Contents

1 GAUSSIAN INTEGRALS

Gaussian measures play a central role in many fields: in probability theory as a consequence of the central limit theorem, in quantum mechanics as we will show and, thus, in quantum field theory, in the theory of phase transitions in statistical physics. Therefore, as a preliminary to the discussion of path integrals, we recall in this chapter a few useful mathematical results about gaussian integrals, and properties of gaussian expectation values. In particular, we prove the corresponding Wick's theorem, a simple result but of major practical significance.

To discuss properties of expectation values with respect to some measure or probability distribution, it is always convenient to introduce a generating function of the moments of the distribution. This also allows defining the generating function of the cumulants of a distribution, which have simpler properties.

The steepest descent method provides an approximation scheme to evaluate certain types of complex integrals. Successive contributions take the form of gaussian expectation values. We explain the method here, for real and complex, simple and multiple integrals, with the aim of eventually applying it to path integrals.

Notation. In this work we use, in general, boldface characters to indicate matrices or vectors, while we use italics (then with indices) to indicate the corresponding matrix elements or vector components.

1.1 Generating function

We consider a positive measure or probability distribution $\Omega(x_1, x_2, \dots, x_n)$ defined on $\mathbb{R}^n$ and properly normalized. We denote by

$$\langle F \rangle \equiv \int \mathrm{d}^n x\, \Omega(\mathbf{x}) F(\mathbf{x})\,,$$

where $\mathrm{d}^n x \equiv \prod_{i=1}^n \mathrm{d}x_i$, the expectation value of a function $F(x_1, \dots, x_n)$. The normalization is chosen such that $\langle 1 \rangle = 1$.

It is generally convenient to introduce the Fourier transform of the distribution. In this work, we consider mainly a special class of distributions such that the Fourier transform is an analytic function that also exists for imaginary arguments. We thus introduce the function

$$\mathcal{Z}(\mathbf{b}) = \left\langle \mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} \right\rangle = \int \mathrm{d}^n x\, \Omega(\mathbf{x})\, \mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} \ \text{ where } \mathbf{b}\cdot\mathbf{x} = \sum_{i=1}^{n} b_i x_i\,. \tag{1.1}$$

The advantage of this definition is that the integrand remains a positive measure for all real values of $\mathbf{b}$.

Expanding the integrand in powers of the variables b_k, one recognizes that the coefficients are expectation values, moments of the distribution:

$$\mathcal{Z}(\mathbf{b}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} \sum_{k_1,k_2,\ldots,k_\ell=1}^{n} b_{k_1} b_{k_2} \ldots b_{k_\ell} \langle x_{k_1} x_{k_2} \ldots x_{k_\ell} \rangle .$$

The function $\mathcal{Z}(\mathbf{b})$ thus is a *generating function* of the moments of the distribution, that is, of the expectation values of monomial functions. The expectation values can be recovered by differentiating the function $\mathcal{Z}(\mathbf{b})$. Quite directly, differentiating both sides of equation (1.1) with respect to b_k, one obtains

$$\frac{\partial}{\partial b_k} \mathcal{Z}(\mathbf{b}) = \int \mathrm{d}^n x \, \Omega(\mathbf{x}) x_k \, \mathrm{e}^{\mathbf{b}\cdot\mathbf{x}} . \tag{1.2}$$

Repeated differentiation then yields, in the limit $\mathbf{b} = 0$,

$$\langle x_{k_1} x_{k_2} \ldots x_{k_\ell} \rangle = \left[\frac{\partial}{\partial b_{k_1}} \frac{\partial}{\partial b_{k_2}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathcal{Z}(\mathbf{b}) \right] \bigg|_{\mathbf{b}=0} . \tag{1.3}$$

This notion of generating functions is very useful and will be extended in Section 2.5.3 to the limit where the number of variables becomes infinite.

1.2 Gaussian expectation values. Wick's theorem

As a consequence of the central limit theorem of probabilities, gaussian distributions play an important role in all stochastic phenomena and, therefore, also in physics. We recall here some algebraic properties of gaussian integrals and gaussian expectation values. Since most algebraic properties generalize to complex gaussian integrals, we consider also below this more general situation.

The gaussian integral

$$\mathcal{Z}(\mathbf{A}) = \int \mathrm{d}^n x \, \exp\left(- \sum_{i,\,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j \right) , \tag{1.4}$$

converges if the matrix $\mathbf{A}$ with elements A_{ij} is a symmetric complex matrix such that the real part of the matrix is non-negative (this implies that all eigenvalues of $\mathrm{Re}\,\mathbf{A}$ are non-negative) and no eigenvalue a_i of $\mathbf{A}$ vanishes:

$$\mathrm{Re}\,\mathbf{A} \geq 0 \, , \;\; a_i \neq 0 \, .$$

Several methods then allow us to prove

$$\mathcal{Z}(\mathbf{A}) = (2\pi)^{n/2} (\det \mathbf{A})^{-1/2} . \tag{1.5}$$

When the matrix is complex, the meaning of the square root and thus the determination of the global sign requires, of course, some special care.

We derive below this result for real positive matrices. In Section 1.7, we give a proof for complex matrices. Another independent proof is indicated in the exercises.

1.2.1 Real matrices: a proof

The general one-dimensional gaussian integral can easily be calculated and one finds ($a > 0$)

$$\int_{-\infty}^{+\infty} \mathrm{d}x \; \mathrm{e}^{-ax^2/2+bx} = \sqrt{2\pi/a}\,\mathrm{e}^{b^2/2a} \;. \tag{1.6}$$

More generally, any real symmetric matrix can be diagonalized by an orthogonal transformation and the matrix $\mathbf{A}$ in (1.4) can thus be written as

$$\mathbf{A} = \mathbf{O}\mathbf{D}\mathbf{O}^{\mathrm{T}} \,, \tag{1.7}$$

where the matrix $\mathbf{O}$ is orthogonal and the matrix $\mathbf{D}$ with elements D_{ij} diagonal:

$$\mathbf{O}^{\mathrm{T}}\mathbf{O} = \mathbf{1} \,, \;\; D_{ij} = a_i \delta_{ij} \;.$$

We thus change variables, $\mathbf{x} \mapsto \mathbf{y}$, in the integral (1.4):

$$x_i = \sum_{j=1}^{n} O_{ij} y_j \;\Rightarrow\; \sum_{i,j} x_i A_{ij} x_j = \sum_{i,j} x_i O_{ik} a_k O_{jk} x_j = \sum_i a_i y_i^2 \;.$$

The corresponding jacobian is $J = |\det \mathbf{O}| = 1$.

The integral then factorizes:

$$\mathcal{Z}(\mathbf{A}) = \prod_{i=1}^{n} \int \mathrm{d}y_i \; \mathrm{e}^{-a_i y_i^2/2} \;.$$

The matrix $\mathbf{A}$ is positive; all eigenvalues a_i are thus positive and each integral converges. From the result (1.6), one then infers

$$\mathcal{Z}(\mathbf{A}) = (2\pi)^{n/2} (a_1 a_2 \ldots a_n)^{-1/2} = (2\pi)^{n/2} (\det \mathbf{A})^{-1/2}.$$

1.2.2 General gaussian integral

We now consider a general gaussian integral

$$\mathcal{Z}(\mathbf{A}, \mathbf{b}) = \int \mathrm{d}^n x \; \exp\left(-\sum_{i,\,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j + \sum_{i=1}^{n} b_i x_i \right) . \tag{1.8}$$

To calculate $\mathcal{Z}(\mathbf{A}, \mathbf{b})$, one looks for the minimum of the quadratic form:

$$\frac{\partial}{\partial x_k} \left(\sum_{i,\,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j - \sum_{i=1}^{n} b_i x_i \right) = \sum_{j=1}^{n} A_{kj} x_j - b_k = 0 \,.$$

Introducing the inverse matrix

$$\mathbf{\Delta} = \mathbf{A}^{-1},$$

one can write the solution as

$$x_i = \sum_{j=1}^{n} \Delta_{ij} b_j \,. \tag{1.9}$$

After the change of variables $x_i \mapsto y_i$,

$$x_i = \sum_{j=1}^{n} \Delta_{ij} b_j + y_i \,, \tag{1.10}$$

the integral becomes

$$\mathcal{Z}(\mathbf{A}, \mathbf{b}) = \exp\left[\sum_{i,j=1}^{n} \tfrac{1}{2} b_i \Delta_{ij} b_j\right] \int \mathrm{d}^n y \, \exp\left(-\sum_{i,j=1}^{n} \tfrac{1}{2} y_i A_{ij} y_j\right). \tag{1.11}$$

The change of variables has reduced the calculation to the integral (1.4). One concludes

$$\mathcal{Z}(\mathbf{A}, \mathbf{b}) = (2\pi)^{n/2} \left(\det \mathbf{A}\right)^{-1/2} \exp\left[\sum_{i,j=1}^{n} \tfrac{1}{2} b_i \Delta_{ij} b_j\right]. \tag{1.12}$$

Remark. Gaussian integrals have a remarkable property: after integration over one variable, one finds again a gaussian integral. This structural stability explains the stability of gaussian probability distributions and is also related to some properties of the harmonic oscillator, which will be discussed in Section 2.6.

1.2.3 Gaussian expectation values and Wick theorem

When the matrix $\mathbf{A}$ is real and positive, the gaussian integrand can be considered as a positive measure or a probability distribution on $\mathbb{R}^n$, which can be used to calculate expectation values of functions of the n variables x_i:

$$\langle F(\mathbf{x}) \rangle \equiv \mathcal{N} \int \mathrm{d}^n x \, F(\mathbf{x}) \exp\left(-\sum_{i,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j\right), \tag{1.13}$$

where the normalization $\mathcal{N}$ is determined by the condition $\langle 1 \rangle = 1$:

$$\mathcal{N} = \mathcal{Z}^{-1}(\mathbf{A}, 0) = (2\pi)^{-n/2} \left(\det \mathbf{A}\right)^{1/2} .$$

The function

$$\mathcal{Z}(\mathbf{A}, \mathbf{b}) / \mathcal{Z}(\mathbf{A}, 0) = \left\langle \mathrm{e}^{\mathbf{b} \cdot \mathbf{x}} \right\rangle , \tag{1.14}$$

where $\mathcal{Z}(\mathbf{A}, \mathbf{b})$ is the function (1.8), is then a *generating function* of the moments of the distribution, that is, of the gaussian expectation values of monomials (see

Section 1.1). Expectation values are then obtained by differentiating equation (1.14) with respect to the variables b_i:

$$\langle x_{k_1} x_{k_2} \dots x_{k_\ell} \rangle = (2\pi)^{-n/2} (\det \mathbf{A})^{1/2} \left[\frac{\partial}{\partial b_{k_1}} \frac{\partial}{\partial b_{k_2}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathcal{Z}(\mathbf{A}, \mathbf{b}) \right] \Bigg|_{\mathbf{b}=0}$$

and, replacing $\mathcal{Z}(\mathbf{A}, \mathbf{b})$ by the explicit expression (1.12),

$$\langle x_{k_1} \dots x_{k_\ell} \rangle = \left\{ \frac{\partial}{\partial b_{k_1}} \cdots \frac{\partial}{\partial b_{k_\ell}} \exp \left[\sum_{i,j=1}^{n} \tfrac{1}{2} b_i \Delta_{ij} b_j \right] \right\} \Bigg|_{\mathbf{b}=0} . \tag{1.15}$$

More generally, if $F(\mathbf{x})$ is a power series in the variables x_i, its expectation value is given by the identity

$$\langle F(\mathbf{x}) \rangle = \left\{ F\left(\frac{\partial}{\partial b} \right) \exp \left[\sum_{i,j} \tfrac{1}{2} b_i \Delta_{ij} b_j \right] \right\} \Bigg|_{\mathbf{b}=0} . \tag{1.16}$$

Wick's theorem. Identity (1.15) leads to Wick's theorem. Each time one differentiates the exponential in the r.h.s., one generates one factor b. One must differentiate this b factor later, otherwise the corresponding contribution vanishes when $\mathbf{b}$ is set to zero. One concludes that the expectation value of the product $x_{k_1} \dots x_{k_\ell}$ with a gaussian weight proportional to $\exp(-\frac{1}{2} x_i A_{ij} x_j)$ is given by the following expression: one pairs in all possible ways the indices $k_1, \dots, k_\ell$ (ℓ must be even, otherwise the moment vanishes). To each pair $k_p k_q$, one associates the element $\Delta_{k_p k_q}$ of the matrix $\mathbf{\Delta} = \mathbf{A}^{-1}$. Then,

$$\langle x_{k_1} \dots x_{k_\ell} \rangle = \sum_{\substack{\text{all possible pairings} \\ P \text{ of } \{k_1 \dots k_\ell\}}} \Delta_{k_{P_1} k_{P_2}} \dots \Delta_{k_{P_{\ell-1}} k_{P_\ell}} , \tag{1.17}$$

$$= \sum_{\substack{\text{all possible pairings} \\ P \text{ of } \{k_1 \dots k_\ell\}}} \langle x_{k_{P_1}} x_{k_{P_2}} \rangle \cdots \langle x_{k_{P_{\ell-1}}} x_{k_{P_\ell}} \rangle . \tag{1.18}$$

Equations (1.17, 1.18) are characteristic properties of all centred (i.e. $\langle x_i \rangle = 0$) gaussian measures. They are known under the name of Wick's theorem. Suitably adapted to quantum mechanics or quantum field theory, they form the basis of perturbation theory. Note that the simplicity of this result should not hide its *major practical significance.* Note also that, since the derivation is purely algebraic, it generalizes to complex integrals. Only the interpretation of gaussian functions as positive measures or probability distributions disappears.

Examples. One finds, successively,

$$\langle x_{i_1} x_{i_2} \rangle = \Delta_{i_1 i_2} ,$$
$$\langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle = \Delta_{i_1 i_2} \Delta_{i_3 i_4} + \Delta_{i_1 i_3} \Delta_{i_2 i_4} + \Delta_{i_1 i_4} \Delta_{i_3 i_2} .$$

More generally, the gaussian expectation value of a product of $2p$ variables is the sum of $(2p-1)(2p-3)\ldots 5\times 3\times 1$ contributions (a simple remark that provides a useful check).

A useful identity. We consider the gaussian expectation value of the product $x_i F(\mathbf{x})$:

$$\langle x_i\, F(\mathbf{x})\rangle = \mathcal{N}\int \mathrm{d}^n x\, x_i\, F(\mathbf{x}) \exp\left(-\sum_{j,k=1}^{n} \tfrac{1}{2} x_j A_{jk} x_k\right). \tag{1.19}$$

Using the identity

$$x_i \exp\left(-\sum_{j,k=1}^{n} \tfrac{1}{2} x_j A_{jk} x_k\right) = -\sum_{\ell} \Delta_{i\ell} \frac{\partial}{\partial x_\ell} \exp\left(-\sum_{j,k=1}^{n} \tfrac{1}{2} x_j A_{jk} x_k\right)$$

inside (1.19), and integrating by parts, one obtains the relation

$$\langle x_i F(\mathbf{x})\rangle = \mathcal{N}\sum_{\ell} \Delta_{i\ell} \int \mathrm{d}^n x \exp\left(-\sum_{j,k=1}^{n} \tfrac{1}{2} x_j A_{jk} x_k\right) \frac{\partial F}{\partial x_\ell}\,,$$

which can also be written as

$$\langle x_i F(x)\rangle = \sum_{\ell} \langle x_i x_\ell\rangle \left\langle \frac{\partial F}{\partial x_\ell}\right\rangle, \tag{1.20}$$

and which can also be derived by applying Wick's theorem.

1.3 Perturbed gaussian measure. Connected contributions

Even in favourable situations where the central limit theorem applies, the gaussian measure is only an asymptotic distribution. Therefore, it is useful to also evaluate expectation values with perturbed gaussian distributions.

1.3.1 Perturbed gaussian measure

We consider a more general normalized distribution $\mathrm{e}^{-A(\mathbf{x},\lambda)}/\mathcal{Z}(\lambda)$, where the function $A(\mathbf{x},\lambda)$ can be written as the sum of a quadratic part and a polynomial $\lambda V(\mathbf{x})$ in the variables x_i:

$$A(\mathbf{x},\lambda) = \frac{1}{2}\sum_{i,j=1}^{n} x_i A_{ij} x_j + \lambda V(\mathbf{x}), \tag{1.21}$$

the parameter λ characterizing the amplitude of the deviation from the gaussian distribution.

The normalization $\mathcal{Z}(\lambda)$ is given by the integral

$$\mathcal{Z}(\lambda) = \int \mathrm{d}^n x\ \mathrm{e}^{-A(\mathbf{x},\lambda)}\,. \tag{1.22}$$

To evaluate it, we expand the integrand in a formal power series in λ and integrate term by term:

$$\begin{aligned}\mathcal{Z}(\lambda) &= \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \int \mathrm{d}^n x\, V^k(\mathbf{x}) \exp\left(-\sum_{i,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j\right) \\ &= \mathcal{Z}(0) \sum_{k=0}^{\infty} \frac{(-\lambda)^k}{k!} \left\langle V^k(\mathbf{x})\right\rangle_0 , \end{aligned} \tag{1.23}$$

where the symbol $\langle \bullet \rangle_0$ refers to the expectation value with respect to the normalized gaussian measure $\exp[-\sum_{i,j} x_i A_{ij} x_j/2]/\mathcal{Z}(0)$. Each term in the expansion, which is a gaussian expectation value of a polynomial, can then be evaluated with the help of Wick's theorem (1.17).

Using equation (1.16) with $F = \mathrm{e}^{-\lambda V}$, one also infers a formal representation of the function (1.22):

$$\mathcal{Z}(\lambda)/\mathcal{Z}(0) = \left.\left\{ \exp\left[-\lambda V\left(\frac{\partial}{\partial b}\right)\right] \exp\left[\sum_{i,j=1}^{n} \tfrac{1}{2} b_i \Delta_{ij} b_j\right]\right\}\right|_{\mathbf{b}=0} . \tag{1.24}$$

Example. In the example of the perturbation

$$V(\mathbf{x}) = \frac{1}{4!} \sum_{i=1}^{n} x_i^4 , \tag{1.25}$$

the expansion to order λ^2 is ($\boldsymbol{\Delta}\mathbf{A} = \mathbf{1}$)

$$\begin{aligned}\mathcal{Z}(\lambda)/\mathcal{Z}(0) &= 1 - \frac{1}{4!}\lambda \sum_i \left\langle x_i^4 \right\rangle_0 + \frac{1}{2!(4!)^2}\lambda^2 \sum_i \sum_j \left\langle x_i^4 x_j^4 \right\rangle_0 + O(\lambda^3) \\ &= 1 - \tfrac{1}{8}\lambda \sum_i \Delta_{ii}^2 + \tfrac{1}{128}\lambda^2 \sum_i \Delta_{ii}^2 \sum_j \Delta_{jj}^2 \\ &\quad + \lambda^2 \sum_{i,j} \left(\tfrac{1}{16}\Delta_{ii}\Delta_{jj}\Delta_{ij}^2 + \tfrac{1}{48}\Delta_{ij}^4\right) + O(\lambda^3). \end{aligned} \tag{1.26}$$

A simple verification of the factors is obtained by specializing to the case of only one variable. Then,

$$\mathcal{Z}(\lambda)/\mathcal{Z}(0) = 1 - \tfrac{1}{8}\lambda + \tfrac{35}{384}\lambda^2 + O(\lambda^3).$$

Note that the first two terms of the expansion (1.26) exponentiate in such a way that $\ln \mathcal{Z}$ has only *connected* contributions, that is, contributions that cannot be factorized into a product of sums:

$$\ln \mathcal{Z}(\lambda) - \ln \mathcal{Z}(0) = -\tfrac{1}{8}\lambda \sum_i \Delta_{ii}^2 + \lambda^2 \sum_{i,j} \left(\tfrac{1}{16}\Delta_{ii}\Delta_{jj}\Delta_{ij}^2 + \tfrac{1}{48}\Delta_{ij}^4\right) + O(\lambda^3).$$

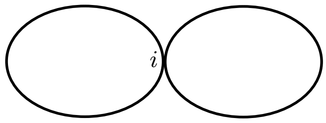

Fig. 1.1 Feynman diagram: the contribution $\left\langle x^4 \right\rangle_0$ at order λ in the example (1.25).

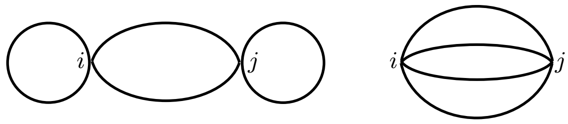

Fig. 1.2 Feynman diagrams: connected contributions from $\left\langle x_i^4 x_j^4 \right\rangle_0$ at order λ^2 in the example (1.25).

1.3.2 Feynman diagrams

To each different contribution generated by Wick's theorem can be associated a graph called a Feynman diagram. Each monomial contributing to $V(\mathbf{x})$ is represented by a point (a vertex) to which are attached a number of lines equal to the degree of the monomial. Each pairing is represented by a line joining the vertices to which the corresponding variables belong.

We have introduced the notion of connected contributions. To this notion corresponds a property of graphs. A connected contribution also corresponds to a connected diagram. Connected contributions to the normalization (1.22) in the example (1.25) are displayed up to order λ^2 in Figs 1.1 and 1.2, the indices i and j indicating the summations in (1.26).

1.3.3 Connected contributions

We now discuss, more generally, the notion of connected contribution that we have just introduced. We use, below, the subscript c to indicate the connected part of an expectation value. Then, for example,

$$\left\langle V(\mathbf{x}) \right\rangle = \left\langle V(\mathbf{x}) \right\rangle_c , \ \left\langle V^2(\mathbf{x}) \right\rangle = \left\langle V^2(\mathbf{x}) \right\rangle_c + \left\langle V(\mathbf{x}) \right\rangle_c^2$$
$$\left\langle V^3(\mathbf{x}) \right\rangle = \left\langle V^3(\mathbf{x}) \right\rangle_c + 3 \left\langle V^2(\mathbf{x}) \right\rangle_c \left\langle V(\mathbf{x}) \right\rangle_c + \left\langle V(\mathbf{x}) \right\rangle_c^3 , \cdots .$$

More generally, at order k, one finds

$$\tfrac{1}{k!} \left\langle V^k(\mathbf{x}) \right\rangle = \tfrac{1}{k!} \left\langle V^k(\mathbf{x}) \right\rangle_c + \text{ non-connected terms.}$$

A non-connected term is a product of the form

$$\left\langle V^{k_1}(\mathbf{x}) \right\rangle_c \left\langle V^{k_2}(\mathbf{x}) \right\rangle_c \cdots \left\langle V^{k_p}(\mathbf{x}) \right\rangle_c , \ k_1 + k_2 + \cdots + k_p = k ,$$

with a weight $1/k!$ coming from the expansion of the exponential function and multiplied by a combinatorial factor corresponding to all possible different ways to group k objects into subsets of $k_1 + k_2 + \cdots + k_p$ objects, if all k_i are distinct. One finds

$$\frac{1}{k!} \times \frac{k!}{k_1!k_2!\ldots k_p!} = \frac{1}{k_1!k_2!\ldots k_p!}\,.$$

If m values k_i are equal, it is necessary to divide by an additional combinatorial factor $1/m!$ because, otherwise, the same term is counted $m!$ times.

One then notices that the perturbative expansion can be written as

$$\mathcal{W}(\lambda) = \ln \mathcal{Z}(\lambda) = \ln \mathcal{Z}(0) + \sum_k \frac{(-\lambda)^k}{k!} \left\langle V^k(\mathbf{x}) \right\rangle_c . \qquad (1.27)$$

1.4 Expectation values. Generating function. Cumulants

We now calculate moments of the distribution $\mathrm{e}^{-A(\mathbf{x},\lambda)}/\mathcal{Z}(\lambda)$, where $A(\mathbf{x},\lambda)$ is a polynomial (1.21):

$$A(\mathbf{x},\lambda) = \sum_{i,j=1}^{n} \tfrac{1}{2} x_i A_{ij} x_j + \lambda V(\mathbf{x}).$$

Expectation values of the form $\langle x_{i_1} x_{i_2} \ldots x_{i_\ell} \rangle_\lambda$, which we will call *$\ell$-point functions* as it is customary in the context of path integrals, are given by the ratios

$$\langle x_{i_1} x_{i_2} \ldots x_{i_\ell} \rangle_\lambda = \mathcal{Z}^{-1}(\lambda) \mathcal{Z}_{i_1 i_2 \ldots i_\ell}(\lambda), \qquad (1.28a)$$

$$\mathcal{Z}_{i_1 i_2 \ldots i_\ell}(\lambda) = \int \mathrm{d}^n x\, x_{i_1} x_{i_2} \ldots x_{i_\ell} \exp\left[-A(\mathbf{x},\lambda)\right]. \qquad (1.28b)$$

1.4.1 The two-point function

As an illustration, we give a few elements of the calculation of the two-point function $\langle x_{i_1} x_{i_2} \rangle_\lambda$ up to order λ^2. One first expands the integral

$$\mathcal{Z}_{i_1 i_2}(\lambda) = \int \mathrm{d}^n x\, x_{i_1} x_{i_2} \exp\left[-A(\mathbf{x},\lambda)\right].$$

In the example (1.25), at order λ^2, one finds

$$\begin{aligned}
\mathcal{Z}_{i_1 i_2}(\lambda)/\mathcal{Z}(0) = {} & \Delta_{i_1 i_2} - \tfrac{1}{24}\lambda \Delta_{i_1 i_2} \sum_i \left\langle x_i^4 \right\rangle_0 - \tfrac{1}{2}\lambda \sum_i \Delta_{i i_1} \Delta_{ii} \Delta_{i i_2} \\
& + \frac{\lambda^2}{2!(4!)^2} \sum_{i,j} \Delta_{i_1 i_2} \left\langle x_i^4 x_j^4 \right\rangle_0 + \frac{\lambda^2}{2!4!} \sum_{i,j} \Delta_{i i_1} \Delta_{ii} \Delta_{i i_2} \left\langle x_j^4 \right\rangle_0 \\
& + \lambda^2 \sum_{i,j} \big(\tfrac{1}{4} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ij}^2 \Delta_{jj} + \tfrac{1}{6} \Delta_{i_1 i} \Delta_{j i_2} \Delta_{ij}^3 \\
& + \tfrac{1}{4} \Delta_{i_1 i} \Delta_{j i_2} \Delta_{ij} \Delta_{ii} \Delta_{jj} \big) + O(\lambda^3).
\end{aligned}$$

One then calculates the ratio

$$\langle x_{i_1} x_{i_2} \rangle_\lambda = \mathcal{Z}_{i_1 i_2}(\lambda)/\mathcal{Z}(\lambda).$$

In the ratio of the two series, the non-connected terms cancel and one left with

$$\begin{aligned}\langle x_{i_1} x_{i_2} \rangle_\lambda = \Delta_{i_1 i_2} - \tfrac{1}{2}\lambda \sum_i \Delta_{i i_1} \Delta_{ii} \Delta_{i i_2} + \lambda^2 \sum_{i,j} \bigl(\tfrac{1}{4} \Delta_{i_1 i} \Delta_{j i_2} \Delta_{ij} \Delta_{ii} \Delta_{jj} \\ + \tfrac{1}{4} \Delta_{i i_1} \Delta_{i i_2} \Delta_{ij}^2 \Delta_{jj} + \tfrac{1}{6} \Delta_{i_1 i} \Delta_{j i_2} \Delta_{ij}^3 \bigr) + O(\lambda^3). \end{aligned} \tag{1.29}$$

In terms of Feynman diagrams, the contributions of order 1, λ and λ^2 are displayed in Figs 1.3 and 1.4, respectively.

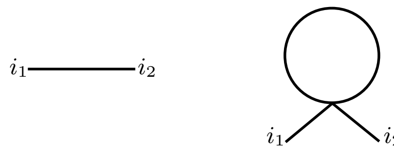

Fig. 1.3 The two-point function: contributions of order 1 and λ.

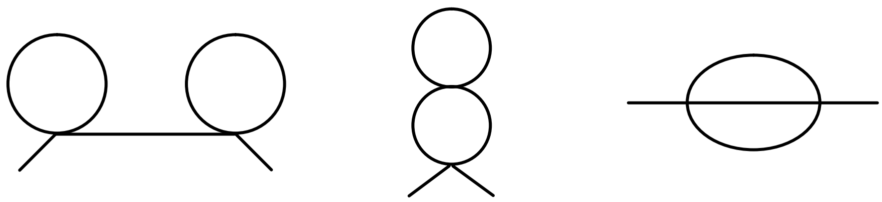

Fig. 1.4 The two-point function: contributions of order λ^2.

One could, similarly, calculate the four-point function, that is, the expectation value of the general monomial of degree 4. One would find many contributions. However, the results simplify if one directly calculates the cumulants of the distribution. For this purpose, it is convenient to first introduce a generating function of the expectation values $\langle x_{i_1} x_{i_2} \ldots x_{i_p} \rangle_\lambda$.

1.4.2 Generating functions. Cumulants

We now introduce the function

$$\mathcal{Z}(\mathbf{b}, \lambda) = \int \mathrm{d}^n x \, \exp\left[-A(\mathbf{x}, \lambda) + \mathbf{b} \cdot \mathbf{x}\right], \tag{1.30}$$

which generalizes the function (1.8) of the gaussian example. It is proportional to the generating function of the expectation values (1.28*a*) (see Section 1.1)

$$\left\langle \mathrm{e}^{\mathbf{b} \cdot \mathbf{x}} \right\rangle_\lambda = \mathcal{Z}(\mathbf{b}, \lambda)/\mathcal{Z}(\lambda)$$

that generalizes the function (1.14). Then, by differentiating one obtains

$$\langle x_{i_1} x_{i_2} \dots x_{i_\ell} \rangle_\lambda = \mathcal{Z}^{-1}(\lambda) \left[\frac{\partial}{\partial b_{i_1}} \frac{\partial}{\partial b_{i_2}} \cdots \frac{\partial}{\partial b_{i_\ell}} \mathcal{Z}(\mathbf{b}, \lambda) \right] \Big|_{\mathbf{b}=0} . \tag{1.31}$$

We now introduce the function

$$\mathcal{W}(\mathbf{b}, \lambda) = \ln \mathcal{Z}(\mathbf{b}, \lambda). \tag{1.32}$$

In a probabilistic interpretation, $\mathcal{W}(\mathbf{b}, \lambda)$ is a generating function of the cumulants of the distribution.

Note that, in the gaussian example, $\mathcal{W}(\mathbf{b})$ reduces to a quadratic form in $\mathbf{b}$. Moreover, it follows from equation (1.27) that the perturbative expansion of cumulants is much simpler because it contains only connected contributions. In particular, all diagrams corresponding to the normalization integral (1.26) can only appear in $\mathcal{W}(0, \lambda)$. Therefore, they cancel in the ratio $\mathcal{Z}(\mathbf{b}, \lambda)/\mathcal{Z}(\lambda)$, as we have already noticed in the calculation of the two-point function in Section 1.4.1.

Remark. In statistical physics, expectation values of products of the form $\langle x_{i_1} x_{i_2} \dots x_{i_\ell} \rangle$ are called ℓ-point correlation functions and the cumulants

$$W^{(\ell)}_{i_1 i_2 \dots i_\ell} = \left[\frac{\partial}{\partial b_{i_1}} \frac{\partial}{\partial b_{i_2}} \cdots \frac{\partial}{\partial b_{i_\ell}} \mathcal{W}(\mathbf{b}, \lambda) \right] \Big|_{\mathbf{b}=0} ,$$

are the *connected* correlation functions.

Examples. Expanding the relation (1.32) in powers of $\mathbf{b}$, one finds that the one-point functions are identical:

$$W^{(1)}_i = \langle x_i \rangle_\lambda \, .$$

For the two-point functions, one finds

$$W^{(2)}_{i_1 i_2} = \langle x_{i_1} x_{i_2} \rangle_\lambda - \langle x_{i_1} \rangle_\lambda \langle x_{i_2} \rangle_\lambda = \langle (x_{i_1} - \langle x_{i_1} \rangle_\lambda)(x_{i_2} - \langle x_{i_2} \rangle_\lambda) \rangle_\lambda \, .$$

Thus, the connected two-point function is the two-point function of the variables to which their expectation values have been subtracted.

In the case of an even perturbation $V(\mathbf{x}) = V(-\mathbf{x})$, as in the example (1.25),

$$\begin{aligned} W^{(2)}_{i_1 i_2} &= \langle x_{i_1} x_{i_2} \rangle_\lambda \, , \\ W^{(4)}_{i_1 i_2 i_3 i_4} &= \langle x_{i_1} x_{i_2} x_{i_3} x_{i_4} \rangle_\lambda - \langle x_{i_1} x_{i_2} \rangle_\lambda \langle x_{i_3} x_{i_4} \rangle_\lambda - \langle x_{i_1} x_{i_3} \rangle_\lambda \langle x_{i_2} x_{i_4} \rangle_\lambda \\ &\quad - \langle x_{i_1} x_{i_4} \rangle_\lambda \langle x_{i_3} x_{i_2} \rangle_\lambda \, . \end{aligned} \tag{1.33}$$

The connected four-point function, which vanishes exactly for a gaussian measure, gives a first evaluation of the deviation from a gaussian measure (Fig. 1.5).

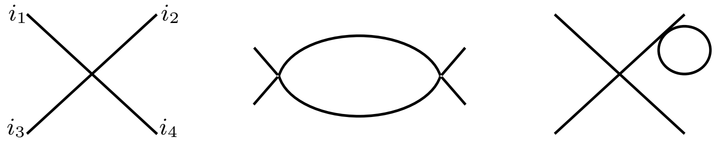

Fig. 1.5 Four-point function: connected contributions of order λ and λ^2 in the example (1.25).

In the example (1.25), at order λ^2, one then finds

$$\begin{aligned} W^{(4)}_{i_1 i_2 i_3 i_4} &= -\lambda \sum_i \Delta_{i_1 i}\Delta_{i_2 i}\Delta_{i_3 i}\Delta_{i_4 i} + \tfrac{1}{2}\lambda^2 \sum_{i,j} \Delta_{i_1 i}\Delta_{i_2 i}\Delta_{i_3 j}\Delta_{i_4 j}\Delta^2_{ij} \\ &\quad + \tfrac{1}{2}\lambda^2 \sum_{i,j} \Delta_{i_1 i}\Delta_{i_3 i}\Delta_{i_2 j}\Delta_{i_4 j}\Delta^2_{ij} + \tfrac{1}{2}\lambda^2 \sum_{i,j} \Delta_{i_1 i}\Delta_{i_4 i}\Delta_{i_3 j}\Delta_{i_2 j}\Delta^2_{ij} \\ &\quad + \tfrac{1}{2}\lambda^2 \sum_{i,j} \left(\Delta_{ii}\Delta_{ij}\Delta_{i_1 i}\Delta_{i_2 j}\Delta_{i_3 j}\Delta_{i_4 j} + 3 \text{ terms}\right) + O(\lambda^3) . \end{aligned} \tag{1.34}$$

1.5 Steepest descent method

Th steepest descent method is an approximation scheme to evaluate certain types of contour integrals in a complex domain. It involves approximating integrals by a sum of gaussian expectation values.

We first describe the method in the case of real integrals over one variable and then generalize to complex integrals. Finally, we generalize the method to an arbitrary number of variables.

1.5.1 Real integrals

We consider the integral

$$\mathcal{I}(\lambda) = \int_a^b \mathrm{d}x\, \mathrm{e}^{-A(x)/\lambda}, \tag{1.35}$$

where the function $A(x)$ is a real function, analytic in a neighbourhood of the real interval (a, b), and λ a positive parameter. We want to evaluate the integral in the limit $\lambda \to 0_+$. In this limit, the integral is dominated by the maxima of the integrand and thus the minima of $A(x)$. Two situations can occur:

(i) The minimum of $A(x)$ corresponds to an end-point of the interval. One then expands $A(x)$ near the end-point and integrates. This is not the case we are interested in here.

(ii) The function $A(x)$ has one, or several, minima in the interval (a, b). The minima correspond to points x^c that are solutions of

$$A'(x^c) = 0\,,$$

where, generically, $A''(x^c) > 0$ (the case $A''(x^c) = 0$ requires a separate analysis). For reasons that will become clearer later, such points are called saddle points (see example (ii)). When several saddle points are found, the largest contribution comes from the absolute minimum of $A(x)$.

Moreover, if corrections of order $\exp[-\text{const.}/\lambda]$ are neglected, the integration domain can be restricted to a neighbourhood $(x^c - \varepsilon, x^c + \varepsilon)$ of x^c, where ε is finite but otherwise arbitrarily small. Indeed, contributions coming from the exterior of the interval are bounded by

$$(b - a)\, \mathrm{e}^{-A''(x^c)\varepsilon^2/2\lambda},$$

where the property $\varepsilon \ll 1$ has been used, in such a way that

$$A(x) - A(x^c) \sim \tfrac{1}{2} A''(x^c)\,(x - x^c)^2 \ .$$

More precisely, the region that contributes is of order $\sqrt{\lambda}$. Thus, it is convenient to change variables, $x \mapsto y$:

$$y = (x - x^c)/\sqrt{\lambda}.$$

The expansion of the function A then reads

$$A/\lambda = A(x^c)/\lambda + \tfrac{1}{2} y^2 A''(x^c) + \tfrac{1}{6}\sqrt{\lambda} A'''(x^c) y^3 + \tfrac{1}{24}\lambda A^{(4)}(x^c) y^4 + O(\lambda^{3/2}).$$

One sees that, at leading order, it is sufficient to keep the quadratic term. This reduces the calculation to the restricted gaussian integral

$$\mathcal{I}(\lambda) \sim \sqrt{\lambda}\, \mathrm{e}^{-A(x^c)/\lambda} \int_{-\varepsilon/\sqrt{\lambda}}^{\varepsilon/\sqrt{\lambda}} \mathrm{d}y \ \mathrm{e}^{-A''(x^c) y^2/2} .$$

The integration range can be extended to the whole real axis $[-\infty, +\infty]$ because, for similar reasons, contributions from outside the integration domain are exponentially small in $1/\lambda$. The leading contribution is thus given by the gaussian integral, which yields

$$\mathcal{I}(\lambda) \sim \sqrt{2\pi\lambda/A''(x^c)}\, \mathrm{e}^{-A(x_c)/\lambda} \ . \tag{1.36}$$

To calculate higher order corrections, one expands the exponential in powers of λ and integrates term by term. Setting

$$\mathcal{I}(\lambda) = \sqrt{2\pi\lambda/A''(x^c)}\, \mathrm{e}^{-A(x_c)/\lambda}\, \mathcal{J}(\lambda),$$

one finds, for example, at next order,

$$\begin{aligned}\mathcal{J}(\lambda) &= 1 - \frac{\lambda}{24} A^{(4)} \left\langle y^4 \right\rangle + \frac{\lambda}{2 \times 6^2} A'''^2 \left\langle y^6 \right\rangle + O(\lambda^2) \\ &= 1 + \frac{\lambda}{24}\left(5 \frac{A'''^2}{A''^3} - 3 \frac{A^{(4)}}{A''^2}\right) + O(\lambda^2),\end{aligned}$$

where $\langle \bullet \rangle$ means gaussian expectation value.

Remarks.

(i) The steepest descent method generates a formal expansion in powers of λ:

$$\mathcal{J}(\lambda) = 1 + \sum_{k=1}^{\infty} J_k \lambda^k,$$

which, in general, diverges for all values of the expansion parameter. The divergence can easily be understood: if one changes the sign of λ in the integral, the maximum of the integrand becomes a minimum, and the saddle point no longer gives the leading contribution to the integral.

Nevertheless, the series is useful because, for small enough λ, partial sums satisfy

$$\exists\,\lambda_0 > 0\,, \{M_K\}: \quad \forall\,K \text{ and } 0_+ \le \lambda \le \lambda_0 \quad \left|\mathcal{J}(\lambda) - \sum_{k=0}^{K} J_k \lambda^k\right| \le M_K \lambda^{K+1},$$

where the coefficients M_k generically grow like $k!$. Such a series is called an asymptotic series. At fixed λ, if the index K is chosen such that the bound is minimum, the function is determined up to corrections of order $\exp[-\text{const.}/\lambda]$. Note that such a bound can be extended to a sector in the λ complex plane, $|\operatorname{Arg}\lambda| < \theta$.

(ii) Often integrals have the more general form

$$\mathcal{I}(\lambda) = \int \mathrm{d}x\,\rho(x)\,\mathrm{e}^{-A(x)/\lambda}\,.$$

Then, provided $\ln\rho(x)$ is analytic at the saddle point, it is not necessary to take into account the factor $\rho(x)$ in the saddle point equation. Indeed, this would induce a shift $x - x_c$ of the position of the saddle point, solution of

$$A''(x_c)(x - x_c) \sim \lambda\rho'(x_c)/\rho(x_c),$$

and, thus, of order λ while the contribution to the integral comes from a much larger region of order $\sqrt{\lambda} \gg \lambda$.

One can, thus, still expand all expressions around the solution of $A'(x) = 0$. At leading order, one then finds

$$\mathcal{I}(\lambda) \sim \sqrt{2\pi\lambda/A''(x^c)}\,\rho(x_c)\,\mathrm{e}^{-A(x_c)/\lambda}\,.$$

Let us now apply the method to two classical examples, the Γ function that generalizes $n!$ to complex arguments, and the modified Bessel function.

Examples.

(i) A classical example is the asymptotic evaluation of the function

$$\Gamma(s) = \int_0^\infty \mathrm{d}x\,x^{s-1}\,\mathrm{e}^{-x},$$

for $s \to +\infty$. The integral does not immediately have the canonical form (1.35), but it takes it after a linear change of variables: $x = (s-1)x'$. One identifies $s - 1 = 1/\lambda$. Then,

$$\Gamma(s) = (s-1)^{s-1} \int_0^\infty \mathrm{d}x\ \mathrm{e}^{-(x-\ln x)/\lambda}$$

and, thus, $A(x) = x - \ln x$. The position of the saddle point is given by

$$A'(x) = 1 - 1/x = 0 \ \Rightarrow\ x^c = 1\,.$$

The second derivative at the saddle point is $A''(x^c) = 1$. The result, at leading order, is

$$\Gamma(s) \underset{s\to\infty}{\sim} \sqrt{2\pi}(s-1)^{s-1/2}\,\mathrm{e}^{1-s} \sim \sqrt{2\pi}s^{s-1/2}\,\mathrm{e}^{-s}, \qquad (1.37)$$

an expression also called Stirling's formula. Note that with the help of the complex generalization of the method, which we explain later, the result can be extended to all complex values of s such that $|\arg s| < \pi$.

(ii) We now evaluate the modified Bessel function

$$I_0(x) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \mathrm{d}\theta\,\mathrm{e}^{x\cos\theta},$$

($= J_0(ix)$) for $x \to +\infty$ (the function is even).

This integral has the canonical form for the application of the steepest descent method ($x = 1/\lambda$), and the integrand is an entire function.

The saddle points are given by

$$\sin\theta = 0 \Rightarrow \theta = 0 \quad (\mathrm{mod}\ \pi)\,.$$

For $x \to +\infty$, the leading saddle point is $\theta = 0$. One expands at the saddle point

$$x\cos\theta = x - \tfrac{1}{2}x\theta^2 + \tfrac{1}{24}x\theta^4 + O(\theta^6).$$

The region that contributes to the integral is of order $\theta = O(1/\sqrt{x})$. Thus,

$$\begin{aligned} I_0(x) &= \frac{1}{2\pi}\,\mathrm{e}^x \int_{-\infty}^{\infty} \mathrm{d}\theta\,\mathrm{e}^{-x\theta^2/2}\left(1 + \tfrac{1}{24}x\theta^4\right) + O(\mathrm{e}^x/x^2) \\ &= \frac{1}{\sqrt{2\pi x}}\,\mathrm{e}^x \left(1 + \frac{1}{8x} + O\left(\frac{1}{x^2}\right)\right). \end{aligned}$$

Let us use this example to justify the denomination *saddle point*. For this purpose, it is necessary to examine the function $\cos\theta$, which appears in the integrand, in the complex plane in the vicinity of the saddle point $\theta = 0$. The curves of constant modulus of the integrand are the curves $\mathrm{Re}\cos\theta$ constant:

$$\mathrm{Re}\cos\theta - 1 \sim -\tfrac{1}{2}\left[(\mathrm{Re}\,\theta)^2 - (\mathrm{Im}\,\theta)^2\right] = \text{const.}$$

Locally, these curves are hyperbolae. They cross only at the saddle point. The hyperbola corresponding to a vanishing constant degenerates into two straight lines (see Fig. 1.6). The modulus of the integrand thus has a saddle point structure.

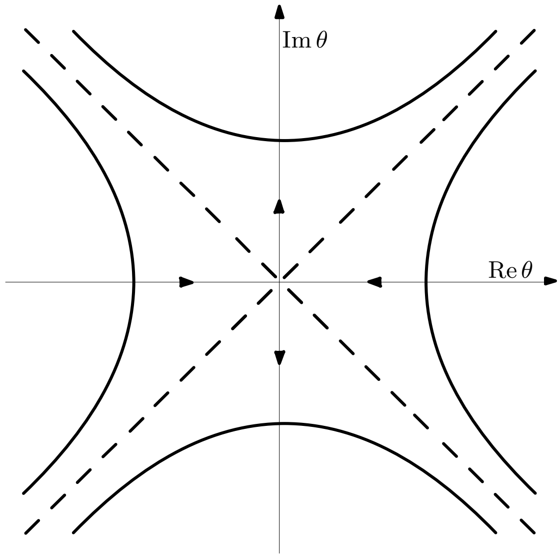

Fig. 1.6 Function I_0: constant modulus curves of the integrand near the saddle point $\theta = 0$.

1.5.2 Complex integrals

One wants now to evaluate the integral

$$\mathcal{I}(\lambda) = \oint_C \mathrm{d}x \; \mathrm{e}^{-A(x)/\lambda}, \tag{1.38}$$

where $A(x)$ is an analytic function of the complex variable x and λ a real positive parameter, for $\lambda \to 0_+$. The contour C goes from a point a to a point b in the complex plane, and is contained within the domain of analyticity of A. As a limiting case, one can consider the situation where the points a and b go to infinity in the complex plane.

At first sight, one could think that the integral is again dominated by the points where the modulus of the integrand is maximum and, thus, the real part of $A(x)$ is minimum. However, the contribution of the neighbourhood of such points in general cancels because the phase also varies rapidly (an argument that leads to the stationary phase method).

The steepest descent method then involves deforming the contour C, within the domain of analyticity of A (and without crossing a singularity), to minimize the maximum modulus of the integrand on the contour, that is, to maximize the minimum of $\operatorname{Re} A(x)$.

If it is possible to deform the contour C into an equivalent contour along which $\operatorname{Re} A(x)$ is monotonous, then the integral is dominated by an end-point. Otherwise, the real part of A has a minimum. On an optimal contour, the minimum corresponds either to a singularity of the function or to a regular point where the derivative of

A vanishes:

$$A'(x) = 0 \,.$$

This is the case we want to study. A point x^c where $A'(x) = 0$ is again, generically, a *saddle point* with respect to the curves of constant $\operatorname{Re} A(x)$ (Fig. 1.6). Such a structure of the integrand can be better understood if one remembers that the curves with $\operatorname{Re} A$ and $\operatorname{Im} A$ constant form two sets of bi-orthogonal curves. The only possible double points of these curves are singularities and saddle points. Indeed, let us expand the function at x^c:

$$A(x) - A(x^c) = \tfrac{1}{2} A''(x^c)\,(x - x^c)^2 + O\left((x - x^c)^3\right).$$

Then, in terms of the real coordinates u, v defined by

$$u + iv = (x - x^c)\, \mathrm{e}^{i \operatorname{Arg} A''(x^c)/2},$$

one finds

$$\operatorname{Re}[A(x) - A(x^c)] \sim \tfrac{1}{2} |A''(x^c)| (u^2 - v^2).$$

Near the saddle point, one can choose a contour that follows a curve with $\operatorname{Im} A$ constant and, thus, along which the phase of the integrand remains constant: no cancellation can occur. The leading contribution to the integral comes from the neighbourhood of the saddle point. Neglected contributions decay faster than any power of λ. The remaining part of the argument and the calculation are the same as in the real case.

Example. Consider the integral representation of the usual Bessel function

$$J_0(x) = \frac{1}{2i\pi} \oint_C \frac{\mathrm{d}z}{z}\, \mathrm{e}^{x(z - 1/z)/2},$$

where C is a simple closed contour, which encloses the origin. We evaluate the integral for x real, $x \to +\infty$, by the steepest descent method.

We set

$$A(z) = (1/z - z)/2 \,.$$

Saddle points are solutions of

$$2A'(z) = -\frac{1}{z^2} - 1 = 0 \;\Rightarrow\; z = \pm i \,.$$

The two saddle points are relevant. For the saddle point $z = i$, we set $z = i + \mathrm{e}^{3i\pi/4}\, s$. Then,

$$A(z) = -i + s^2/2 + O(s^3).$$

The contribution of the saddle point is

$$\frac{1}{2\pi}\, \mathrm{e}^{ix - i\pi/4} \int_{-\infty}^{+\infty} \mathrm{d}s\; \mathrm{e}^{-xs^2/2} = \frac{1}{\sqrt{2\pi x}}\, \mathrm{e}^{ix - i\pi/4}$$

The second saddle point gives the complex conjugate contribution. One thus finds

$$J_0(x) \underset{x \to +\infty}{\sim} \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4).$$

1.6 Steepest descent method: Several variables, generating functions

We now consider the general n-dimensional integral

$$\mathcal{I}(\lambda) = \int \mathrm{d}^n x \, \exp\left[-\frac{1}{\lambda} A(x_1, \ldots, x_n)\right], \tag{1.39}$$

where, for simplicity, we assume that A is an entire function and the integration domain is $\mathbb{R}^n$.

In the limit $\lambda \to 0_+$, the integral is dominated by saddle points, solutions of

$$\frac{\partial}{\partial x_i} A(x_1, x_2, \ldots, x_n) = 0\,, \ \forall i\,. \tag{1.40}$$

When several saddle points are found, one orders them according to the values of $\operatorname{Re} A$. Often the relevant saddle point corresponds to the minimum value of $\operatorname{Re} A$, but this is not necessarily the case since the saddle point may not reachable by a deformation of the initial domain of integration. Unfortunately, in the case of several complex variables, deforming contours is, generally, not a simple exercise.

To evaluate the leading contribution of a saddle point $\mathbf{x}^c$, it is convenient to change variables, setting

$$\mathbf{x} = \mathbf{x}^c + \mathbf{y}\sqrt{\lambda}\,.$$

One then expands $A(\mathbf{x})$ in powers of λ (and thus y):

$$\frac{1}{\lambda} A(x_1, \ldots, x_n) = \frac{1}{\lambda} A(\mathbf{x}^c) + \frac{1}{2!} \sum_{i,j} \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i x_j} y_i y_j + R(\mathbf{y}) \tag{1.41}$$

with

$$R(\mathbf{y}) = \sum_{k=3}^{\infty} \frac{\lambda^{k/2-1}}{k!} \sum_{i_1, i_2, \ldots, i_k} \frac{\partial^k A(\mathbf{x}^c)}{\partial x_{i_1} \ldots \partial x_{i_k}} y_{i_1} \ldots y_{i_k}\,. \tag{1.42}$$

After the change of variables, the term quadratic in $\mathbf{y}$ becomes independent of λ. The integral then reads

$$\mathcal{I}(\lambda) = \lambda^{n/2} \, \mathrm{e}^{-A(\mathbf{x}^c)/\lambda} \int \mathrm{d}^n y \, \exp\left[-\frac{1}{2!} \sum_{i,j} \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j} y_i y_j - R(\mathbf{y})\right]. \tag{1.43}$$

Once the integrand is expanded in powers of $\sqrt{\lambda}$, the calculation of each term is reduced to the gaussian expectation value of a polynomial. At leading order, one finds

$$\mathcal{I}(\lambda) \underset{\lambda \to 0}{\sim} (2\pi\lambda)^{n/2} \left[\det \mathbf{A}^{(2)}\right]^{-1/2} \mathrm{e}^{-A(\mathbf{x}^c)/\lambda}\,, \tag{1.44}$$

where $\mathbf{A}^{(2)}$ is the matrix of second partial derivatives:

$$[\mathbf{A}^{(2)}]_{ij} \equiv \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j}\,.$$

1.6.1 Generating function and steepest descent method

We now introduce the function

$$\mathcal{Z}(\mathbf{b}, \lambda) = \int \mathrm{d}^n x \, \exp\left[-\frac{1}{\lambda}\left(A(\mathbf{x}) - \mathbf{b}\cdot\mathbf{x}\right)\right], \tag{1.45}$$

where $A(\mathbf{x})$ is now a regular function of the x_i. We also define

$$\mathcal{N} = 1/\mathcal{Z}(0, \lambda).$$

The function $\mathcal{Z}(\mathbf{b}, \lambda)$ has the general form (1.30), and is proportional to the generating function of the moments of a distribution $\mathcal{N}\,\mathrm{e}^{-A(\mathbf{x})/\lambda}$.

Expectation values of polynomials with the weight $\mathcal{N}\,\mathrm{e}^{-A(\mathbf{x})/\lambda}$,

$$\langle x_{k_1} x_{k_2} \ldots x_{k_\ell} \rangle \equiv \mathcal{N} \int \mathrm{d}^n x \, x_{k_1} x_{k_2} \ldots x_{k_\ell} \, \mathrm{e}^{-A(\mathbf{x})/\lambda}, \tag{1.46}$$

are obtained by differentiating $\mathcal{Z}(\mathbf{b}, \lambda)$ (see equation (1.31)):

$$\langle x_{k_1} x_{k_2} \ldots x_{k_\ell} \rangle = \lambda^\ell \mathcal{N} \left[\frac{\partial}{\partial b_{k_1}} \frac{\partial}{\partial b_{k_2}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathcal{Z}(\mathbf{b}, \lambda)\right]\bigg|_{\mathbf{b}=0}.$$

Steepest descent method. We now apply the steepest descent method to the integral (1.45). The saddle point equation is

$$b_i = \frac{\partial A}{\partial x_i}, \ \forall\, i\,. \tag{1.47}$$

We expand $A(\mathbf{x})$ at a saddle point $\mathbf{x}^c$, as explained in Section 1.5, and use the leading order result (1.44):

$$\mathcal{Z}(\mathbf{b}, \lambda) \underset{\lambda \to 0}{\sim} (2\pi\lambda)^{n/2} \left[\det \mathbf{A}^{(2)}\right]^{-1/2} \exp\left[-\frac{1}{\lambda}\left(A(\mathbf{x}^c) - \mathbf{b}\cdot\mathbf{x}^c\right)\right]$$

with

$$[\mathbf{A}^{(2)}]_{ij} \equiv \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j}\,.$$

We now introduce $\mathcal{W}(\mathbf{b}, \lambda)$, the generating function of the cumulants of the distribution, which are also the connected correlation functions (equation (1.32) but with a different normalization)

$$\mathcal{W}(\mathbf{b}, \lambda) = \lambda \ln \mathcal{Z}(\mathbf{b}, \lambda).$$

Using identity (3.51): $\ln\det\mathbf{M} = \operatorname{tr}\ln\mathbf{M}$, valid for any matrix $\mathbf{M}$, we can write the first order terms of the expansion of $\mathcal{W}$ as

$$\mathcal{W}(\mathbf{b}, \lambda) = -A(\mathbf{x}_c) + \mathbf{b}\cdot\mathbf{x}^c + \tfrac{1}{2} n \ln(2\pi\lambda) - \tfrac{1}{2}\lambda \operatorname{tr}\ln \frac{\partial^2 A(\mathbf{x}^c)}{\partial x_i \partial x_j} + O(\lambda^2). \tag{1.48}$$

Since

$$\langle x_{k_1} x_{k_2} \ldots x_{k_\ell} \rangle_c = \lambda^{\ell-1} \left[\frac{\partial}{\partial b_{k_1}} \frac{\partial}{\partial b_{k_2}} \cdots \frac{\partial}{\partial b_{k_\ell}} \mathcal{W}(\mathbf{b}, \lambda)\right]\bigg|_{\mathbf{b}=0},$$

successive derivatives of the expansion (1.48) with respect to $\mathbf{b}$ (taking into account that $\mathbf{x}_c$ is a function of $\mathbf{b}$ through equation (1.47)), calculated at $\mathbf{b} = 0$, yield the corresponding expansions of the cumulants of the distribution.

1.7 Gaussian integrals: Complex matrices

In Section 1.2, we have proved the result (1.5) only for real matrices. Here, we extend the proof to complex matrices.

The proof based on diagonalization, used for real matrices, has a complex generalization. Indeed, any complex symmetric matrix $\mathbf{A}$ has a decomposition of the form

$$\mathbf{A} = \mathbf{U}\mathbf{D}\mathbf{U}^{\mathrm{T}}\,, \tag{1.49}$$

where $\mathbf{U}$ is a unitary matrix and $\mathbf{D}$ a diagonal positive matrix. In the integral (1.4), one then changes variables $\mathbf{x} \mapsto \mathbf{y}$:

$$x_i = \sum_{i=j}^{n} U_{ij} y_j \,.$$

This change of variables is a direct complex generalization of the orthogonal transformation (1.7). The integral (1.4) then factorizes and the result is the product of the integral and the (here non-trivial) jacobian of the change of variables. Thus,

$$\mathcal{Z}(\mathbf{A}) = (2\pi)^{n/2} (\det \mathbf{D})^{-1/2} / \det \mathbf{U}\,.$$

Since

$$\det \mathbf{A} = \det \mathbf{D} (\det \mathbf{U})^2,$$

one recovers the result (1.5).

Complex matrix representations. Since the existence of the representation (1.49) may not be universally known, as an exercise, we give here a general proof.

(i) *Polar decomposition.* A complex matrix $\mathbf{M}$ has a 'polar' decomposition:

$$\mathbf{M} = \mathbf{U}\mathbf{H} \text{ with } \mathbf{U}^\dagger \mathbf{U} = \mathbf{1}\,,\ \mathbf{H} = \mathbf{H}^\dagger. \tag{1.50}$$

If a matrix has no vanishing eigenvalue the proof is simple. The representation (1.50) implies a relation between hermitian positive matrices:

$$\mathbf{M}^\dagger \mathbf{M} = \mathbf{H}^2.$$

One chooses for $\mathbf{H}$ the matrix $(\mathbf{M}^\dagger\mathbf{M})^{1/2}$ with positive eigenvalues. One then verifies immediately that the matrix $\mathbf{U} = \mathbf{M}\mathbf{H}^{-1}$ is unitary:

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{1}\,.$$

This decomposition implies another, equivalent one. A hermitian matrix can be diagonalized by a unitary transformation and thus can be written as

$$\mathbf{H} = \mathbf{V}^\dagger \mathbf{D} \mathbf{V}\,,\ \mathbf{V}^\dagger \mathbf{V} = \mathbf{1}$$

with $\mathbf{D}$ diagonal: $D_{ij} = h_i \delta_{ij}$, $h_i > 0$. It follows that

$$\mathbf{M} = \mathbf{U}\mathbf{V}^\dagger \mathbf{D}\mathbf{V}\,,$$

or, in a simpler notation,

$$\mathbf{M} = \mathbf{U}_2^\dagger \mathbf{D}\mathbf{U}_1\,, \tag{1.51}$$

where $\mathbf{U}_1$ and $\mathbf{U}_2$ are two unitary matrices.

(ii) *Symmetric unitary matrices.* We now prove a decomposition of symmetric unitary matrices. We thus consider a matrix $\mathbf{U}$ satisfying

$$\mathbf{U}^\dagger \mathbf{U} = \mathbf{1}\,, \quad \mathbf{U} = \mathbf{U}^{\mathrm{T}}.$$

We decompose $\mathbf{U}$ into real and imaginary parts:

$$\mathbf{U} = \mathbf{X} + i\mathbf{Y}\,.$$

The two matrices $\mathbf{X}$ and $\mathbf{Y}$ are real and symmetric and satisfy, as a consequence of the unitarity relation,

$$\mathbf{X}^2 + \mathbf{Y}^2 = \mathbf{1}\,, \quad \mathbf{X}\mathbf{Y} - \mathbf{Y}\mathbf{X} = 0\,.$$

Two commuting real symmetric matrices can be diagonalized simultaneously. The corresponding eigenvalues $\{x_i, y_i\}$ (which are real) satisfy

$$x_i^2 + y_i^2 = 1\,,$$

which we parametrize as

$$x_i = r_i \cos\theta_i\,, \quad y_i = r_i \sin\theta_i\,, \quad r_i > 0\,.$$

If $\mathbf{O}$ is the common orthogonal matrix that diagonalizes $\mathbf{X}$ and $\mathbf{Y}$ and thus $\mathbf{U}$, the matrix $\mathbf{U}$ can be written as

$$\mathbf{U} = \mathbf{O}^{\mathrm{T}} \mathbf{R}\mathbf{W}\mathbf{O}$$

with

$$W_{ij} = \mathrm{e}^{i\theta_i}\,\delta_{ij}\,, \quad R_{ij} = r_i \delta_{ij}\,.$$

One then sets

$$V_{ij} = \mathrm{e}^{i\theta_i/2}\,O_{ij} \;\Leftrightarrow \mathbf{V} = \mathbf{W}^{1/2}\mathbf{O}\,,$$

and obtains the expected representation (1.49) for unitary matrices:

$$\mathbf{U} = \mathbf{V}^{\mathrm{T}}\mathbf{R}\mathbf{V}\,, \quad \mathbf{R} \text{ diagonal } > 0\,, \quad \mathbf{V}^{\mathrm{T}}\mathbf{V} = \mathbf{1}\,. \tag{1.52}$$

(iii) *Complex symmetric matrices.* We now prove the representation for general complex symmetric matrices

$$\mathbf{M} = \mathbf{M}^{\mathrm{T}}.$$

Representation (1.51) must then satisfy

$$\mathbf{U}_2^\dagger \mathbf{D} \mathbf{U}_1 = \mathbf{U}_1^{\mathrm{T}} \mathbf{D} \mathbf{U}_2^*,$$

where * denotes complex conjugation. Introducing the unitary matrix

$$\mathbf{W} = \mathbf{U}_2 \mathbf{U}_1^{\mathrm{T}},$$

one obtains the constraint

$$\mathbf{D} = \mathbf{W}\mathbf{D}\mathbf{W}^* \ \Leftrightarrow \ \mathbf{D} = \mathbf{W}^{\mathrm{T}}\mathbf{D}\mathbf{W}^\dagger.$$

Multiplying the r.h.s. of the second equation by the r.h.s. of the first equation (in this order), one finds

$$\mathbf{D}^2 = \mathbf{W}^{\mathrm{T}}\mathbf{D}^2\mathbf{W}^* \ \Leftrightarrow \ \mathbf{D}^2 = \mathbf{W}^\dagger\mathbf{D}^2\mathbf{W} \ \Leftrightarrow \ [\mathbf{W}, \mathbf{D}^2] = 0\,.$$

The latter equation, in component form reads

$$\left(h_i^2 - h_j^2\right) W_{ij} = 0\,,$$

and thus

$$W_{ij} = 0\,, \text{ for } h_i \neq h_j\,.$$

If all eigenvalues of $\mathbf{D}$ are simple, then $\mathbf{W}$ is a diagonal unitary matrix:

$$W_{ij} = \mathrm{e}^{i\theta_i}\,\delta_{ij}.$$

Introducing this result into the representation (1.51), eliminating $\mathbf{U_2}$ in terms of $\mathbf{W}$, one finds

$$\mathbf{M} = \mathbf{U}_1^{\mathrm{T}}\mathbf{W}^*\mathbf{D}\mathbf{U}_1\,.$$

Finally, setting

$$\mathbf{U}_0 = [\mathbf{W}^{1/2}]^*\mathbf{U}_1\,,$$

one obtains the representation of a complex symmetric matrix in terms of a positive diagonal matrix $\mathbf{D}$ and a unitary matrix $\mathbf{U}_0$:

$$\mathbf{M} = \mathbf{U}_0^{\mathrm{T}}\mathbf{D}\mathbf{U}_0\,. \tag{1.53}$$

If $\mathbf{D}$ has degenerate eigenvalues, from (1.51) in the corresponding subspace the matrix $\mathbf{M}$ is proportional to a symmetric unitary matrix. One then uses the result (1.52) and this shows that the decomposition (1.53) still holds.

Exercises

Exercise 1.1

One considers two stochastic correlated variables x, y with gaussian probability distribution. One finds the five expectation values

$$\langle x \rangle = \langle y \rangle = 0\,, \ \left\langle x^2 \right\rangle = 5\,, \ \langle xy \rangle = 3\,, \ \left\langle y^2 \right\rangle = 2\,.$$

Calculate the expectation values $\left\langle x^4 \right\rangle$, $\left\langle x^3 y \right\rangle$, $\left\langle x^2 y^2 \right\rangle$, $\left\langle x y^5 \right\rangle$, $\left\langle y^6 \right\rangle$, $\left\langle x^3 y^3 \right\rangle$, using Wick's theorem.

Determine the corresponding gaussian distribution.

Solution.

$$75, 45, 28, 180, 120, 432\,.$$

The gaussian distribution is proportional to

$$\mathrm{e}^{-(2x^2 - 6xy + 5y^2)/2}\,.$$

Exercise 1.2

One considers three stochastic correlated variables x, y, z with gaussian probability distribution. One finds the nine expectation values

$$\langle x \rangle = \langle y \rangle = \langle z \rangle = 0\,, \ \left\langle x^2 \right\rangle = \left\langle y^2 \right\rangle = \left\langle z^2 \right\rangle = a\,, \ \langle xy \rangle = b\,, \ \langle xz \rangle = \langle zy \rangle = c\,.$$

Calculate the expectation values $\left\langle x^4 \right\rangle$, $\left\langle x^6 \right\rangle$, $\left\langle x^3 y \right\rangle$, $\left\langle x^2 y^2 \right\rangle$, $\left\langle x^2 yz \right\rangle$ as functions of a, b, c.

Determine for $a = 2, b = 1, c = 0$ the corresponding gaussian distribution.

Solution.

$$\left\langle x^4 \right\rangle = 3a^2, \ \left\langle x^6 \right\rangle = 15a^3, \ \left\langle x^3 y \right\rangle = 3ab\,, \ \left\langle x^2 y^2 \right\rangle = a^2 + 2b^2, \ \left\langle x^2 yz \right\rangle = ac + 2bc\,.$$

For $a = 2, b = 1, c = 0$ the gaussian distribution is proportional to

$$\exp\left[-\tfrac{1}{12}\left(4x^2 + 4y^2 + 3z^2 - 4xy\right)\right].$$

Exercise 1.3

*Inductive algebraic proof of result (*1.5*).* The determinant of a general matrix $\mathbf{A}^{(n)}$ $n \times n$, of elements $A_{ij}^{(n)}$ can be calculated inductively by subtracting from all rows a multiple of the last row in order to cancel the last column (assuming that $A_{nn}^{(n)} \neq 0$, otherwise one interchanges rows or columns). This method leads to a relation between determinants:

$$\det \mathbf{A}^{(n)} = A_{nn}^{(n)} \det \mathbf{A}^{(n-1)},$$

where $\mathbf{A}^{(n-1)}$ is a matrix $(n-1)\times(n-1)$ with elements

$$A_{ij}^{(n-1)} = A_{ij}^{(n)} - A_{in}^{(n)} A_{nj}^{(n)} / A_{nn}^{(n)} , \quad i,j=1,\ldots,n-1 . \tag{1.54}$$

Show that the result (1.6), combined with this identity, leads to the general result (1.5).

Solution. One considers the integral (1.4) and integrates over one variable, which one can call x_n (assuming that $\operatorname{Re} A_{nn} > 0$), using the result (1.6):

$$\int \mathrm{d}x_n \exp\left(-\tfrac{1}{2}A_{nn}x_n^2 - x_n \sum_{i=1}^{n-1} A_{ni}x_i\right) = \sqrt{\frac{2\pi}{A_{nn}}} \exp\left(\tfrac{1}{2}\sum_{i,\,j=1}^{n-1} \frac{A_{in}A_{nj}}{A_{nn}} x_i x_j\right).$$

The remaining integral is a gaussian integral over $n-1$ variables:

$$\mathcal{Z}(\mathbf{A}) = \sqrt{\frac{2\pi}{A_{nn}}} \int \left(\prod_{i=1}^{n-1} \mathrm{d}x_i\right) \exp\left(-\sum_{i,\,j=1}^{n-1} \tfrac{1}{2} x_i \left(A_{ij} - A_{in}A_{nn}^{-1}A_{nj}\right) x_j\right).$$

One then notes that, by iterating this partial integration, one obtains a form of $1/\sqrt{\det \mathbf{A}}$ as generated by identity (1.54). One concludes

$$\mathcal{Z}(\mathbf{A}) = (2\pi)^{n/2} (\det \mathbf{A})^{-1/2}. \tag{1.55}$$

Exercise 1.4

Use the steepest descent method to evaluate the integral

$$I_n(\alpha) = \int_0^1 \mathrm{d}x\, x^{\alpha n}(1-x)^{\beta n},$$

with $\beta = 1-\alpha$, $\alpha > 0$, $\beta > 0$, in the limit $n \to \infty$.

Solution. The saddle point is $x_c = \alpha$ and thus

$$I_n(\alpha) \sim \sqrt{2\pi\alpha(1-\alpha)/n}\,\alpha^{n\alpha}(1-\alpha)^{n(1-\alpha)}.$$

Exercise 1.5

One considers the integral

$$Z(g) = \int \mathrm{d}^3 q \, \exp\left[\frac{1}{g}\left(\frac{\mathbf{q}^2}{2} - \frac{(\mathbf{q}^2)^2}{4}\right)\right],$$

where $\mathbf{q}$ is a two-component vector $\mathbf{q} = (q_1, q_2)$. Evaluate the integral for $g \to 0_+$ by the steepest descent method (this exercise involves a subtle point).

Solution. For some indications, see Section 8.3.1:

$$Z(g) \sim 4\pi^{3/2} g^{1/2}\, \mathrm{e}^{1/4g} .$$

Exercise 1.6

Hermite polynomials $\mathcal{H}_n$ appear in the eigenfunctions of the quantum harmonic oscillator. One integral representation is

$$\mathcal{H}_n(z) = \sqrt{\frac{n}{2\pi}} \int_{-\infty}^{+\infty} \mathrm{d}s \; \mathrm{e}^{-ns^2/2} (z - is)^n .$$

Evaluate the polynomials for $n \to \infty$ and z real by the steepest descent method.

Solution. The polynomials $\mathcal{H}_n(z)$ are alternatively even or odd:

$$\mathcal{H}_n(-z) = (-1)^n \mathcal{H}_n(z).$$

Thus, one can restrict the problem to the values $z \geq 0$.

One sets

$$A(s) = \tfrac{1}{2}s^2 - \ln(z - is).$$

The saddle points are given by

$$A'(s) = s - \frac{1}{s + iz} = 0 \;\Rightarrow\; s_\pm = -\tfrac{1}{2}iz \pm \sqrt{1 - z^2/4}.$$

Moreover,

$$A''(s) = 1 + \frac{1}{(s + iz)^2} = s^2 + 1\,.$$

It is then necessary to distinguish between the two cases $0 \leq z < 2$ and $z > 2$ ($z = 2$ requires a special treatment).

(ii) $z > 2$. It is convenient to set $z = 2\cosh\theta$ with $\theta > 0$. Then,

$$s_\pm = -i\,\mathrm{e}^{\pm\theta} \;\Rightarrow\; \mathrm{e}^{-nA} = \exp\left[\tfrac{1}{2}n\,\mathrm{e}^{\pm 2\theta} \mp n\theta\right].$$

It is simple to verify by contour deformation that the relevant saddle point is s_- (s_+ is a saddle point between the hole at $s = -iz$ and the saddle point s_-) and thus

$$\mathcal{H}_n(z) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{1 - \mathrm{e}^{-2\theta}}} \exp\left[\tfrac{1}{2}n\,\mathrm{e}^{-2\theta} + n\theta\right].$$

In contrast, for $|z| < 2$, the two saddle points are relevant. Setting $z = 2\cos\theta$, one finds

$$\mathcal{H}_n(z) \underset{n\to\infty}{\sim} \frac{1}{\sqrt{1 - \mathrm{e}^{-2i\theta}}} \mathrm{e}^{n\,\mathrm{e}^{-2i\theta}/2 + ni\theta} + \text{complex conjugate}.$$

Exercise 1.7

Evaluate, using the steepest descent method, the integral

$$I_n(a) = \int_{-\infty}^{+\infty} \mathrm{d}x \; \mathrm{e}^{-nx^2/2+nax} \cosh^n x$$

as a function of the real parameter a in the limit $n \to \infty$. Express the result in a parametric form as a function of the saddle point position.

Solution. One notices that the integral can be written as

$$I_n(a) = \int_{-\infty}^{+\infty} \mathrm{d}x \; \mathrm{e}^{nf(x)},$$
$$f(x) = -x^2/2 + ax + \ln\cosh x\,.$$

The saddle point position is given by

$$f'(x) = 0 = -x + a + \tanh x\,,$$

with

$$f''(x) = -\tanh^2(x).$$

The saddle point equation thus has a unique solution $x(a)$ for all a. To parametrize the result in terms of $x(a)$, one substitutes

$$a = x - \tanh x\,.$$

At the saddle point

$$f(x) = x^2/2 - x\tanh x + \ln\cosh x\,,$$

and thus

$$I_n(a) = \frac{(2\pi)^{1/2}}{|\tanh x|} \,\mathrm{e}^{nf(x)}\,.$$

Note that the steepest descent method does not apply for $a = 0$ where $f''(x)$ vanishes. It is then necessary to expand $f(x)$ to order x^4 and integrate directly.