said to be *linearly dependent*, though when they are not proportional, the equations are said to be *linearly independent*. Linear dependence and independence between systems of linear algebraic equations is important, irrespective of the number of equations and unknowns that are involved. Later, when the most important properties of determinants have been established, a determinant test for the linear independence of n homogeneous linear equations in n unknowns will be derived.

When the equations in system (2.1) are linearly dependent, the system only contains one equation relating x_1 and x_2, so one of the equations can be discarded, say the second equation. This means that one of the variables, say x_1, can only be determined in terms of the other variable x_2, so in this sense the values of x_1 and x_2, although related, become indeterminate because then x_2 is arbitrary. To discover the effect this has on the solutions in (2.2), suppose the second equation is λ times the first equation, so that $a_{21} = \lambda a_{11}$, $a_{22} = \lambda a_{12}$ and $b_2 = \lambda b_1$.

Substituting these results into (2.2), and canceling the nonzero scale factor λ, gives

$$x_1 = \frac{b_1 a_{12} - b_1 a_{12}}{a_{11} a_{12} - a_{12} a_{11}} \text{ and } x_2 = \frac{b_1 a_{11} - b_1 a_{11}}{a_{11} a_{12} - a_{12} a_{11}},$$

showing that both the numerators and the denominator in the expressions for x_1 and x_2 vanish, confirming that x_1 and x_2 are indeterminate. A comparison of this result with (2.6) shows that when two rows of a determinant are proportional, its value is zero. This is, in fact, a result that is true for all determinants and not just for second-order determinants.

The indeterminacy of the solution set is hardly surprising, because one of the equations in system (2.1) is redundant, and assigning x_2 an arbitrary value $x_2 = k$, say, will determine x_1 in terms of k as $x_1 = (b_1 - a_{12}k)/a_{11}$, so the solution set $\{x_1, x_2\}$ then takes the form $\{(b_1 - a_{12}k)/a_{11}, k\}$, where k is a parameter. Thus, when the two equations are linearly dependent, that is when $\Delta = 0$, a solution set will exist but it will not be unique, because the solution set will depends on the parameter k, which may be assigned any nonzero value. If, however, $\Delta \neq 0$ the equations will be linearly independent, and the solution set in (2.2) will exist and be unique.

A different situation arises if the left sides of the equations in (2.1) are proportional, but the constants on the right do not share the same proportionality constant, because then the equations imply a contradiction, and no solution set exists. When this happens the equations are said to be *inconsistent*. A final, very important result follows from the solution set (2.6) when the system of Eq. (2.1) is *homogeneous*; which occurs when $b_1 = b_2 = 0$. The consequence of this is most easily seen from (2.2), which is equivalent to (2.6). When the equations are homogeneous, the numerators in (2.2) both vanish because each term in the expansion of the determinant contains a zero factor, so if $\Delta = \det \mathbf{A} \neq 0$, it follows that the solution $x_1 = x_2 = 0$ is unique. This zero solution is called the *null solution*, or the *trivial solution*. Thus the only solution of a *linearly independent* set of homogeneous equations in system (2.1) is the null solution. However, if $\Delta = 0$ the equations will be *linearly dependent* (proportional), and then a solution will exist but, as has been shown, it will be such that x_1 will depend on the variable x_2, which may be assigned arbitrarily. These results will be encountered again when general systems of equations are considered that may be homogeneous or nonhomogeneous.

2.3 Properties of Determinants and the Laplace Expansion Theorem

Having seen something of the way determinants enter into the solution of the system of Eq. (2.1), it is time to return to the study of determinants. The definition of $\det \mathbf{A}$ in (2.4) can be used to establish the following general properties of second-order determinants which, it turns out, are also properties common to determinants of all orders, though determinants of order greater than two have still to be defined.

Theorem 2.1 *Properties of det A.*

1. *Multiplication of the elements of any one row (column) of det* A *by a constant k changes the value of the determinant to k det* **A**. *Equivalently, multiplication of det* **A** *by k can be replaced by multiplying the elements of any one row (column) of det* **A** *by k.*
2. *If every element in a row (column) of det* **A** *is zero, then det* $\mathbf{A} = 0$.
3. *If two rows (columns) of det* **A** *are the identical, or proportional, then det* $\mathbf{A} = 0$.
4. *The value of a determinant is unchanged if a constant multiple of each element in a row (column) is added the corresponding element in another row (column).*
5. *If two rows (columns) in det* **A** *are interchanged, the sign of det* **A** *is changed.*
6. *det* $\mathbf{A} =$ *det* $\mathbf{A}^T$.
7. *If det* **A** *and det* **B** *are determinants of equal order, then det(***AB***) = det* **A***det* **B**.

Proof. Result 1 follows directly from definition (2.4), because each product in the definition of det **A** is multiplied by k. Result 2 also follows directly from definition (2.4), because then a coefficient in each of the products in the definition of det **A** is zero. Result 3 is an extension of the result considered previously where a row was proportional to another row. The result follows from the fact that if two rows (columns) in det **A** are equal, or proportional, the two products in the definition of det **A** cancel. To prove result 4 suppose, for example, that k times each element in the first row of det **A** is added to the corresponding element in the second row, to give det **B** where

$$\det \mathbf{B} = \begin{vmatrix} a_{11} & a_{12} \\ ka_{11} + a_{21} & ka_{12} + a_{22} \end{vmatrix}.$$

Expanding det **B** and canceling terms gives

$$\det \mathbf{B} = a_{11}(ka_{12} + a_{22}) - a_{12}(ka_{11} + a_{21}) = a_{11}a_{22} - a_{12}a_{21} = \det \mathbf{A}.$$

Similar reasoning establishes the equivalent results concerning the other row of the determinant, and also its two columns. Result 5 follows because interchanging two rows (or columns) in det **A** reverses the order of the products in the definition of det **A** in (2.4), and so changes the sign of det **A**. The proof of result 6 is left as Exercise 2.3, and the proof of result 7 will be postponed until Chapter 3, where it is given in Section 3.4 for second-order determinants, using an argument that extends directly to determinates of any order.

♦

In this account of determinants the ***n*th order determinant** associated with an $n \times n$ coefficient matrix $\mathbf{A} = [a_{ij}]$ will be defined in terms of determinants of order $n - 1$ and then, after stepping down recursively to still lower-order determinants, to a definition in terms of a sum of second-order determinants. To proceed to a definition of an nth order determinant, the definition is first extended to a third-order determinant

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_1 & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}. \tag{2.7}$$

The third-order determinant in (2.7) is defined in terms of second-order determinants as

$$\det \mathbf{A} = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{2.8}$$

To remember this definition, notice how the terms are obtained. The first term is the product of a_{11} times the second-order determinant obtained from **A** by omitting the row and column containing a_{11}, the second term is $(-1) \times a_{12}$ times the determinant obtained from **A** by omitting the row and column containing a_{12} and, finally, the third term is a_{13} times the determinant obtained from **A** by omitting the row and column containing a_{13}.

Reasoning as in the proof of Theorem 2.1 and using the fact that a third-order determinant is expressible as a sum of multiples of second-order determinants, it is a straightforward though slightly tedious matter to show that the properties of second-order determinants listed in Theorem 2.1 also apply to third-order determinants, though the proofs of these results are left as exercises.

Determinants of order greater than three will be defined after the cofactors of a determinant have been defined. As already mentioned, the statements in Theorem 2.1 are true for determinants of all orders, though their proof for higher-order determinants will be omitted.

The three determinants

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

that occurred in (2.8) are called, respectively, the minors associated with the elements a_{11}, a_{12} and a_{13} in the first row of det **A**. These minors will be denoted by M_{11}, M_{12} and M_{13}, using the same suffixes as the elements a_{11}, a_{12} and a_{13} to which they correspond, so that

$$M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}, \quad M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}, \quad M_{13} = \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}. \tag{2.9}$$

Remember, that the elements of the minors M_{1i} for $i = 1, 2$ and 3, are obtained from the elements of **A** by omitting the elements in row 1 and column i.

Corresponding to the minors M_{11}, M_{12} and M_{13}, are what are called the cofactors C_{11}, C_{12} and C_{13} associated with the elements a_{11}, a_{12} and a_{13}, and these are defined in terms of the minors as

$$C_{11} = (-)^{1+1}M_{11},\ C_{12} = (-1)^{1+2}M_{12} \ \text{and}\ C_{13} = (-1)^{1+3}. \tag{2.10}$$

The effect of the factors $(-1)^{1+i}$ for $i = 1, 2, 3$ in the definitions of the cofactors C_{11}, C_{12} and C_{13} is to introduce an alternation of sign in the pattern of the minors. Using (2.6) and (2.9) allows us to write $\det \mathbf{A} = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13}$, so from (2.10) this becomes

$$\det \mathbf{A} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}. \tag{2.11}$$

This result is called the expansion of det **A** in terms of the cofactors of the elements of its first row.

There is a minor is associated with every element of a determinant, and not only the elements of its first row. The minor associated with the general element a_{ij}, for $i, j = 1, 2, 3$ is denoted by M_{ij}, and for a third-order determinant it is the numerical value of the 2×2 determinant derived from det **A** by deleting the elements in its ith row and jth column.

Example 2.2. Find the minors and cofactors of the elements of the first row of det **A**, and also the value of det **A**, given that

$$\mathbf{A} = \begin{bmatrix} -4 & 3 & -1 \\ -2 & 4 & 2 \\ 1 & 10 & 1 \end{bmatrix}.$$

Solution. We have

$$M_{11} = \begin{vmatrix} 4 & 2 \\ 10 & 1 \end{vmatrix} = -16, \quad M_{12} = \begin{vmatrix} -2 & 2 \\ 1 & 1 \end{vmatrix} = -4, \quad M_{13} = \begin{vmatrix} -2 & 4 \\ 1 & 10 \end{vmatrix} = -24,$$

so the corresponding cofactors are

$$C_{11} = (-1)^{1+1}(-16) = -16, \; C_{12} = (-1)^{1+2}(-4) = 4, \; C_{13} = (-1)^{1+3}(-24) = -24.$$

From (2.11), when the determinant is expanded in terms of the elements of the first row,

$$\begin{aligned} \det \mathbf{A} &= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \\ &= (-4) \times (-16) + 3 \times 4 + (-1) \times (-24) = 100. \end{aligned}$$

♦

To extend the role of the cofactor associated with the minor of any element of det **A** we start by expanding the expression for a third-order determinant in (2.8), to obtain

$$\det \mathbf{A} = \mathrm{D} = a_{11}a_{22}a_{33} - a_{11}a_{23}a_{32} + a_{12}a_{23}a_{31} - a_{12}a_{21}a_{33} + a_{13}a_{21}a_{32} - a_{13}a_{22}a_{31}. \tag{2.12}$$

Next, we define the cofactor C_{ij} associated with the general element a_{ij} in $\mathbf{A}$ to be

$$C_{ij} = (-1)^{i+j} M_{ij}, \tag{2.13}$$

where for this third-order determinant M_{ij} is the 2×2 minor obtained from det $\mathbf{A}$ by deleting the elements in its ith row and jth column. Using this definition of a general cofactor, and rearranging the terms in (2.12) to give results similar to (2.8), but this time with terms a_{i1}, a_{i2} and a_{i3} multiplying the determinants, it is easily shown that

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + a_{i3}C_{i3}, \quad \text{for } i = 1,\ 2 \text{ or } 3. \tag{2.14}$$

This result provides three different, but equivalent, ways of calculating det $\mathbf{A}$, the first of which was encountered in (2.11). Expressed in words, result (2.14) says that det $\mathbf{A}$ is equal to the sum of the products of the elements and their respective cofactors in any row of the determinant. The result is important, and it is called the expansion of det $\mathbf{A}$ in terms of the elements and cofactors of the ith row of the determinant. So (2.11) is seen to be the expansion of det $\mathbf{A}$ in terms of the elements and cofactors of its first row.

A different rearrangement of the terms in (2.12) shows that

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + a_{3j}C_{3j}, \quad \text{for } j = 1,\ 2 \text{ or } 3, \tag{2.15}$$

providing three more ways of expanding det $\mathbf{A}$. When expressed in words, this expansion says that det $\mathbf{A}$ can be calculated as the sum of the products of the elements and their respective cofactors in any column of the determinant. Result (2.15) is called the expansion of det $\mathbf{A}$ in terms of the elements and cofactors of the jth column of the determinant.

It remains for us to determine the effect of forming the sum of the products of the elements of a row, or column, with the corresponding cofactors of a different row, or column. To resolve this, let δ be the sum of the products of the elements of row i with the cofactors of row s, so that $\delta = \sum_{j=1}^{3} a_{ij}C_{sj}$ for $s \neq j$. Now δ can be interpreted as a third-order determinant with the elements a_{ij} forming its ith row, and the remaining elements taken to be the cofactors C_{sj}. As $s \neq j$, it follows that each cofactor will contain elements from row i, so when the third-order determinant is reconstructed, it will contain another row equal to the ith row except, possibly, for a change of sign throughout the row. Thus the determinant δ will either have two identical rows, or two rows which are identical apart from a change of sign. So by an extension of the results of Theorem 2.1 (see Example 2.3), the determinant must vanish. A similar argument shows that the sum of products formed by multiplying the elements of a column with the corresponding cofactors of a different column is also zero, so that $\sum_{i=1}^{3} a_{ij}C_{ik} = 0$ for $k \neq j$.

The extension of these expansions to include nth-order determinants follows from (2.13) and (2.14) by defining the nth order determinant as either

$$\det \mathbf{A} = \sum_{j=1}^{n} a_{ij}C_{ij} \quad \text{for } i = 1, 2, \ldots, n \tag{2.16}$$

or as

$$\det \mathbf{A} = \sum_{i=1}^{n} a_{ij} C_{ij} \quad \text{for } j = 1, 2, ..., n. \tag{2.17}$$

Notice that now the cofactors C_{ij} are determinants of order $n - 1$. These expressions provide equivalent recursive definitions for an nth-order determinant in terms of second-order determinants, because any determinant of order $n \geq 3$ can always be reduced to a sum of products involving second-order determinants. A determinant det **A** is said to be singular if det $\mathbf{A} = 0$, and nonsingular if det $\mathbf{A} \neq 0$. Chapter 3 will show it is necessary that det $\mathbf{A} \neq 0$ when defining an important matrix $\mathbf{A}^{-1}$ called the inverse matrix associated with a square matrix **A**, or more simply the inverse of **A**.

To avoid the tedious algebraic manipulations involved when extending the results of Theorem 2.1 to determinants of order n, we again mention that the properties listed in the theorem apply to determinants of all orders. However, some of the properties in Theorem 2.1 are almost self-evident for determinants of all orders, as for example the properties 1, 2 and 3.

The extension of the previous results to an nth-order determinant yields the following fundamental expansion theorem due to Laplace.

Theorem 2.2 *The Laplace Expansion of a Determinant.*
Let $\mathbf{A} = [\mathrm{a_{ij}}]$ *be an* $n \times n$ *matrix, and let the cofactor associated with* a_{ij} *be* C_{ij}*. Then, for any i,*

$$\det \mathbf{A} = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in} \ \ (\textit{expansion by elements of the ith row}),$$

and for any j,

$$\det \mathbf{A} = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj} \ \ (\textit{expansion by elements of the jth column})$$

while for any i with $\mathrm{s} \neq i$

$$a_{i1}C_{s1} + a_{i2}C_{s2} + \cdots + a_{in}C_{sn} = 0 \ \ (\textit{expansion using different rows})$$

or for any j with $k \neq j$

$$a_{1j}C_{1k} + a_{2j}C_{2k} + \cdots + a_{nj}C_{nk} = 0 \ \ (\textit{expansion using different columns}).$$

♦

Example 2.3. (a) Expand the determinant in Example 2.2 in terms of elements and cofactors of the third column. (b) Compute the sum of the products of the elements in the first row and the corresponding cofactors of the second row, and hence confirm that the result is zero. (c) Reconstruct the determinant corresponding to the calculation in (b), and hence show why the result is zero.

Solution.

(a) To expand the determinant using elements and cofactors of the third column it is necessary to compute C_{13}, C_{23} and C_{33}. We have

$$\mathbf{A} = \begin{bmatrix} -4 & 3 & -1 \\ -2 & 4 & 2 \\ 1 & 10 & 1 \end{bmatrix}, \text{ so } C_{13} = (-)^{1+3}\begin{vmatrix} -2 & 4 \\ 1 & 10 \end{vmatrix} = -24,$$

$$C_{23} = (-)^{2+3}\begin{vmatrix} -4 & 3 \\ 1 & 10 \end{vmatrix} = 43, \; C_{33} = (-)^{3+3}\begin{vmatrix} -4 & 3 \\ -2 & 4 \end{vmatrix} = -10.$$

Expanding det $\mathbf{A}$ in terms of the elements and cofactors of the third column gives det $\mathbf{A} = (-1) \times (-24) + 2 \times 43 + 1 \times (-10) = 100$, in agreement with Example 2.2.

(b) To form the sum of the products of the elements of the first row with the corresponding cofactors of the second row it is necessary to compute C_{21}, C_{22} and C_{23}. We have

$$C_{21} = (-1)^{2+1}\begin{vmatrix} 3 & -1 \\ 10 & 1 \end{vmatrix} = -13\,, \quad C_{22} = (-1)^{2+2}\begin{vmatrix} -4 & -1 \\ 1 & 1 \end{vmatrix} = -3\,,$$

$$C_{23} = (-1)^{2+3}\begin{vmatrix} -4 & 3 \\ 1 & 10 \end{vmatrix} = 43\,.$$

So the required expansion in terms of elements of the first row and the corresponding cofactors in the second row becomes

$$(-4) \times (-13) + 3 \times (-3) + (-1) \times 43 = 0\,,$$

confirming the third property in Theorem 2.2.

(c) To reconstruct the third-order determinant δ corresponding to the sum of products of the elements in the first row and the cofactors in the second row used in (b) we first write δ as

$$\delta = (-4) \times C_{21} + 3 \times C_{22} + (-1) \times C_{23}.$$

Substituting for the cofactors this becomes

$$\delta = 4 \times \begin{vmatrix} 3 & -1 \\ 10 & 1 \end{vmatrix} + 3 \times \begin{vmatrix} -4 & -1 \\ 1 & 1 \end{vmatrix} + 1 \times \begin{vmatrix} -4 & 3 \\ 1 & 10 \end{vmatrix}.$$

To express this result as the appropriate expansion of a determinant it is necessary restore the correct signs to the multipliers 4, 3 and 1 in the above expression to make them equal to the elements in the first row of **A**, namely -4, 3 and -1. To do this we use result 1 from Theorem 2.1 which shows that when a determinant is multiplied by -1, this multiplier can be taken inside the determinant and used as a multiplier for any one of its rows. To be consistent, we will change the signs of the terms in the last rows of the determinants, so that δ becomes

$$\delta = -\left\{(-4)\times\begin{vmatrix} 3 & -1 \\ -10 & -1 \end{vmatrix} + (3)\times\begin{vmatrix} -4 & -1 \\ -1 & -1 \end{vmatrix} + (-1)\times\begin{vmatrix} -4 & -3 \\ -1 & -1 \end{vmatrix}\right\}.$$

Recognizing that these three determinants are now the cofactors of the elements -4, 3 and -1 in the first row of the determinant that is to be reconstructed, allows the result can be written

$$\delta = -\begin{vmatrix} -4 & 3 & -1 \\ -4 & 3 & -1 \\ -1 & -10 & -1 \end{vmatrix}.$$

This determinant has two identical rows, and so vanishes, showing why result (b) yields the value zero.

♦

The equivalent definitions of an nth order determinant in Theorem 2.2 permit the immediate evaluation of some important and frequently occurring types of determinants. The first case to be considered occurs when det **A** is the nth-order diagonal determinant

$$\det \mathbf{A} = \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}. \tag{2.18}$$

This follows because expanding the determinant in terms of elements of the first row, gives $\det \mathbf{A} = a_{11}C_{11}$, where the cofactor C_{11} is the determinant of order $n-1$ with the same diagonal structure as det **A**. Expanding C_{11} in terms of the elements of its first row gives $\det \mathbf{A} = a_{11}a_{22}C_{11}^{(1)}$, where $C_{11}^{(1)}$ is now the cofactor belonging to determinant C_{11} corresponding to the first element a_{22} in its first row. Continuing this process n times gives the stated result $\det \mathbf{A} = a_{11}a_{22}a_{33}\cdots a_{nn}$.

Two other determinants whose values can be written down at sight are the determinants det **L** and det **U** associated, respectively, with the upper and lower triangular $n \times n$ matrices **L** and **U**. We have

$$\det \mathbf{L} = \begin{vmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn} \tag{2.19}$$

and

$$\det \mathbf{U} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{vmatrix} = a_{11}a_{22}a_{33}\cdots a_{nn}. \tag{2.20}$$

Result (2.18) is obtained in a manner similar to the derivation of (2.18), by repeated expansion of det **L** in terms of the elements of its first row, while result (2.20) follows by a similar repeated expansion of det **U** in terms of elements of its first column.

The next example illustrates how the properties of Theorem 2.1 can sometimes be used to evaluate a determinant without first expanding it with respect to either the elements in its rows or the elements in its columns. The determinant involved has a special form, and it is called an alternant, also known as a Vandermonde determinant.

Example 2.4. Show without direct expansion that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)\,.$$

Solution. Using property 4 of Theorem 2.1, which leaves the value of a determinant unchanged, we subtract column 1 from columns 2 and 3 to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & (b-a) & (c-a) \\ a^2 & (b^2-a^2) & (c^2-a^2) \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ a & (b-a) & (c-a) \\ a^2 & (b+a)(b-a) & (c+a)(c-a) \end{vmatrix}.$$

Next we use property 1 of Theorem 2.1 to remove factors $(b-a)$ and $(c-a)$ from the second and third columns to obtain

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 1 \\ a^2 & (b+a) & (c+a) \end{vmatrix}.$$

Finally, subtracting column two from column three we find that

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)\begin{vmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ a^2 & (b+a) & (c-b) \end{vmatrix}.$$

The determinant is now of lower triangular form, so from (2.19) its value is $(c - b)$. So, as required, we have shown the value of this alternant to be

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{vmatrix} = (b-a)(c-a)(c-b)\,.$$

♦

2.4 Gaussian Elimination and Determinants

The expansion of a determinant using Theorem 2.2 is mainly of theoretical interest, because to evaluate a determinant of order n requires $n!$ multiplications. So, evaluating a determinant of order 8 requires 40,320 multiplications, while evaluating a determinant of order 15 requires approximately 1.31×10^9 multiplications. If, for example, this method of evaluating a determinant were to be performed on a computer where one multiplication takes 1/1,000 s, the evaluation of a determinant of order 15 would take approximately 41.5 years. Clearly, when the order is large, some other way must be found by which to evaluate determinants if this prohibitive number of multiplications is to be avoided, not to mention the buildup of round-off errors that would result. A better method is essential, because many applications of mathematics lead to determinants with orders far larger than 15.

The way around this difficulty is found in property 4 of Theorem 2.1. Subtracting a_{21}/a_{11} times the first row of the determinant from the second row reduces to zero the element immediately below a_{11}. Similarly, subtracting a_{31}/a_{11} times the first row of the determinant from the third row reduces to zero the element in row three below a_{11}, while neither of these operations changes the value of the determinant. So, proceeding down the first column in this manner leads to a new determinant in which the only nonzero entry in its first column is a_{11}. If this procedure is now applied to the second column of the modified determinant, starting with the new coefficient $\tilde{a}_{22}$ that is now in row 2 and column 2, it will reduce to zero all entries below the element $\tilde{a}_{22}$. Proceeding in this way, column by column, the determinant will eventually be replaced by an equivalent nth-order determinant of upper triangular form, the value of which follows, as in (2.20), by forming the product of all the elements in its leading diagonal. This way of evaluating a determinant, called the Gaussian elimination method, or sometimes the Gaussian reduction method, converts a determinant to upper triangular form, whose value is simply the products of the elements on its leading diagonal. This method requires significantly fewer multiplications than the direct expansion used in the definition, and so is efficient when applied to determinants of large order. Software programs are based on a refinement of this method, and even on a relatively slow PC the evaluation of a determinant of order 50 may take only a few seconds.

It can happen that at the ith stage of this reduction process a zero element occurs on the leading diagonal, thereby preventing further reduction of the determinant. This difficulty is easily overcome by interchanging the ith row with a row below it in which the ith element is not zero, after which the reduction continues as before. However, after such an interchange of rows, the sign of the determinant must be changed as required by property 5 of Theorem 2.1. If, on the other hand, at some stage of the reduction process a complete row of zeros is produced, further simplification is impossible, and this shows the value of the determinant is zero or, in other words, that the determinant is singular. The following Example shows how such a reduction proceeds in a typical case when a row interchange becomes necessary.

Remember that an interchange of rows changes the sign of a determinant, so if p interchanges become necessary during the Gaussian elimination process used to calculate the value of determinant, then the sign of the upper triangular determinant that is obtained must be multiplied by $(-1)^p$ in order to arrive at the value of the original determinant.

Example 2.5. Evaluate the following determinant by reducing it to upper triangular form:

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & 1 & 1 \end{vmatrix}.$$

Solution. Subtracting row 1 from row 2 gives

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 2 & 1 \\ 1 & 3 & 6 & 3 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & 1 & 1 \end{vmatrix}.$$

The second element in row 2 is zero, so subtracting multiples of row 2 from rows 3 and 4 cannot reduce to zero the elements in the column below this zero element. To overcome this difficulty we interchange rows 2 and 3, because row 3 has a nonzero element in its second position, and compensate for the row interchange by changing the sign of the determinant, to obtain

$$\det \mathbf{A} = \begin{vmatrix} 1 & 3 & 2 & 1 \\ 0 & 0 & 4 & 2 \\ 0 & 2 & 1 & 5 \\ 0 & 2 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 4 & 2 \\ 0 & 2 & 1 & 1 \end{vmatrix}.$$

Finally, subtracting the new row 2 from row 4 produces the required upper triangular form

$$\det \mathbf{A} = -\begin{vmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 4 & 2 \\ 0 & 2 & 1 & 1 \end{vmatrix} = -\begin{vmatrix} 1 & 3 & 2 & 1 \\ 0 & 2 & 1 & 5 \\ 0 & 0 & 4 & 2 \\ 0 & 0 & 0 & -4 \end{vmatrix},$$

so from (2.20),

$$\det \mathbf{A} = -(1) \times (2) \times (4) \times (-4) = 32\,.$$

♦

Once the inverse matrix has been introduced, matrix algebra will be used to prove the following generalization of Cramer's rule to a nonhomogeneous system of n linear equations in the n unknowns $x_1, x_2, \ldots, x_n$. However, it will be useful to state this generalization in advance of its proof.

Theorem 2.3 *The Generalized Cramer's Rule.*
The system of n *nonhomogeneous linear equations in the variables* $x_1, x_2, \ldots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2, \\ &\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_b &= b_n \end{aligned} \tag{2.21}$$

has the solution set $\{x_1, x_2, \ldots, x_n\}$ *given by*

$$x_1 = \frac{\Delta_1}{\Delta}\,,\ x_2 = \frac{\Delta_2}{\Delta},\ \cdots\ ,\ x_n = \frac{\Delta_n}{\Delta}\ , \tag{2.22}$$

provided $\Delta \neq 0$, *where*

$$\Delta = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix},\ \Delta_1 = \begin{vmatrix} b_1 & a_{12} & \cdots & a_{1n} \\ b_2 & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ b_n & a_{n1} & a_{n2} & a_{nn} \end{vmatrix}, \ldots,$$
$$\Delta_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & b_1 \\ a_{21} & a_{22} & \cdots & b_2 \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & b_n \end{vmatrix}. \tag{2.23}$$

♦

Notice that in (2.23) $\Delta = \det \mathbf{A}$ is the determinant of the coefficient matrix $\mathbf{A}$, and the determinant Δ_i for $i = 1, 2, \ldots, n$ is derived from Δ by replacing its ith column by the column vector containing the nonhomogeneous terms $b_1, b_2, \ldots, b_n$.

2.5 Homogeneous Systems of Equations and a Test for Linear Independence

Consider the system of n homogeneous linear equations in the n independent variables $x_1, x_2, \ldots, x_n$:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= 0, \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= 0, \\ &\ldots\ldots\ldots\ldots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= 0. \end{aligned} \tag{2.24}$$

Accepting the validity of this generalization of Cramer's rule, it follows that if the determinant of the coefficients $\det \mathbf{A} \neq 0$, the only possible solution of (2.24) is the null solution $x_1 = x_2 = \cdots = x_n = 0$. This means that no equation in (2.24) can be expressed as the sum of multiples of other equations belonging to the system, so the equations in the system are linearly independent. Suppose, however, that one of the equations is formed by the addition of multiples of some of the remaining equations, making it linearly dependent on other equations in the system. Subtracting these same multiples of equations from the linearly dependent equation will reduce it to an equation of the form $0x_1 + 0x_2 + \cdots + 0x_n = 0$, leading to a row of zeros in the equivalent coefficient matrix. It then follows immediately that $\det \mathbf{A} = 0$, and the same conclusion follows if more than one of the equations in (2.24) is linearly dependent on the other equations. We have established the following useful result.

Theorem 2.4 *Determinant Test for Linear Independence.*
A necessary and sufficient condition that the n homogeneous equations in (2.24) with the coefficient matrix $\mathbf{A}$ *are linearly independent is that* $\det \mathbf{A} \neq 0$. *Conversely, if* $\det \mathbf{A} = 0$, *the equations are linearly dependent.*

♦

It follows from Theorem 2.4 that if $m < n$ of the equations in (2.24) are linearly independent, it is only possible to solve for m of the unknown variables $x_1, x_2, \ldots, x_n$ in terms of the of the remaining $n - m$ variables that can then be regarded as arbitrary parameters.

The next example shows how this situation arises when dealing with a system of four equations, only two of which are linearly independent.

Example 2.6. Show only two of the following four linear homogeneous equations are linearly independent, and find the solution set if two of the unknowns are assigned arbitrary values.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + 2x_4 &= 0, \\ 3x_1 + 2x_2 - 3x_3 - x_4 &= 0, \\ x_1 + 5x_2 - 4x_3 - 3x_4 &= 0, \\ 5x_1 - x_2 - 2x_3 + x_4 &= 0. \end{aligned}$$

Solution. Later a simple way will be found of determining which equations may be taken to be linearly independent. However, for the moment, it will suffice to notice that the third equation is obtained by subtracting the first equation from the second equation, and the fourth equation is obtained by adding the first and second equations. So we may take the first and second equations as being linearly independent, and the last two equations as being redundant because of their linear dependence on the first two equations. The linear dependence of this system of equations is easily checked by using the determinant test in Theorem 2.4, because

$$\det \mathbf{A} = \begin{vmatrix} 2 & -3 & 1 & 2 \\ 3 & 2 & -3 & -1 \\ 1 & 5 & -4 & -3 \\ 5 & -1 & -2 & 1 \end{vmatrix} = 0\,.$$

While the determinant test establishes the existence of linear dependence amongst the equations in system (2.24), it does not show how many of the equations are linearly independent.

As we know by inspection that the first two equations contain all of the information in this system, the last two equations can be disregarded, and we can work with the first two equations

$$\begin{aligned} 2x_1 - 3x_2 + x_3 + 2x_4 &= 0, \\ 3x_1 + 2x_2 - 3x_3 - x_4 &= 0. \end{aligned}$$

If we set $x_3 = k_1$ and $x_4 = k_2$, each of which is arbitrary, the system reduces to the two equations for x_1 and x_2,

$$\begin{aligned} 2x_1 - 3x_2 &= -k_1 - 2k_2, \\ 3x_1 + 2x_2 &= 3k_1 + k_2. \end{aligned}$$

Solving these equations for x_1 and x_2 shows the solution set $\{x_1, x_2, x_3, x_4\}$ for the system has for its elements

$$x_1 = \tfrac{7}{13}k_1 - \tfrac{1}{13}k_2\,, \quad x_2 = \tfrac{9}{13}k_1 + \tfrac{8}{13}k_2\,, \quad x_3 = k_1\,, \quad x_4 = k_2\,,$$

where the quantities k_1 and k_2 are to be regarded as arbitrary parameters.

♦

Corollary 2.5. *Linear Dependence of the Columns of a Determinant.*
If in system (2.24) det **A** *= 0, then the columns of the determinant are linearly dependent.*

Proof. The result is almost immediate, and it follows from the fact that the rows of det $\mathbf{A}^{\mathrm{T}}$ are the columns of det **A**. The vanishing of det **A** implies linear dependence between the rows of det **A**, but det $\mathbf{A} = \det \mathbf{A}^{\mathrm{T}}$, so the vanishing of det **A** implies linear dependence between the columns of det **A**.

♦

2.6 Determinants and Eigenvalues: A First Look

An important type of determinant associated with an $n \times n$ matrix $\mathbf{A} = [a_{ij}]$ has the form det$[\mathbf{A} - \lambda\mathbf{I}]$, where λ is a scalar parameter. To interpret the matrix expression $\mathbf{A} - \lambda\mathbf{I}$ we need to anticipate the definition of the multiplication of a matrix by a scalar. This is accomplished by defining the matrix $\lambda\mathbf{I}$ to be the matrix obtained from the unit matrix **I** by multiplying each of its elements by λ, so if **I** is the 3×3 unit matrix,

$$\lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix}.$$

Example 2.7. Given

$$\mathbf{A} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix},$$

find $\mathbf{A} - \lambda\mathbf{I}$ and write down det$[\mathbf{A} - \lambda\mathbf{I}]$.

Solution. We have

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix} - \lambda\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & -2 \\ 0 & -2 & 1 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix},$$

from which it follows that

$$\mathbf{A} - \lambda\mathbf{I} = \begin{bmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{bmatrix}, \text{ and so } \det[\mathbf{A} - \lambda\mathbf{I}] = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{vmatrix}.$$

♦

If **A** is an $n \times n$ matrix, when expanded det $[\mathbf{A} - \lambda \mathbf{I}]$ yields a polynomial $p(\lambda)$ of degree n in λ, where n is the order of det **A**. In Example 2.7 the polynomial $p(\lambda)$ given by

$$p(\lambda) = \det[\mathbf{A} - \lambda \mathbf{I}] = \begin{vmatrix} 1-\lambda & 2 & 0 \\ 2 & -1-\lambda & -2 \\ 0 & -2 & 1-\lambda \end{vmatrix} = -\lambda^3 . + \lambda^2 + 9\lambda - 9.$$

The roots of det $[\mathbf{A} - \lambda \mathbf{I}] = 0$, that is the zeros of $p(\lambda)$, are called the eigenvalues of the matrix **A**, so in Example 2.7 the polynomial $p(\lambda) = 0$ becomes the cubic equation $\lambda^3 - \lambda^2 - 9\lambda + 9 = 0$. This has the roots $\lambda = 1$, $\lambda = -3$ and $\lambda = 3$, so these are the eigenvalues of matrix **A**. The expression $p(\lambda)$ is called the characteristic polynomial of matrix **A**, and $p(\lambda) = 0$ is called the characteristic equation of matrix **A**. As the eigenvalues of a square matrix **A** are the roots of a polynomial it is possible for the eigenvalues of **A** to be complex numbers, even when all of the elements of **A** are real. It is also important to recognize that only square matrices have eigenvalues, because when **A** is an $m \times n$ matrix with $m \neq n$, det **A** has no meaning.

Theorem 2.5 *The eigenvalues of A and* A^T.
The matrix **A** *and its transpose* $\mathbf{A}^T$ *have the same characteristic polynomial, and the same eigenvalues.*
Proof. The results follow directly from Property 6 of Theorem 2.1, because **A** and $\mathbf{A}^{\mathrm{T}}$ have the same characteristic polynomial, and hence the same eigenvalues.

♦

Example 2.8. If $\mathbf{A} = \begin{bmatrix} 1 & 3 & 2 \\ -1 & 2 & 4 \\ 1 & 0 & -1 \end{bmatrix}$, then $\mathbf{A}^{\mathrm{T}} = \begin{bmatrix} 1 & -1 & 1 \\ 3 & 2 & 0 \\ 2 & 4 & -1 \end{bmatrix}$, and routine calculations confirm that

$$p(\lambda) = \det[\mathbf{A} - \lambda \mathbf{I}] = \det[\mathbf{A}^{\mathrm{T}} - \lambda \mathbf{I}] = \lambda^3 - 2\lambda^2 - 3,$$

so the characteristic polynomials are identical. The eigenvalues determined by $p(\lambda) = 0$ are

$$\lambda_1 = 2.48558, \quad \lambda_2 = 0.24279 - 1.07145i \;\text{ and }\; \lambda_3 = \bar{\lambda}_2 = 0.24279 + 1.07145i,$$

so in this case one eigenvalue is real and the other two are complex conjugates.

♦

Exercises

1. Evaluate the determinants

$$\text{(a) } \det \mathbf{A} = \begin{vmatrix} 7 & 3 & 4 \\ 1 & 2 & 1 \\ 3 & 0 & 2 \end{vmatrix}, \text{ (b) } \det \mathbf{B} = \begin{vmatrix} 1 & -3 & 2 \\ 4 & 5 & 6 \\ 5 & 2 & 8 \end{vmatrix}, \text{ (c) } \det \mathbf{C} = \begin{vmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{vmatrix}.$$

2. Evaluate the determinants

$$\text{(a)} \quad \det \mathbf{A} = \begin{vmatrix} \sin t & \cos t & 1 \\ -\cos t & \sin t & 0 \\ e^t & 0 & 0 \end{vmatrix}, \text{(b)} \det \mathbf{B} = \begin{vmatrix} e^{-t}\sin t & e^{-t}\cos t & 0 \\ -e^{-t}\cos t & e^{-t}\sin t & 1 \\ e^t & 0 & 1 \end{vmatrix}.$$

3. Construct a 3×3 matrix $\mathbf{A}$ of your own choice, and by expanding the determinants det $\mathbf{A}$ and det $\mathbf{A}^{\mathrm{T}}$ show that det $\mathbf{A} =$ det $\mathbf{A}^{\mathrm{T}}$. Prove that if $\mathbf{A}$ is any $n \times n$ matrix, then it is always true that det $\mathbf{A} =$ det $\mathbf{A}^{\mathrm{T}}$.
4. Evaluate the determinant

$$\det \mathbf{A} = \begin{vmatrix} 2 & 0 & -1 & 3 \\ 1 & 4 & 9 & 0 \\ -2 & 1 & 3 & -1 \\ 4 & 0 & 3 & 2 \end{vmatrix}.$$

5. Show without expanding the determinant that

$$\begin{vmatrix} 1+a & a & a \\ b & 1+b & b \\ b & b & 1+b \end{vmatrix} = (1+a+2b).$$

6. Show without expanding the determinant that

$$\begin{vmatrix} x^3+1 & 1 & 1 \\ 1 & x^3+1 & 1 \\ 1 & 1 & x^3+1 \end{vmatrix} = x^6(x^3+3).$$

7. Evaluate the following determinant by reducing it to upper triangular form

$$\Delta = \begin{vmatrix} 2 & 1 & 0 & 1 \\ 3 & 2 & 4 & 2 \\ 1 & 2 & 1 & 3 \\ 0 & 3 & 1 & 1 \end{vmatrix}.$$

8. Use Cramer's rule to solve the system of equations

$$x_1 + 2x_2 - x_3 = 9,$$
$$2x_1 - 3x_2 + 5x_3 = -2,$$
$$4x_1 - 2x_2 - 3x_3 = 7.$$

9. Are the equations in the following two systems linearly dependent?

$$\text{(a)}\quad \begin{aligned} x_1 - 2x_2 + 4x_3 &= 0 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 7x_1 + 22x_2 - 2x_3 &= 0, \end{aligned} \qquad \text{(b)}\quad \begin{aligned} 3x_1 - x_2 + 2x_3 &= 0 \\ x_1 + 4x_2 + 6x_3 &= 0 \\ 3x_1 - x_2 + 4x_3 &= 0\ . \end{aligned}$$

10. Are the equations in the following system linearly independent? Give a reason for your answer.

$$\begin{aligned} x_1 + 2x_2 - x_3 - x_4 &= 0, \\ 2x_1 - x_2 + 2x_3 + 2x_4 &= 0, \\ 4x_1 - 7x_2 + 8x_3 + 8x_4 &= 0, \\ 3x_1 - x_2 + 3x_3 - 2x_4 &= 0\ . \end{aligned}$$

11. Given that

$$\mathbf{A} = \begin{bmatrix} 2 & 0 & -1 \\ -1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix},$$

confirm by direct computation that if a constant k is subtracted from each element on the leading diagonal of matrix **A**, the eigenvalues of the modified matrix are the eigenvalues of matrix **A** from each of which is subtracted the constant k. Could this result have been deduced without direct computation, and if so how? Is this result only true for this matrix **A**, or is it a general property of the eigenvalues of $n \times n$ matrices?

12. Construct a square matrix of your choice, and verify by direct expansion that the characteristic polynomials of **A** and $\mathbf{A}^{\mathrm{T}}$ are identical.

The calculation of integrals over areas and volumes is often simplified by changing the variables involved to ones that are more natural for the geometry of the problem. When an integral is expressed in terms of the Cartesian coordinates x, y and z, a change of the coordinates to u_1, u_2 and u_3 involves making a transformation of the form

$$x = f(u_1,\ u_2,\ u_3),\quad y = g(u_1,\ u_2,\ u_3),\ z = h(u_1,\ u_2,\ u_3),$$

and when this is done a scale factor J enters the transformed integrand to compensate for the change of scales. The factor J is a functional determinant denoted by $\frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)}$, where

$$J = \frac{\partial(x,y,z)}{\partial(u_1,u_2,u_3)} = \begin{vmatrix} \frac{\partial x}{\partial u_1} & \frac{\partial x}{\partial u_2} & \frac{\partial x}{\partial u_3} \\ \frac{\partial y}{\partial u_1} & \frac{\partial y}{\partial u_2} & \frac{\partial y}{\partial u_3} \\ \frac{\partial z}{\partial u_1} & \frac{\partial z}{\partial u_2} & \frac{\partial z}{\partial u_3} \end{vmatrix},$$

and J is called the Jacobian of the transformation or, more simply, just the Jacobian. If the Jacobian vanishes at any point P, the transformation fails to establish a unique correspondence at that point between the point (x_P, y_P, z_P) and the transformed point

$$(u_{1\mathrm{P}}, u_{2P}, u_{3\mathrm{P}}).$$

In Exercises 13 and 14, find the Jacobian of the given transformation, and determine when $J = 0$. Give a geometrical reason why the transformation fails when $J = 0$.

13. Find the Jacobian for the cylindrical polar coordinates $x = r\cos\phi$, $y = r\sin\phi$, $z = z$ where the coordinate system is shown in Fig. 2.1.

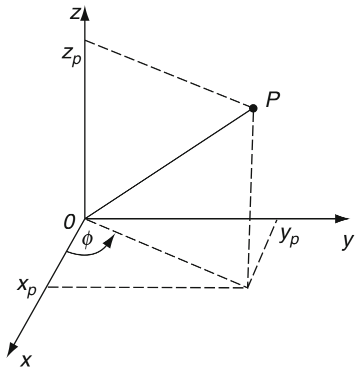

Fig. 2.1 The cylindrical polar coordinate system

14. Find the Jacobian for the *spherical polar coordinates* $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$ where the coordinate system is shown in Fig. 2.2.

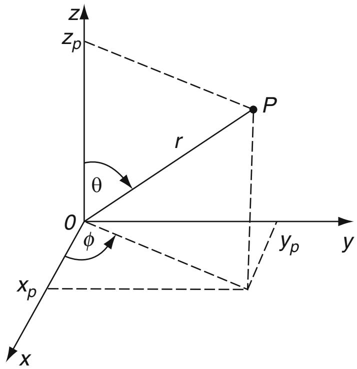

Fig. 2.2 The spherical polar coordinate system