Chapter 2

Manifolds

Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of $\mathbb{R}^n$. The coordinates on a chart allow one to carry out computations as though in a Euclidean space, so that many concepts from $\mathbb{R}^n$, such as differentiability, point-derivations, tangent spaces, and differential forms, carry over to a manifold.

Bernhard Riemann

(1826–1866)

Like most fundamental mathematical concepts, the idea of a manifold did not originate with a single person, but is rather the distillation of years of collective activity. In his masterpiece *Disquisitiones generales circa superficies curvas* ("General Investigations of Curved Surfaces") published in 1827, Carl Friedrich Gauss freely used local coordinates on a surface, and so he already had the idea of charts. Moreover, he appeared to be the first to consider a surface as an abstract space existing in its own right, independent of a particular embedding in a Euclidean space. Bernhard Riemann's inaugural lecture *Über die Hypothesen, welche der Geometrie zu Grunde liegen* ("On the hypotheses that underlie geometry") in Göttingen in 1854 laid the foundations of higher-dimensional differential geometry. Indeed, the word "manifold" is a direct translation of the German word "Mannigfaltigkeit," which Riemann used to describe the objects of his inquiry. This was followed by the work of Henri Poincaré in the late nineteenth century on homology, in which locally Euclidean spaces figured prominently. The late nineteenth and early twentieth centuries were also a period of feverish development in point-set topology. It was not until 1931 that one finds the modern definition of a manifold based on point-set topology and a group of transition functions [37].

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In this chapter we give the basic definitions and properties of a smooth manifold and of smooth maps between manifolds. Initially, the only way we have to verify that a space is a manifold is to exhibit a collection of C^∞ compatible charts covering the space. In Section 7 we describe a set of sufficient conditions under which a quotient topological space becomes a manifold, giving us a second way to construct manifolds.

§5 Manifolds

While there are many kinds of manifolds—for example, topological manifolds, C^k-manifolds, analytic manifolds, and complex manifolds—in this book we are concerned mainly with smooth manifolds. Starting with topological manifolds, which are Hausdorff, second countable, locally Euclidean spaces, we introduce the concept of a maximal C^∞ atlas, which makes a topological manifold into a smooth manifold. This is illustrated with a few simple examples.

5.1 Topological Manifolds

We first recall a few definitions from point-set topology. For more details, see Appendix A. A topological space is *second countable* if it has a countable basis. A *neighborhood* of a point p in a topological space M is any open set containing p. An *open cover* of M is a collection $\{U_\alpha\}_{\alpha\in\mathrm{A}}$ of open sets in M whose union $\bigcup_{\alpha\in\mathrm{A}} U_\alpha$ is M.

Definition 5.1. A topological space M is *locally Euclidean of dimension n* if every point p in M has a neighborhood U such that there is a homeomorphism ϕ from U onto an open subset of $\mathbb{R}^n$. We call the pair $(U, \phi\colon U \to \mathbb{R}^n)$ a *chart*, U a *coordinate neighborhood* or a *coordinate open set*, and ϕ a *coordinate map* or a *coordinate system* on U. We say that a chart (U,ϕ) is *centered* at $p \in U$ if $\phi(p) = 0$.

Definition 5.2. A *topological manifold* is a Hausdorff, second countable, locally Euclidean space. It is said to be of *dimension n* if it is locally Euclidean of dimension n.

For the dimension of a topological manifold to be well defined, we need to know that for $n \neq m$ an open subset of $\mathbb{R}^n$ is not homeomorphic to an open subset of $\mathbb{R}^m$. This fact, called *invariance of dimension*, is indeed true, but is not easy to prove directly. We will not pursue this point, since we are mainly interested in *smooth* manifolds, for which the analogous result is easy to prove (Corollary 8.7). Of course, if a topological manifold has several connected components, it is possible for each component to have a different dimension.

Example. The Euclidean space $\mathbb{R}^n$ is covered by a single chart $(\mathbb{R}^n, \mathbb{1}_{\mathbb{R}^n})$, where $\mathbb{1}_{\mathbb{R}^n}: \mathbb{R}^n \to \mathbb{R}^n$ is the identity map. It is the prime example of a topological manifold. Every open subset of $\mathbb{R}^n$ is also a topological manifold, with chart $(U, \mathbb{1}_U)$.

Recall that the Hausdorff condition and second countability are "hereditary properties"; that is, they are inherited by subspaces: a subspace of a Hausdorff space is Hausdorff (Proposition A.19) and a subspace of a second-countable space is second countable (Proposition A.14). So any subspace of $\mathbb{R}^n$ is automatically Hausdorff and second countable.

Example 5.3 (*A cusp*). The graph of $y = x^{2/3}$ in $\mathbb{R}^2$ is a topological manifold (Figure 5.1(a)). By virtue of being a subspace of $\mathbb{R}^2$, it is Hausdorff and second countable. It is locally Euclidean, because it is homeomorphic to $\mathbb{R}$ via $(x, x^{2/3}) \mapsto x$.

(a) Cusp (b) Cross

Fig. 5.1.

Example 5.4 (*A cross*). Show that the cross in $\mathbb{R}^2$ in Figure 5.1 with the subspace topology is not locally Euclidean at the intersection p, and so cannot be a topological manifold.

Solution. Suppose the cross is locally Euclidean of dimension n at the point p. Then p has a neighborhood U homeomorphic to an open ball $B := B(0, \varepsilon) \subset \mathbb{R}^n$ with p mapping to 0. The homeomorphism $U \to B$ restricts to a homeomorphism $U - \{p\} \to B - \{0\}$. Now $B - \{0\}$ is either connected if $n \geq 2$ or has two connected components if $n = 1$. Since $U - \{p\}$ has four connected components, there can be no homeomorphism from $U - \{p\}$ to $B - \{0\}$. This contradiction proves that the cross is not locally Euclidean at p. □

5.2 Compatible Charts

Suppose $(U, \phi: U \to \mathbb{R}^n)$ and $(V, \psi: V \to \mathbb{R}^n)$ are two charts of a topological manifold. Since $U \cap V$ is open in U and $\phi: U \to \mathbb{R}^n$ is a homeomorphism onto an open subset of $\mathbb{R}^n$, the image $\phi(U \cap V)$ will also be an open subset of $\mathbb{R}^n$. Similarly, $\psi(U \cap V)$ is an open subset of $\mathbb{R}^n$.

Definition 5.5. Two charts $(U, \phi: U \to \mathbb{R}^n)$, $(V, \psi: V \to \mathbb{R}^n)$ of a topological manifold are C^∞-*compatible* if the two maps

$$\phi \circ \psi^{-1} \colon \psi(U \cap V) \to \phi(U \cap V), \quad \psi \circ \phi^{-1} \colon \phi(U \cap V) \to \psi(U \cap V)$$

are C^∞ (Figure 5.2). These two maps are called the *transition functions* between the charts. If $U \cap V$ is empty, then the two charts are automatically C^∞-compatible. To simplify the notation, we will sometimes write $U_{\alpha\beta}$ for $U_\alpha \cap U_\beta$ and $U_{\alpha\beta\gamma}$ for $U_\alpha \cap U_\beta \cap U_\gamma$.

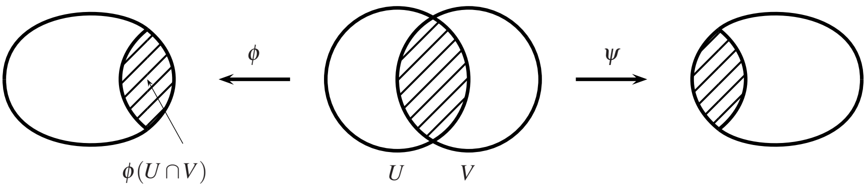


Fig. 5.2. The transition function $\psi \circ \phi^{-1}$ is defined on $\phi(U \cap V)$.

Since we are interested only in C^∞-compatible charts, we often omit mention of "C^∞" and speak simply of compatible charts.

Definition 5.6. A C^∞ *atlas* or simply an *atlas* on a locally Euclidean space M is a collection $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ of pairwise C^∞-compatible charts that cover M, i.e., such that $M = \bigcup_\alpha U_\alpha$.

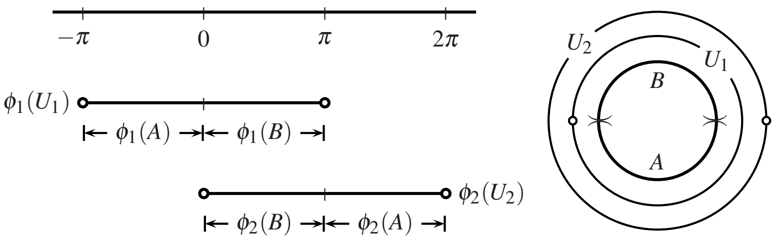


Fig. 5.3. A C^∞ atlas on a circle.

Example 5.7 (*A C^∞ atlas on a circle*). The unit circle S^1 in the complex plane $\mathbb{C}$ may be described as the set of points $\{e^{it} \in \mathbb{C} \mid 0 \le t \le 2\pi\}$. Let U_1 and U_2 be the two open subsets of S^1 (see Figure 5.3)

$$U_1 = \{e^{it} \in \mathbb{C} \mid -\pi < t < \pi\},$$
$$U_2 = \{e^{it} \in \mathbb{C} \mid 0 < t < 2\pi\},$$

and define $\phi_\alpha \colon U_\alpha \to \mathbb{R}$ for $\alpha = 1,2$ by

$$\begin{aligned} \phi_1(e^{it}) &= t, \quad -\pi < t < \pi, \\ \phi_2(e^{it}) &= t, \quad 0 < t < 2\pi. \end{aligned}$$

Both ϕ_1 and ϕ_2 are branches of the complex log function $(1/i)\log z$ and are homeomorphisms onto their respective images. Thus, (U_1, ϕ_1) and (U_2, ϕ_2) are charts on S^1. The intersection $U_1 \cap U_2$ consists of two connected components,

$$\begin{aligned} A &= \{e^{it} \mid -\pi < t < 0\}, \\ B &= \{e^{it} \mid 0 < t < \pi\}, \end{aligned}$$

with

$$\begin{aligned} \phi_1(U_1 \cap U_2) &= \phi_1(A \amalg B) = \phi_1(A) \amalg \phi_1(B) = \,]-\pi, 0[\, \amalg \,]0, \pi[, \\ \phi_2(U_1 \cap U_2) &= \phi_2(A \amalg B) = \phi_2(A) \amalg \phi_2(B) = \,]\pi, 2\pi[\, \amalg \,]0, \pi[. \end{aligned}$$

Here we use the notation $A \amalg B$ to indicate a union in which the two subsets A and B are disjoint. The transition function $\phi_2 \circ \phi_1^{-1} \colon \phi_1(A \amalg B) \to \phi_2(A \amalg B)$ is given by

$$(\phi_2 \circ \phi_1^{-1})(t) = \begin{cases} t + 2\pi & \text{for } t \in \,]-\pi, 0[, \\ t & \text{for } t \in \,]0, \pi[. \end{cases}$$

Similarly,

$$(\phi_1 \circ \phi_2^{-1})(t) = \begin{cases} t - 2\pi & \text{for } t \in \,]\pi, 2\pi[, \\ t & \text{for } t \in \,]0, \pi[. \end{cases}$$

Therefore, (U_1, ϕ_1) and (U_2, ϕ_2) are C^∞-compatible charts and form a C^∞ atlas on S^1.

Although the C^∞ compatibility of charts is clearly reflexive and symmetric, it is not transitive. The reason is as follows. Suppose (U_1, ϕ_1) is C^∞-compatible with (U_2, ϕ_2), and (U_2, ϕ_2) is C^∞-compatible with (U_3, ϕ_3). Note that the three coordinate functions are simultaneously defined only on the triple intersection U_{123}. Thus, the composite

$$\phi_3 \circ \phi_1^{-1} = (\phi_3 \circ \phi_2^{-1}) \circ (\phi_2 \circ \phi_1^{-1})$$

is C^∞, but only on $\phi_1(U_{123})$, not necessarily on $\phi_1(U_{13})$ (Figure 5.4). A priori we know nothing about $\phi_3 \circ \phi_1^{-1}$ on $\phi_1(U_{13} - U_{123})$ and so we cannot conclude that (U_1, ϕ_1) and (U_3, ϕ_3) are C^∞-compatible.

We say that a chart (V, ψ) is *compatible with an atlas* $\{(U_\alpha, \phi_\alpha)\}$ if it is compatible with all the charts (U_α, ϕ_α) of the atlas.

Lemma 5.8. *Let $\{(U_\alpha, \phi_\alpha)\}$ be an atlas on a locally Euclidean space. If two charts (V, ψ) and (W, σ) are both compatible with the atlas $\{(U_\alpha, \phi_\alpha)\}$, then they are compatible with each other.*

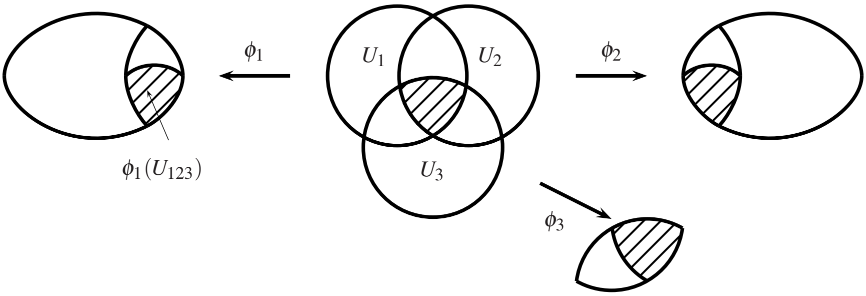


Fig. 5.4. The transition function $\phi_3 \circ \phi_1^{-1}$ is C^∞ on $\phi_1(U_{123})$.

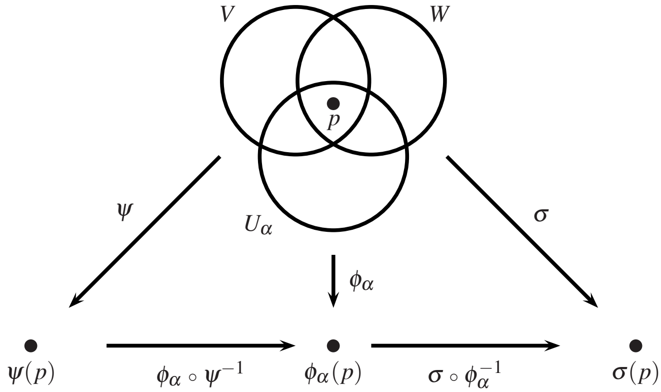


Fig. 5.5. Two charts (V, ψ), (W, σ) compatible with an atlas.

Proof. (See Figure 5.5.) Let $p \in V \cap W$. We need to show that $\sigma \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, $p \in U_\alpha$ for some α. Then p is in the triple intersection $V \cap W \cap U_\alpha$.

By the remark above, $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ is C^∞ on $\psi(V \cap W \cap U_\alpha)$, hence at $\psi(p)$. Since p was an arbitrary point of $V \cap W$, this proves that $\sigma \circ \psi^{-1}$ is C^∞ on $\psi(V \cap W)$. Similarly, $\psi \circ \sigma^{-1}$ is C^∞ on $\sigma(V \cap W)$. □

Note that in an equality such as $\sigma \circ \psi^{-1} = (\sigma \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \psi^{-1})$ in the proof above, the maps on the two sides of the equality sign have different domains. What the equality means is that the two maps are equal on their common domain.

5.3 Smooth Manifolds

An atlas $\mathfrak{M}$ on a locally Euclidean space is said to be *maximal* if it is not contained in a larger atlas; in other words, if $\mathfrak{U}$ is any other atlas containing $\mathfrak{M}$, then $\mathfrak{U} = \mathfrak{M}$.

Definition 5.9. A *smooth* or C^∞ *manifold* is a topological manifold M together with a maximal atlas. The maximal atlas is also called a *differentiable structure* on M. A manifold is said to have dimension n if all of its connected components have dimension n. A 1-dimensional manifold is also called a *curve*, a 2-dimensional manifold a *surface*, and an n-dimensional manifold an *n-manifold*.

In Corollary 8.7 we will prove that if an open set $U \subset \mathbb{R}^n$ is diffeomorphic to an open set $V \subset \mathbb{R}^m$, then $n = m$. As a consequence, the dimension of a manifold at a point is well defined.

In practice, to check that a topological manifold M is a smooth manifold, it is not necessary to exhibit a maximal atlas. The existence of *any* atlas on M will do, because of the following proposition.

Proposition 5.10. *Any atlas* $\mathfrak{U} = \{(U_\alpha, \phi_\alpha)\}$ *on a locally Euclidean space is contained in a unique maximal atlas.*

Proof. Adjoin to the atlas $\mathfrak{U}$ all charts (V_i, ψ_i) that are compatible with $\mathfrak{U}$. By Proposition 5.8 the charts (V_i, ψ_i) are compatible with one another. So the enlarged collection of charts is an atlas. Any chart compatible with the new atlas must be compatible with the original atlas $\mathfrak{U}$ and so by construction belongs to the new atlas. This proves that the new atlas is maximal.

Let $\mathfrak{M}$ be the maximal atlas containing $\mathfrak{U}$ that we have just constructed. If $\mathfrak{M}'$ is another maximal atlas containing $\mathfrak{U}$, then all the charts in $\mathfrak{M}'$ are compatible with $\mathfrak{U}$ and so by construction must belong to $\mathfrak{M}$. This proves that $\mathfrak{M}' \subset \mathfrak{M}$. Since both are maximal, $\mathfrak{M}' = \mathfrak{M}$. Therefore, the maximal atlas containing $\mathfrak{U}$ is unique. □

In summary, to show that a topological space M is a C^∞ manifold, it suffices to check that

(i) M is Hausdorff and second countable,
(ii) M has a C^∞ atlas (not necessarily maximal).

From now on, a "manifold" will mean a C^∞ manifold. We use the terms "smooth" and "C^∞" interchangeably. In the context of manifolds, we denote the standard coordinates on $\mathbb{R}^n$ by $r^1, \ldots, r^n$. If $(U, \phi\colon U \to \mathbb{R}^n)$ is a chart of a manifold, we let $x^i = r^i \circ \phi$ be the ith component of ϕ and write $\phi = (x^1, \ldots, x^n)$ and $(U, \phi) = (U, x^1, \ldots, x^n)$. Thus, for $p \in U$, $(x^1(p), \ldots, x^n(p))$ is a point in $\mathbb{R}^n$. The functions $x^1, \ldots, x^n$ are called *coordinates* or *local coordinates* on U. By abuse of notation, we sometimes omit the p. So the notation $(x^1, \ldots, x^n)$ stands alternately for local coordinates on the open set U and for a point in $\mathbb{R}^n$. By a *chart* (U, ϕ) *about* p in a manifold M, we will mean a chart in the differentiable structure of M such that $p \in U$.

5.4 Examples of Smooth Manifolds

Example 5.11 (*Euclidean space*). The Euclidean space $\mathbb{R}^n$ is a smooth manifold with a single chart $(\mathbb{R}^n, r^1, \ldots, r^n)$, where $r^1, \ldots, r^n$ are the standard coordinates on $\mathbb{R}^n$.

Example 5.12 (*Open subset of a manifold*). Any open subset V of a manifold M is also a manifold. If $\{(U_\alpha, \phi_\alpha)\}$ is an atlas for M, then $\{(U_\alpha \cap V, \phi_\alpha|_{U_\alpha \cap V}\}$ is an atlas for V, where $\phi_\alpha|_{U_\alpha\cap V}: U_\alpha \cap V \to \mathbb{R}^n$ denotes the restriction of ϕ_α to the subset $U_\alpha \cap V$.

Example 5.13 (*Manifolds of dimension zero*). In a manifold of dimension zero, every singleton subset is homeomorphic to $\mathbb{R}^0$ and so is open. Thus, a zero-dimensional manifold is a discrete set. By second countability, this discrete set must be countable.

Example 5.14 (*Graph of a smooth function*). For a subset of $A \subset \mathbb{R}^n$ and a function $f: A \to \mathbb{R}^m$, the *graph* of f is defined to be the subset (Figure 5.6)

$$\Gamma(f) = \{(x, f(x)) \in A \times \mathbb{R}^m\}.$$

If U is an open subset of $\mathbb{R}^n$ and $f: U \to \mathbb{R}^n$ is C^∞, then the two maps

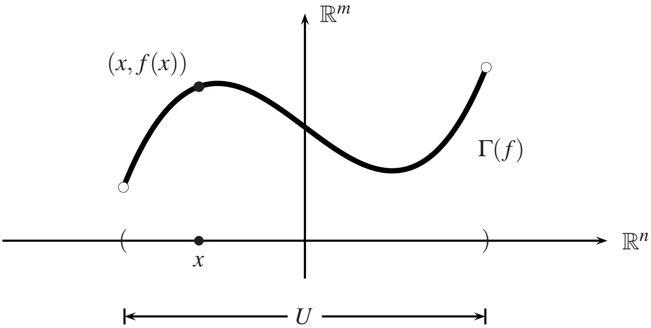


Fig. 5.6. The graph of a smooth function $f: \mathbb{R}^n \supset U \to \mathbb{R}^m$.

$$\phi: \Gamma(f) \to U, \qquad (x, f(x)) \mapsto x,$$

and

$$(1, f): U \to \Gamma(f), \qquad x \mapsto (x, f(x)),$$

are continuous and inverse to each other, and so are homeomorphisms. The graph $\Gamma(f)$ of a C^∞ function $f: U \to \mathbb{R}^m$ has an atlas with a single chart $(\Gamma(f), \phi)$, and is therefore a C^∞ manifold. This shows that many of the familiar surfaces of calculus, for example an elliptic paraboloid or a hyperbolic paraboloid, are manifolds.

Example 5.15 (*General linear groups*). For any two positive integers m and n let $\mathbb{R}^{m\times n}$ be the vector space of all $m \times n$ matrices. Since $\mathbb{R}^{m\times n}$ is isomorphic to $\mathbb{R}^{mn}$, we give it the topology of $\mathbb{R}^{mn}$. The *general linear group* $\mathrm{GL}(n, \mathbb{R})$ is by definition

$$\mathrm{GL}(n, \mathbb{R}) := \{A \in \mathbb{R}^{n\times n} \mid \det A \neq 0\} = \det^{-1}(\mathbb{R} - \{0\}).$$

Since the determinant function

$$\det\colon \mathbb{R}^{n\times n} \to \mathbb{R}$$

is continuous, $\mathrm{GL}(n,\mathbb{R})$ is an open subset of $\mathbb{R}^{n\times n} \simeq \mathbb{R}^{n^2}$ and is therefore a manifold.

The *complex general linear group* $\mathrm{GL}(n,\mathbb{C})$ is defined to be the group of nonsingular $n\times n$ complex matrices. Since an $n\times n$ matrix A is nonsingular if and only if $\det A \neq 0$, $\mathrm{GL}(n,\mathbb{C})$ is an open subset of $\mathbb{C}^{n\times n} \simeq \mathbb{R}^{2n^2}$, the vector space of $n\times n$ complex matrices. By the same reasoning as in the real case, $\mathrm{GL}(n,\mathbb{C})$ is a manifold of dimension $2n^2$.

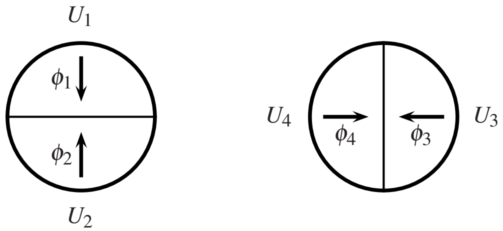


Fig. 5.7. Charts on the unit circle.

Example 5.16 (*Unit circle in the* (x,y)*-plane*). In Example 5.7 we found a C^∞ atlas with two charts on the unit circle S^1 in the complex plane $\mathbb{C}$. It follows that S^1 is a manifold. We now view S^1 as the unit circle in the real plane $\mathbb{R}^2$ with defining equation $x^2+y^2=1$, and describe a C^∞ atlas with four charts on it.

We can cover S^1 with four open sets: the upper and lower semicircles U_1, U_2, and the right and left semicircles U_3, U_4 (Figure 5.7). On U_1 and U_2, the coordinate function x is a homeomorphism onto the open interval $]-1,1[$ on the x-axis. Thus, $\phi_i(x,y) = x$ for $i=1,2$. Similarly, on U_3 and U_4, y is a homeomorphism onto the open interval $]-1,1[$ on the y-axis, and so $\phi_i(x,y)=y$ for $i=3,4$.

It is easy to check that on every nonempty pairwise intersection $U_\alpha \cap U_\beta$, $\phi_\beta\circ\phi_\alpha^{-1}$ is C^∞. For example, on $U_1\cap U_3$,

$$(\phi_3\circ\phi_1^{-1})(x) = \phi_3\left(x, \sqrt{1-x^2}\right) = \sqrt{1-x^2},$$

which is C^∞. On $U_2\cap U_4$,

$$(\phi_4\circ\phi_2^{-1})(x) = \phi_4\left(x, -\sqrt{1-x^2}\right) = -\sqrt{1-x^2},$$

which is also C^∞. Thus, $\{(U_i,\phi_i)\}_{i=1}^4$ is a C^∞ atlas on S^1.

Example 5.17 (*Product manifold*). If M and N are C^∞ manifolds, then $M\times N$ with its product topology is Hausdorff and second countable (Corollary A.21 and Proposition A.22). To show that $M\times N$ is a manifold, it remains to exhibit an atlas on it. Recall that the product of two set maps $f\colon X\to X'$ and $g\colon Y\to Y'$ is

$$f\times g\colon X\times Y \to X'\times Y', \quad (f\times g)(x,y) = (f(x), g(y)).$$

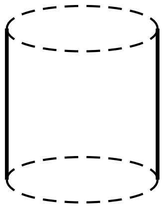
Proposition 5.18 (An atlas for a product manifold). *If $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_i, \psi_i)\}$ are C^∞ atlases for the manifolds M and N of dimensions m and n, respectively, then the collection*

$$\{(U_\alpha \times V_i, \phi_\alpha \times \psi_i \colon U_\alpha \times V_i \to \mathbb{R}^m \times \mathbb{R}^n)\}$$

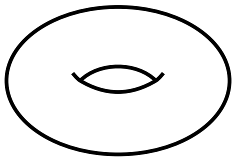
of charts is a C^∞ atlas on $M \times N$. Therefore, $M \times N$ is a C^∞ manifold of dimension $m+n$.

Proof. Problem 5.5. □

Example. It follows from Proposition 5.18 that the infinite cylinder $S^1 \times \mathbb{R}$ and the torus $S^1 \times S^1$ are manifolds (Figure 5.8).



Infinite cylinder.



Torus.

Fig. 5.8.

Since $M \times N \times P = (M \times N) \times P$ is the successive product of pairs of spaces, if M, N, and P are manifolds, then so is $M \times N \times P$. Thus, the n-dimensional torus $S^1 \times \cdots \times S^1$ (n times) is a manifold.

Remark. Let S^n be the unit sphere

$$(x^1)^2 + (x^2)^2 + \cdots + (x^{n+1})^2 = 1$$

in $\mathbb{R}^{n+1}$. Using Problem 5.3 as a guide, it is easy to write down a C^∞ atlas on S^n, showing that S^n has a differentiable structure. The manifold S^n with this differentiable structure is called the *standard n-sphere*.

One of the most surprising achievements in topology was John Milnor's discovery [27] in 1956 of exotic 7-spheres, smooth manifolds homeomorphic but not diffeomorphic to the standard 7-sphere. In 1963, Michel Kervaire and John Milnor [24] determined that there are exactly 28 nondiffeomorphic differentiable structures on S^7.

It is known that in dimensions < 4 every topological manifold has a unique differentiable structure and in dimensions > 4 every compact topological manifold has a finite number of differentiable structures. Dimension 4 is a mystery. It is not known whether S^4 has a finite or infinite number of differentiable structures. The statement that S^4 has a unique differentiable structure is called the *smooth Poincaré conjecture*. As of this writing in 2010, the conjecture is still open.

There are topological manifolds with no differentiable structure. Michel Kervaire was the first to construct an example [23].

Problems

5.1. The real line with two origins
Let A and B be two points not on the real line $\mathbb{R}$. Consider the set $S = (\mathbb{R} - \{0\}) \cup \{A, B\}$ (see Figure 5.9).

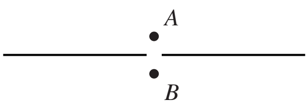


Fig. 5.9. Real line with two origins.

For any two positive real numbers c, d, define

$$I_A(-c,d) =]-c,0[\cup \{A\} \cup]0,d[$$

and similarly for $I_B(-c,d)$, with B instead of A. Define a topology on S as follows: On $(\mathbb{R} - \{0\})$, use the subspace topology inherited from $\mathbb{R}$, with open intervals as a basis. A basis of neighborhoods at A is the set $\{I_A(-c,d) \mid c,d > 0\}$; similarly, a basis of neighborhoods at B is $\{I_B(-c,d) \mid c,d > 0\}$.

(a) Prove that the map $h\colon I_A(-c,d) \to]-c,d[$ defined by

$$\begin{aligned} h(x) &= x \quad \text{for } x \in]-c,0[\cup]0,d[, \\ h(A) &= 0 \end{aligned}$$

is a homeomorphism.

(b) Show that S is locally Euclidean and second countable, but not Hausdorff.

5.2. A sphere with a hair
A fundamental theorem of topology, the theorem on invariance of dimension, states that if two nonempty open sets $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^m$ are homeomorphic, then $n = m$ (for a proof, see [18, p. 126]). Use the idea of Example 5.4 as well as the theorem on invariance of dimension to prove that the sphere with a hair in $\mathbb{R}^3$ (Figure 5.10) is not locally Euclidean at q. Hence it cannot be a topological manifold.

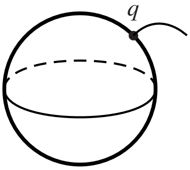


Fig. 5.10. A sphere with a hair.

5.3. Charts on a sphere
Let S^2 be the unit sphere

$$x^2+y^2+z^2=1$$

in $\mathbb{R}^3$. Define in S^2 the six charts corresponding to the six hemispheres—the front, rear, right, left, upper, and lower hemispheres (Figure 5.11):

$$\begin{aligned}
U_1 &= \{(x,y,z)\in S^2 \mid x>0\}, &\quad \phi_1(x,y,z) &= (y,z),\\
U_2 &= \{(x,y,z)\in S^2 \mid x<0\}, &\quad \phi_2(x,y,z) &= (y,z),\\
U_3 &= \{(x,y,z)\in S^2 \mid y>0\}, &\quad \phi_3(x,y,z) &= (x,z),\\
U_4 &= \{(x,y,z)\in S^2 \mid y<0\}, &\quad \phi_4(x,y,z) &= (x,z),\\
U_5 &= \{(x,y,z)\in S^2 \mid z>0\}, &\quad \phi_5(x,y,z) &= (x,y),\\
U_6 &= \{(x,y,z)\in S^2 \mid z<0\}, &\quad \phi_6(x,y,z) &= (x,y).
\end{aligned}$$

Describe the domain $\phi_4(U_{14})$ of $\phi_1\circ\phi_4^{-1}$ and show that $\phi_1\circ\phi_4^{-1}$ is C^∞ on $\phi_4(U_{14})$. Do the same for $\phi_6\circ\phi_1^{-1}$.

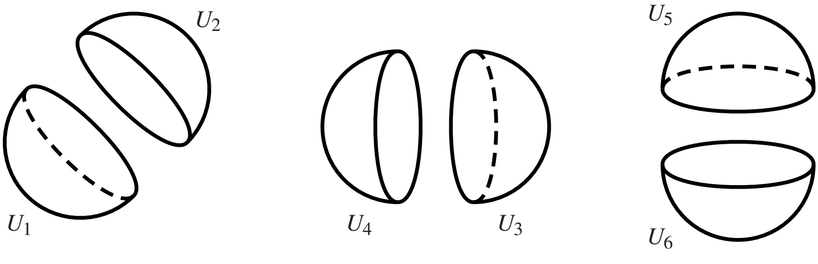


Fig. 5.11. Charts on the unit sphere.

5.4.* Existence of a coordinate neighborhood
Let $\{(U_\alpha,\phi_\alpha)\}$ be the maximal atlas on a manifold M. For any open set U in M and a point $p\in U$, prove the existence of a coordinate open set U_α such that $p\in U_\alpha\subset U$.

5.5. An atlas for a product manifold
Prove Proposition 5.18.

§6 Smooth Maps on a Manifold

Now that we have defined smooth manifolds, it is time to consider maps between them. Using coordinate charts, one can transfer the notion of smooth maps from Euclidean spaces to manifolds. By the C^∞ compatibility of charts in an atlas, the smoothness of a map turns out to be independent of the choice of charts and is therefore well defined. We give various criteria for the smoothness of a map as well as examples of smooth maps.

Next we transfer the notion of partial derivatives from Euclidean space to a coordinate chart on a manifold. Partial derivatives relative to coordinate charts allow us to generalize the inverse function theorem to manifolds. Using the inverse function theorem, we formulate a criterion for a set of smooth functions to serve as local coordinates near a point.

6.1 Smooth Functions on a Manifold

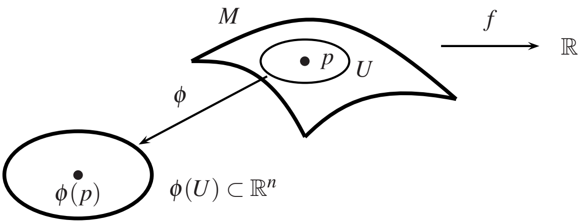


Fig. 6.1. Checking that a function f is C^∞ at p by pulling back to $\mathbb{R}^n$.

Definition 6.1. Let M be a smooth manifold of dimension n. A function $f\colon M \to \mathbb{R}$ is said to be C^∞ or *smooth at a point* p in M if there is a chart (U,ϕ) about p in M such that $f \circ \phi^{-1}$, a function defined on the open subset $\phi(U)$ of $\mathbb{R}^n$, is C^∞ at $\phi(p)$ (see Figure 6.1). The function f is said to be C^∞ *on* M if it is C^∞ at every point of M.

Remark 6.2. The definition of the smoothness of a function f at a point is independent of the chart (U,ϕ), for if $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$ and (V,ψ) is any other chart about p in M, then on $\psi(U \cap V)$,

$$f \circ \psi^{-1} = (f \circ \phi^{-1}) \circ (\phi \circ \psi^{-1}),$$

which is C^∞ at $\psi(p)$ (see Figure 6.2).

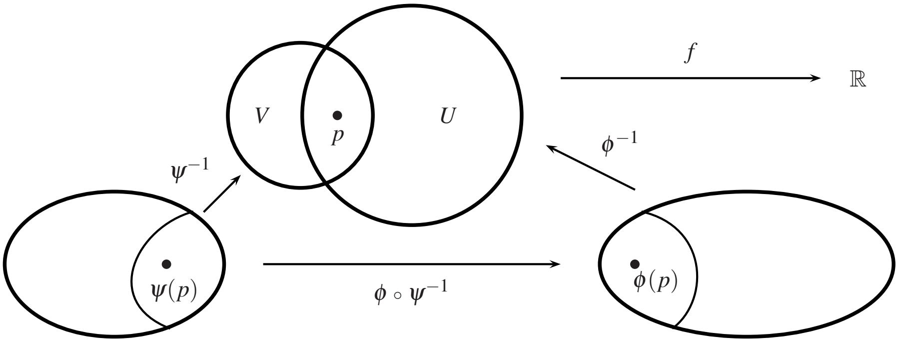


Fig. 6.2. Checking that a function f is C^∞ at p via two charts.

In Definition 6.1, $f\colon M \to \mathbb{R}$ is not assumed to be continuous. However, if f is C^∞ at $p \in M$, then $f \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}$, being a C^∞ function at the point $\phi(p)$ in an open subset of $\mathbb{R}^n$, is continuous at $\phi(p)$. As a composite of continuous functions, $f = (f \circ \phi^{-1}) \circ \phi$ is continuous at p. Since we are interested only in functions that are smooth on an open set, there is no loss of generality in assuming at the outset that f is continuous.

Proposition 6.3 (Smoothness of a real-valued function). *Let M be a manifold of dimension n, and $f\colon M \to \mathbb{R}$ a real-valued function on M. The following are equivalent:*

(i) *The function $f\colon M \to \mathbb{R}$ is C^∞.*
(ii) *The manifold M has an atlas such that for every chart (U,ϕ) in the atlas, $f \circ \phi^{-1}\colon \mathbb{R}^n \supset \phi(U) \to \mathbb{R}$ is C^∞.*
(iii) *For every chart (V,ψ) on M, the function $f \circ \psi^{-1}\colon \mathbb{R}^n \supset \psi(V) \to \mathbb{R}$ is C^∞.*

Proof. We will prove the proposition as a cyclic chain of implications.
(ii) $\Rightarrow$ (i): This follows directly from the definition of a C^∞ function, since by (ii) every point $p \in M$ has a coordinate neighborhood (U,ϕ) such that $f \circ \phi^{-1}$ is C^∞ at $\phi(p)$.
(i) $\Rightarrow$ (iii): Let (V,ψ) be an arbitrary chart on M and let $p \in V$. By Remark 6.2, $f \circ \psi^{-1}$ is C^∞ at $\psi(p)$. Since p was an arbitrary point of V, $f \circ \psi^{-1}$ is C^∞ on $\psi(V)$.
(iii) $\Rightarrow$ (ii): Obvious. □

The smoothness conditions of Proposition 6.3 will be a recurrent motif throughout the book: to prove the smoothness of an object, it is sufficient that a smoothness criterion hold on the charts of some atlas. Once the object is shown to be smooth, it then follows that the same smoothness criterion holds on *every* chart on the manifold.

Definition 6.4. Let $F\colon N \to M$ be a map and h a function on M. The *pullback* of h by F, denoted by F^*h, is the composite function $h \circ F$.

In this terminology, a function f on M is C^∞ on a chart (U,ϕ) if and only if its pullback $(\phi^{-1})^*f$ by ϕ^{-1} is C^∞ on the subset $\phi(U)$ of Euclidean space.

6.2 Smooth Maps Between Manifolds

We emphasize again that unless otherwise specified, by a manifold we always mean a C^∞ manifold. We use the terms "C^∞" and "smooth" interchangeably. An atlas or a chart on a smooth manifold means an atlas or a chart contained in the differentiable structure of the smooth manifold. We generally denote a manifold by M and its dimension by n. However, when speaking of two manifolds simultaneously, as in a map $f\colon N \to M$, we will let the dimension of N be n and that of M be m.

Definition 6.5. Let N and M be manifolds of dimension n and m, respectively. A continuous map $F\colon N \to M$ is C^∞ *at a point* p in N if there are charts (V,ψ) about $F(p)$ in M and (U,ϕ) about p in N such that the composition $\psi \circ F \circ \phi^{-1}$, a map from the open subset $\phi(F^{-1}(V)\cap U)$ of $\mathbb{R}^n$ to $\mathbb{R}^m$, is C^∞ at $\phi(p)$ (see Figure 6.3). The continuous map $F\colon N \to M$ is said to be C^∞ if it is C^∞ at every point of N.

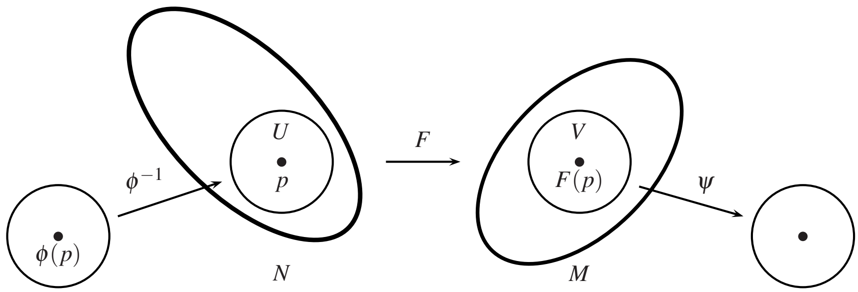


Fig. 6.3. Checking that a map $F\colon N \to M$ is C^∞ at p.

In Definition 6.5, we assume $F\colon N \to M$ continuous to ensure that $F^{-1}(V)$ is an open set in N. Thus, C^∞ maps between manifolds are by definition continuous.

Remark 6.6 (*Smooth maps into* $\mathbb{R}^m$). In case $M = \mathbb{R}^m$, we can take $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ as a chart about $F(p)$ in $\mathbb{R}^m$. According to Definition 6.5, $F\colon N \to \mathbb{R}^m$ is C^∞ at $p \in N$ if and only if there is a chart (U,ϕ) about p in N such that $F \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}^m$ is C^∞ at $\phi(p)$. Letting $m = 1$, we recover the definition of a function being C^∞ at a point.

We show now that the definition of the smoothness of a map $F\colon N \to M$ at a point is independent of the choice of charts. This is analogous to how the smoothness of a function $N \to \mathbb{R}$ at $p \in N$ is independent of the choice of a chart on N about p.

Proposition 6.7. *Suppose $F\colon N \to M$ is C^∞ at $p \in N$. If (U,ϕ) is any chart about p in N and (V,ψ) is any chart about $F(p)$ in M, then $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$.*

Proof. Since F is C^∞ at $p \in N$, there are charts (U_α,ϕ_α) about p in N and (V_β,ψ_β) about $F(p)$ in M such that $\psi_\beta \circ F \circ \phi_\alpha^{-1}$ is C^∞ at $\phi_\alpha(p)$. By the C^∞ compatibility of charts in a differentiable structure, both $\phi_\alpha \circ \phi^{-1}$ and $\psi \circ \psi_\beta^{-1}$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ F \circ \phi^{-1} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ F \circ \phi_\alpha^{-1}) \circ (\phi_\alpha \circ \phi^{-1})$$

is C^∞ at $\phi(p)$. □

The next proposition gives a way to check smoothness of a map without specifying a point in the domain.

Proposition 6.8 (Smoothness of a map in terms of charts). *Let N and M be smooth manifolds, and $F: N \to M$ a continuous map. The following are equivalent:*

(i) *The map $F: N \to M$ is C^∞.*

(ii) *There are atlases $\mathfrak{U}$ for N and $\mathfrak{V}$ for M such that for every chart (U,ϕ) in $\mathfrak{U}$ and (V,ψ) in $\mathfrak{V}$, the map*

$$\psi \circ F \circ \phi^{-1}\colon \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is C^∞.

(iii) *For every chart (U,ϕ) on N and (V,ψ) on M, the map*

$$\psi \circ F \circ \phi^{-1}\colon \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is C^∞.

Proof. (ii) ⇒ (i): Let $p \in N$. Suppose (U,ϕ) is a chart about p in $\mathfrak{U}$ and (V,ψ) is a chart about $F(p)$ in $\mathfrak{V}$. By (ii), $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. By the definition of a C^∞ map, $F: N \to M$ is C^∞ at p. Since p was an arbitrary point of N, the map $F: N \to M$ is C^∞.

(i) ⇒ (iii): Suppose (U,ϕ) and (V,ψ) are charts on N and M respectively such that $U \cap F^{-1}(V) \neq \varnothing$. Let $p \in U \cap F^{-1}(V)$. Then (U,ϕ) is a chart about p and (V,ψ) is a chart about $F(p)$. By Proposition 6.7, $\psi \circ F \circ \phi^{-1}$ is C^∞ at $\phi(p)$. Since $\phi(p)$ was an arbitrary point of $\phi(U \cap F^{-1}(V))$, the map $\psi \circ F \circ \phi^{-1}\colon \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$ is C^∞.

(iii) ⇒ (ii): Clear. □

Proposition 6.9 (Composition of C^∞ maps). *If $F: N \to M$ and $G: M \to P$ are C^∞ maps of manifolds, then the composite $G \circ F: N \to P$ is C^∞.*

Proof. Let (U,ϕ), (V,ψ), and (W,σ) be charts on N, M, and P respectively. Then

$$\sigma \circ (G \circ F) \circ \phi^{-1} = (\sigma \circ G \circ \psi^{-1}) \circ (\psi \circ F \circ \phi^{-1}).$$

Since F and G are C^∞, by Proposition 6.8(i)⇒(iii), $\sigma \circ G \circ \psi^{-1}$ and $\psi \circ F \circ \phi^{-1}$ are C^∞. As a composite of C^∞ maps of open subsets of Euclidean spaces, $\sigma \circ (G \circ F) \circ \phi^{-1}$ is C^∞. By Proposition 6.8(iii)⇒(i), $G \circ F$ is C^∞. □

6.3 Diffeomorphisms

A *diffeomorphism* of manifolds is a bijective C^∞ map $F\colon N \to M$ whose inverse F^{-1} is also C^∞. According to the next two propositions, coordinate maps are diffeomorphisms, and conversely, every diffeomorphism of an open subset of a manifold with an open subset of a Euclidean space can serve as a coordinate map.

Proposition 6.10. *If (U,ϕ) is a chart on a manifold M of dimension n, then the coordinate map $\phi\colon U \to \phi(U) \subset \mathbb{R}^n$ is a diffeomorphism.*

Proof. By definition, ϕ is a homeomorphism, so it suffices to check that both ϕ and ϕ^{-1} are smooth. To test the smoothness of $\phi\colon U \to \phi(U)$, we use the atlas $\{(U,\phi)\}$ with a single chart on U and the atlas $\{(\phi(U), \mathbb{1}_{\phi(U)})\}$ with a single chart on $\phi(U)$. Since $\mathbb{1}_{\phi(U)} \circ \phi \circ \phi^{-1}\colon \phi(U) \to \phi(U)$ is the identity map, it is C^∞. By Proposition 6.8(ii)$\Rightarrow$(i), ϕ is C^∞.

To test the smoothness of $\phi^{-1}\colon \phi(U) \to U$, we use the same atlases as above. Since $\phi \circ \phi^{-1} \circ \mathbb{1}_{\phi(U)} = \mathbb{1}_{\phi(U)}\colon \phi(U) \to \phi(U)$, the map ϕ^{-1} is also C^∞. □

Proposition 6.11. *Let U be an open subset of a manifold M of dimension n. If $F\colon U \to F(U) \subset \mathbb{R}^n$ is a diffeomorphism onto an open subset of $\mathbb{R}^n$, then (U,F) is a chart in the differentiable structure of M.*

Proof. For any chart (U_α, ϕ_α) in the maximal atlas of M, both ϕ_α and ϕ_α^{-1} are C^∞ by Proposition 6.10. As composites of C^∞ maps, both $F \circ \phi_\alpha^{-1}$ and $\phi_\alpha \circ F^{-1}$ are C^∞. Hence, (U,F) is compatible with the maximal atlas. By the maximality of the atlas, the chart (U,F) is in the atlas. □

6.4 Smoothness in Terms of Components

In this subsection we derive a criterion that reduces the smoothness of a map to the smoothness of real-valued functions on open sets.

Proposition 6.12 (Smoothness of a vector-valued function). *Let N be a manifold and $F\colon N \to \mathbb{R}^m$ a continuous map. The following are equivalent:*

(i) *The map $F\colon N \to \mathbb{R}^m$ is C^∞.*
(ii) *The manifold N has an atlas such that for every chart (U,ϕ) in the atlas, the map $F \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}^m$ is C^∞.*
(iii) *For every chart (U,ϕ) on N, the map $F \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}^m$ is C^∞.*

Proof. (ii) $\Rightarrow$ (i): In Proposition 6.8(ii), take $\mathfrak{V}$ to be the atlas with the single chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $M = \mathbb{R}^m$.
(i) $\Rightarrow$ (iii): In Proposition 6.8(iii), let (V,ψ) be the chart $(\mathbb{R}^m, \mathbb{1}_{\mathbb{R}^m})$ on $M = \mathbb{R}^m$.
(iii) $\Rightarrow$ (ii): Obvious. □

Proposition 6.13 (Smoothness in terms of components). *Let N be a manifold. A vector-valued function $F\colon N \to \mathbb{R}^m$ is C^∞ if and only if its component functions $F^1, \dots, F^m\colon N \to \mathbb{R}$ are all C^∞.*

Proof.

The map $F\colon N \to \mathbb{R}^m$ is C^∞

$\iff$ for every chart (U,ϕ) on N, the map $F \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}^m$ is C^∞ (by Proposition 6.12)

$\iff$ for every chart (U,ϕ) on N, the functions $F^i \circ \phi^{-1}\colon \phi(U) \to \mathbb{R}$ are all C^∞ (definition of smoothness for maps of Euclidean spaces)

$\iff$ the functions $F^i\colon N \to \mathbb{R}$ are all C^∞ (by Proposition 6.3). □

Exercise 6.14 (Smoothness of a map to a circle).* Prove that the map $F\colon \mathbb{R} \to S^1$, $F(t) = (\cos t, \sin t)$ is C^∞.

Proposition 6.15 (Smoothness of a map in terms of vector-valued functions). *Let $F\colon N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:*

(i) *The map $F\colon N \to M$ is C^∞.*
(ii) *The manifold M has an atlas such that for every chart $(V,\psi) = (V, y^1, \dots, y^m)$ in the atlas, the vector-valued function $\psi \circ F\colon F^{-1}(V) \to \mathbb{R}^m$ is C^∞.*
(iii) *For every chart $(V,\psi) = (V, y^1, \dots, y^m)$ on M, the vector-valued function $\psi \circ F\colon F^{-1}(V) \to \mathbb{R}^m$ is C^∞.*

Proof. (ii) $\Rightarrow$ (i): Let $\mathfrak{V}$ be the atlas for M in (ii), and let $\mathfrak{U} = \{(U,\phi)\}$ be an arbitrary atlas for N. For each chart (V,ψ) in the atlas $\mathfrak{V}$, the collection $\{(U \cap F^{-1}(V), \phi|_{U\cap F^{-1}(V)})\}$ is an atlas for $F^{-1}(V)$. Since $\psi \circ F\colon F^{-1}(V) \to \mathbb{R}^m$ is C^∞, by Proposition 6.12(i)$\Rightarrow$(iii),

$$\psi \circ F \circ \phi^{-1}\colon \phi(U \cap F^{-1}(V)) \to \mathbb{R}^m$$

is C^∞. It then follows from Proposition 6.8(ii)$\Rightarrow$(i) that $F\colon N \to M$ is C^∞.
(i) $\Rightarrow$ (iii): Being a coordinate map, ψ is C^∞ (Proposition 6.10). As the composite of two C^∞ maps, $\psi \circ F$ is C^∞.
(iii) $\Rightarrow$ (ii): Obvious. □

By Proposition 6.13, this smoothness criterion for a map translates into a smoothness criterion in terms of the components of the map.

Proposition 6.16 (Smoothness of a map in terms of components). *Let $F\colon N \to M$ be a continuous map between two manifolds of dimensions n and m respectively. The following are equivalent:*

(i) *The map $F\colon N \to M$ is C^∞.*
(ii) *The manifold M has an atlas such that for every chart $(V,\psi) = (V, y^1, \dots, y^m)$ in the atlas, the components $y^i \circ F\colon F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.*
(iii) *For every chart $(V,\psi) = (V, y^1, \dots, y^m)$ on M, the components $y^i \circ F\colon F^{-1}(V) \to \mathbb{R}$ of F relative to the chart are all C^∞.*

6.5 Examples of Smooth Maps

We have seen that coordinate maps are smooth. In this subsection we look at a few more examples of smooth maps.

Example 6.17 (*Smoothness of a projection map*). Let M and N be manifolds and $\pi\colon M \times N \to M$, $\pi(p,q) = p$ the projection to the first factor. Prove that π is a C^∞ map.

Solution. Let (p,q) be an arbitrary point of $M \times N$. Suppose $(U,\phi) = (U,x^1,\dots,x^m)$ and $(V,\psi) = (V,y^1,\dots,y^n)$ are coordinate neighborhoods of p and q in M and N respectively. By Proposition 5.18, $(U \times V, \phi \times \psi) = (U \times V, x^1,\dots,x^m,y^1,\dots,y^n)$ is a coordinate neighborhood of (p,q). Then

$$\left(\phi \circ \pi \circ (\phi \times \psi)^{-1}\right)(a^1,\dots,a^m,b^1,\dots,b^n) = (a^1,\dots,a^m),$$

which is a C^∞ map from $(\phi \times \psi)(U \times V)$ in $\mathbb{R}^{m+n}$ to $\phi(U)$ in $\mathbb{R}^m$, so π is C^∞ at (p,q). Since (p,q) was an arbitrary point in $M \times N$, π is C^∞ on $M \times N$.

Exercise 6.18 (Smoothness of a map to a Cartesian product).* Let M_1, M_2, and N be manifolds of dimensions m_1, m_2, and n respectively. Prove that a map $(f_1,f_2)\colon N \to M_1 \times M_2$ is C^∞ if and only if $f_i\colon N \to M_i$, $i = 1,2$, are both C^∞.

Example 6.19. In Examples 5.7 and 5.16 we showed that the unit circle S^1 defined by $x^2 + y^2 = 1$ in $\mathbb{R}^2$ is a C^∞ manifold. Prove that a C^∞ function $f(x,y)$ on $\mathbb{R}^2$ restricts to a C^∞ function on S^1.

Solution. To avoid confusing functions with points, we will denote a point on S^1 as $p = (a,b)$ and use x, y to mean the standard coordinate functions on $\mathbb{R}^2$. Thus, $x(a,b) = a$ and $y(a,b) = b$. Suppose we can show that x and y restrict to C^∞ functions on S^1. By Exercise 6.18, the inclusion map $i\colon S^1 \to \mathbb{R}^2$, $i(p) = (x(p),y(p))$ is then C^∞ on S^1. As the composition of C^∞ maps, $f|_{S^1} = f \circ i$ will be C^∞ on S^1 (Proposition 6.9).

Consider first the function x. We use the atlas (U_i,ϕ_i) from Example 5.16. Since x is a coordinate function on U_1 and on U_2, by Proposition 6.10 it is C^∞ on $U_1 \cup U_2 = S^1 - \{(\pm 1, 0)\}$. To show that x is C^∞ on U_3, it suffices to check the smoothness of $x \circ \phi_3^{-1}\colon \phi_3(U_3) \to \mathbb{R}$:

$$\left(x \circ \phi_3^{-1}\right)(b) = x\left(\sqrt{1-b^2}, b\right) = \sqrt{1-b^2}.$$

On U_3, we have $b \neq \pm 1$, so that $\sqrt{1-b^2}$ is a C^∞ function of b. Hence, x is C^∞ on U_3.

On U_4,

$$\left(x \circ \phi_4^{-1}\right)(b) = x\left(-\sqrt{1-b^2}, b\right) = -\sqrt{1-b^2},$$

which is C^∞ because b is not equal to ± 1. Since x is C^∞ on the four open sets U_1, U_2, U_3, and U_4, which cover S^1, x is C^∞ on S^1.

The proof that y is C^∞ on S^1 is similar.

Armed with the definition of a smooth map between manifolds, we can define a Lie group.

Definition 6.20. A *Lie group*[1] is a C^∞ manifold G having a group structure such that the multiplication map

$$\mu \colon G \times G \to G$$

and the inverse map

$$\iota \colon G \to G, \quad \iota(x) = x^{-1},$$

are both C^∞.

Similarly, a *topological group* is a topological space having a group structure such that the multiplication and inverse maps are both continuous. Note that a topological group is required to be a topological space, but not a topological manifold.

Examples.

(i) The Euclidean space $\mathbb{R}^n$ is a Lie group under addition.
(ii) The set $\mathbb{C}^\times$ of nonzero complex numbers is a Lie group under multiplication.
(iii) The unit circle S^1 in $\mathbb{C}^\times$ is a Lie group under multiplication.
(iv) The Cartesian product $G_1 \times G_2$ of two Lie groups (G_1, μ_1) and (G_2, μ_2) is a Lie group under coordinatewise multiplication $\mu_1 \times \mu_2$.

Example 6.21 (*General linear group*). In Example 5.15 we defined the general linear group

$$\mathrm{GL}(n,\mathbb{R}) = \{A = [a_{ij}] \in \mathbb{R}^{n\times n} \mid \det A \neq 0\}.$$

As an open subset of $\mathbb{R}^{n\times n}$, it is a manifold. Since the (i,j)-entry of the product of two matrices A and B in $\mathrm{GL}(n,\mathbb{R})$,

$$(AB)_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj},$$

is a polynomial in the coordinates of A and B, matrix multiplication

$$\mu \colon \mathrm{GL}(n,\mathbb{R}) \times \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$$

is a C^∞ map.

Recall that the (i,j)-*minor* of a matrix A is the determinant of the submatrix of A obtained by deleting the ith row and the jth column of A. By Cramer's rule from linear algebra, the (i,j)-entry of A^{-1} is

$$(A^{-1})_{ij} = \frac{1}{\det A} \cdot (-1)^{i+j}((j,i)\text{-minor of } A),$$

which is a C^∞ function of the a_{ij}'s provided $\det A \neq 0$. Therefore, the inverse map $\iota \colon \mathrm{GL}(n,\mathbb{R}) \to \mathrm{GL}(n,\mathbb{R})$ is also C^∞. This proves that $\mathrm{GL}(n,\mathbb{R})$ is a Lie group.

[1] Lie groups and Lie algebras are named after the Norwegian mathematician Sophus Lie (1842–1899). In this context, "Lie" is pronounced "lee," not "lye."

In Section 15 we will study less obvious examples of Lie groups.

NOTATION. The notation for matrices presents a special challenge. An $n \times n$ matrix A can represent a linear transformation $y = Ax$, with $x, y \in \mathbb{R}^n$. In this case, $y^i = \sum_j a^i_j x^j$, so $A = [a^i_j]$. An $n \times n$ matrix can also represent a bilinear form $\langle x, y \rangle = x^T A y$ with $x, y \in \mathbb{R}^n$. In this case, $\langle x, y \rangle = \sum_{i,j} x^i a_{ij} y^j$, so $A = [a_{ij}]$. In the absence of any context, we will write a matrix as $A = [a_{ij}]$, using a lowercase letter a to denote an entry of a matrix A and using a double subscript $(\ )_{ij}$ to denote the (i, j)-entry.

6.6 Partial Derivatives

On a manifold M of dimension n, let (U, ϕ) be a chart and f a C^∞ function As a function into $\mathbb{R}^n$, ϕ has n components $x^1, \dots, x^n$. This means that if $r^1, \dots, r^n$ are the standard coordinates on $\mathbb{R}^n$, then $x^i = r^i \circ \phi$. For $p \in U$, we define the *partial derivative* $\partial f/\partial x^i$ *of* f *with respect to* x^i *at* p to be

$$\left.\frac{\partial}{\partial x^i}\right|_p f := \frac{\partial f}{\partial x^i}(p) := \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}(\phi(p)) := \left.\frac{\partial}{\partial r^i}\right|_{\phi(p)} \left(f \circ \phi^{-1}\right).$$

Since $p = \phi^{-1}(\phi(p))$, this equation may be rewritten in the form

$$\frac{\partial f}{\partial x^i}\left(\phi^{-1}(\phi(p))\right) = \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}(\phi(p)).$$

Thus, as functions on $\phi(U)$,

$$\frac{\partial f}{\partial x^i} \circ \phi^{-1} = \frac{\partial\left(f \circ \phi^{-1}\right)}{\partial r^i}.$$

The partial derivative $\partial f/\partial x^i$ is C^∞ on U because its pullback $(\partial f/\partial x^i) \circ \phi^{-1}$ is C^∞ on $\phi(U)$.

In the next proposition we see that partial derivatives on a manifold satisfy the same duality property $\partial r^i/\partial r^j = \delta^i_j$ as the coordinate functions r^i on $\mathbb{R}^n$.

Proposition 6.22. *Suppose* $(U, x^1, \dots, x^n)$ *is a chart on a manifold. Then* $\partial x^i/\partial x^j = \delta^i_j$.

Proof. At a point $p \in U$, by the definition of $\partial/\partial x^j|_p$,

$$\frac{\partial x^i}{\partial x^j}(p) = \frac{\partial\left(x^i \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial\left(r^i \circ \phi \circ \phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial r^i}{\partial r^j}(\phi(p)) = \delta^i_j. \quad \square$$

Definition 6.23. Let $F\colon N \to M$ be a smooth map, and let $(U, \phi) = (U, x^1, \dots, x^n)$ and $(V, \psi) = (V, y^1, \dots, y^m)$ be charts on N and M respectively such that $F(U) \subset V$. Denote by

$$F^i := y^i \circ F = r^i \circ \psi \circ F\colon U \to \mathbb{R}$$

the ith component of F in the chart (V,ψ). Then the matrix $[\partial F^i/\partial x^j]$ is called the *Jacobian matrix* of F relative to the charts (U,ϕ) and (V,ψ). In case N and M have the same dimension, the determinant $\det[\partial F^i/\partial x^j]$ is called the *Jacobian determinant* of F relative to the two charts. The Jacobian determinant is also written as $\partial(F^1,\dots,F^n)/\partial(x^1,\dots,x^n)$.

When M and N are open subsets of Euclidean spaces and the charts are $(U,r^1,\dots,r^n)$ and $(V,r^1,\dots,r^m)$, the Jacobian matrix $[\partial F^i/\partial r^j]$, where $F^i = r^i \circ F$, is the usual Jacobian matrix from calculus.

Example 6.24 (*Jacobian matrix of a transition map*). Let $(U,\phi) = (U,x^1,\dots,x^n)$ and $(V,\psi) = (V,y^1,\dots,y^n)$ be overlapping charts on a manifold M. The transition map $\psi\circ\phi^{-1}\colon \phi(U\cap V) \to \psi(U\cap V)$ is a diffeomorphism of open subsets of $\mathbb{R}^n$. Show that its Jacobian matrix $J(\psi\circ\phi^{-1})$ at $\phi(p)$ is the matrix $[\partial y^i/\partial x^j]$ of partial derivatives at p.

Solution. By definition, $J(\psi\circ\phi^{-1}) = [\partial(\psi\circ\phi^{-1})^i/\partial r^j]$, where

$$\frac{\partial\left(\psi\circ\phi^{-1}\right)^i}{\partial r^j}(\phi(p)) = \frac{\partial\left(r^i\circ\psi\circ\phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial\left(y^i\circ\phi^{-1}\right)}{\partial r^j}(\phi(p)) = \frac{\partial y^i}{\partial x^j}(p).$$

6.7 The Inverse Function Theorem

By Proposition 6.11, any diffeomorphism $F\colon U \to F(U) \subset \mathbb{R}^n$ of an open subset U of a manifold may be thought of as a coordinate system on U. We say that a C^∞ map $F\colon N \to M$ is *locally invertible* or a *local diffeomorphism* at $p \in N$ if p has a neighborhood U on which $F|_U\colon U \to F(U)$ is a diffeomorphism.

Given n smooth functions $F^1,\dots,F^n$ in a neighborhood of a point p in a manifold N of dimension n, one would like to know whether they form a coordinate system, possibly on a smaller neighborhood of p. This is equivalent to whether $F = (F^1,\dots,F^n)\colon N \to \mathbb{R}^n$ is a local diffeomorphism at p. The inverse function theorem provides an answer.

Theorem 6.25 (Inverse function theorem for $\mathbb{R}^n$). *Let $F\colon W \to \mathbb{R}^n$ be a C^∞ map defined on an open subset W of $\mathbb{R}^n$. For any point p in W, the map F is locally invertible at p if and only if the Jacobian determinant* $\det[\partial F^i/\partial r^j(p)]$ *is not zero.*

This theorem is usually proved in an undergraduate course on real analysis. See Appendix B for a discussion of this and related theorems. Because the inverse function theorem for $\mathbb{R}^n$ is a local result, it easily translates to manifolds.

Theorem 6.26 (Inverse function theorem for manifolds). *Let $F\colon N \to M$ be a C^∞ map between two manifolds of the same dimension, and $p \in N$. Suppose for some charts $(U,\phi) = (U,x^1,\dots,x^n)$ about p in N and $(V,\psi) = (V,y^1,\dots,y^n)$ about $F(p)$ in M, $F(U) \subset V$. Set $F^i = y^i \circ F$. Then F is locally invertible at p if and only if its Jacobian determinant* $\det[\partial F^i/\partial x^j(p)]$ *is nonzero.*

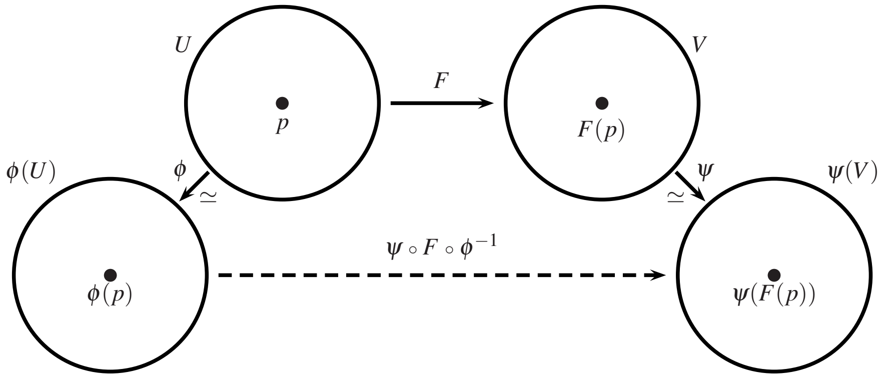


Fig. 6.4. The map F is locally invertible at p because $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$.

Proof. Since $F^i = y^i \circ F = r^i \circ \psi \circ F$, the Jacobian matrix of F relative to the charts (U, ϕ) and (V, ψ) is

$$\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \left[\frac{\partial (r^i \circ \psi \circ F)}{\partial x^j}(p)\right] = \left[\frac{\partial (r^i \circ \psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right],$$

which is precisely the Jacobian matrix at $\phi(p)$ of the map

$$\psi \circ F \circ \phi^{-1} \colon \mathbb{R}^n \supset \phi(U) \to \psi(V) \subset \mathbb{R}^n$$

between two open subsets of $\mathbb{R}^n$. By the inverse function theorem for $\mathbb{R}^n$,

$$\det\left[\frac{\partial F^i}{\partial x^j}(p)\right] = \det\left[\frac{\partial r^i \circ (\psi \circ F \circ \phi^{-1})}{\partial r^j}(\phi(p))\right] \neq 0$$

if and only if $\psi \circ F \circ \phi^{-1}$ is locally invertible at $\phi(p)$. Since ψ and ϕ are diffeomorphisms (Proposition 6.10), this last statement is equivalent to the local invertibility of F at p (see Figure 6.4). □

We usually apply the inverse function theorem in the following form.

Corollary 6.27. *Let N be a manifold of dimension n. A set of n smooth functions $F^1, \ldots, F^n$ defined on a coordinate neighborhood $(U, x^1, \ldots, x^n)$ of a point $p \in N$ forms a coordinate system about p if and only if the Jacobian determinant* $\det[\partial F^i / \partial x^j(p)]$ *is nonzero.*

Proof. Let $F = (F^1, \ldots, F^n) \colon U \to \mathbb{R}^n$. Then

$\det[\partial F^i / \partial x^j(p)] \neq 0$

$\iff$ $F \colon U \to \mathbb{R}^n$ is locally invertible at p (by the inverse function theorem)

$\iff$ there is a neighborhood W of p in N such that $F \colon W \to F(W)$ is a diffeomorphism (by the definition of local invertibility)

$\iff$ $(W, F^1, \ldots, F^n)$ is a coordinate chart about p in the differentiable structure of N (by Proposition 6.11). $\square$

Example. Find all points in $\mathbb{R}^2$ in a neighborhood of which the functions x^2+y^2-1, y can serve as a local coordinate system.

Solution. Define $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ by

$$F(x,y) = \left(x^2+y^2-1, y\right).$$

The map F can serve as a coordinate map in a neighborhood of p if and only if it is a local diffeomorphism at p. The Jacobian determinant of F is

$$\frac{\partial\left(F^1, F^2\right)}{\partial(x,y)} = \det\begin{bmatrix} 2x & 2y \\ 0 & 1 \end{bmatrix} = 2x.$$

By the inverse function theorem, F is a local diffeomorphism at $p = (x,y)$ if and only if $x \neq 0$. Thus, F can serve as a coordinate system at any point p not on the y-axis.

Problems

6.1. Differentiable structures on $\mathbb{R}$

Let $\mathbb{R}$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \phi = \mathbb{1}\colon \mathbb{R} \to \mathbb{R})$, and let $\mathbb{R}'$ be the real line with the differentiable structure given by the maximal atlas of the chart $(\mathbb{R}, \psi\colon \mathbb{R} \to \mathbb{R})$, where $\psi(x) = x^{1/3}$.

(a) Show that these two differentiable structures are distinct.

(b) Show that there is a diffeomorphism between $\mathbb{R}$ and $\mathbb{R}'$. (*Hint*: The identity map $\mathbb{R} \to \mathbb{R}$ is not the desired diffeomorphism; in fact, this map is not smooth.)

6.2. The smoothness of an inclusion map

Let M and N be manifolds and let q_0 be a point in N. Prove that the inclusion map $i_{q_0}\colon M \to M \times N$, $i_{q_0}(p) = (p, q_0)$, is C^∞.

6.3.* Group of automorphisms of a vector space

Let V be a finite-dimensional vector space over $\mathbb{R}$, and $\mathrm{GL}(V)$ the group of all linear automorphisms of V. Relative to an ordered basis $e = (e_1, \ldots, e_n)$ for V, a linear automorphism $L \in \mathrm{GL}(V)$ is represented by a matrix $[a^i_j]$ defined by

$$L(e_j) = \sum_i a^i_j e_i.$$

The map

$$\phi_e\colon \mathrm{GL}(V) \to \mathrm{GL}(n, \mathbb{R}),$$
$$L \mapsto [a^i_j],$$

is a bijection with an open subset of $\mathbb{R}^{n\times n}$ that makes $\mathrm{GL}(V)$ into a C^∞ manifold, which we denote temporarily by $\mathrm{GL}(V)_e$. If $\mathrm{GL}(V)_u$ is the manifold structure induced from another ordered basis $u = (u_1, \ldots, u_n)$ for V, show that $\mathrm{GL}(V)_e$ is the same as $\mathrm{GL}(V)_u$.

6.4. Local coordinate systems

Find all points in $\mathbb{R}^3$ in a neighborhood of which the functions x, $x^2+y^2+z^2-1$, z can serve as a local coordinate system.

§7 Quotients

Gluing the edges of a malleable square is one way to create new surfaces. For example, gluing together the top and bottom edges of a square gives a cylinder; gluing together the boundaries of the cylinder with matching orientations gives a torus (Figure 7.1). This gluing process is called an *identification* or a *quotient construction*.

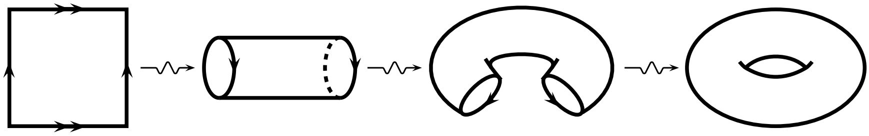


Fig. 7.1. Gluing the edges of a malleable square.

The quotient construction is a process of simplification. Starting with an equivalence relation on a set, we identify each equivalence class to a point. Mathematics abounds in quotient constructions, for example, the quotient group, quotient ring, or quotient vector space in algebra. If the original set is a topological space, it is always possible to give the quotient set a topology so that the natural projection map becomes continuous. However, even if the original space is a manifold, a quotient space is often not a manifold. The main results of this section give conditions under which a quotient space remains second countable and Hausdorff. We then study real projective space as an example of a quotient manifold.

Real projective space can be interpreted as a quotient of a sphere with antipodal points identified, or as the set of lines through the origin in a vector space. These two interpretations give rise to two distinct generalizations—covering maps on the one hand and Grassmannians of k-dimensional subspaces of a vector space on the other. In one of the exercises, we carry out an extensive investigation of $G(2,4)$, the Grassmannian of 2-dimensional subspaces of $\mathbb{R}^4$.

7.1 The Quotient Topology

Recall that an equivalence relation on a set S is a reflexive, symmetric, and transitive relation. The *equivalence class* $[x]$ of $x \in S$ is the set of all elements in S equivalent to x. An equivalence relation on S partitions S into disjoint equivalence classes. We denote the set of equivalence classes by $S/\sim$ and call this set the *quotient* of S by the equivalence relation $\sim$. There is a natural *projection map* $\pi\colon S \to S/\sim$ that sends $x \in S$ to its equivalence class $[x]$.

Assume now that S is a topological space. We define a topology on $S/\sim$ by declaring a set U in $S/\sim$ to be *open* if and only if $\pi^{-1}(U)$ is open in S. Clearly, both the empty set $\varnothing$ and the entire quotient $S/\sim$ are open. Further, since

$$\pi^{-1}\left(\bigcup_\alpha U_\alpha\right) = \bigcup_\alpha \pi^{-1}(U_\alpha)$$

and

$$\pi^{-1}\left(\bigcap_i U_i\right) = \bigcap_i \pi^{-1}(U_i),$$

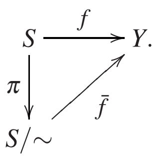
the collection of open sets in $S/\sim$ is closed under arbitrary unions and finite intersections, and is therefore a topology. It is called the *quotient topology* on $S/\sim$. With this topology, $S/\sim$ is called the *quotient space* of S by the equivalence relation $\sim$. With the quotient topology on $S/\sim$, the projection map $\pi\colon S \to S/\sim$ is automatically continuous, because the inverse image of an open set in $S/\sim$ is by definition open in S.

7.2 Continuity of a Map on a Quotient

Let $\sim$ be an equivalence relation on the topological space S and give $S/\sim$ the quotient topology. Suppose a function $f\colon S \to Y$ from S to another topological space Y is constant on each equivalence class. Then it induces a map $\bar{f}\colon S/\sim \to Y$ by

$$\bar{f}([p]) = f(p) \quad \text{for } p \in S.$$

In other words, there is a commutative diagram



Proposition 7.1. *The induced map $\bar{f}\colon S/\sim \to Y$ is continuous if and only if the map $f\colon S \to Y$ is continuous.*

Proof.
($\Rightarrow$) If $\bar{f}$ is continuous, then as the composite $\bar{f}\circ\pi$ of continuous functions, f is also continuous.

($\Leftarrow$) Suppose f is continuous. Let V be open in Y. Then $f^{-1}(V) = \pi^{-1}(\bar{f}^{-1}(V))$ is open in S. By the definition of quotient topology, $\bar{f}^{-1}(V)$ is open in $S/\sim$. Since V was arbitrary, $\bar{f}\colon S/\sim \to Y$ is continuous. $\square$

This proposition gives a useful criterion for checking whether a function $\bar{f}$ on a quotient space $S/\sim$ is continuous: simply lift the function $\bar{f}$ to $f := \bar{f}\circ\pi$ on S and check the continuity of the lifted map f on S. For examples of this, see Example 7.2 and Proposition 7.3.

7.3 Identification of a Subset to a Point

If A is a subspace of a topological space S, we can define a relation $\sim$ on S by declaring

$$x \sim x \quad \text{for all } x \in S$$

(so the relation is reflexive) and

$$x \sim y \quad \text{for all } x, y \in A.$$

This is an equivalence relation on S. We say that the quotient space $S/\!\sim$ is obtained from S by *identifying A to a point*.

Example 7.2. Let I be the unit interval $[0,1]$ and $I/\!\sim$ the quotient space obtained from I by identifying the two points $\{0,1\}$ to a point. Denote by S^1 the unit circle in the complex plane. The function $f\colon I \to S^1$, $f(x) = \exp(2\pi i x)$, assumes the same value at 0 and 1 (Figure 7.2), and so induces a function $\bar{f}\colon I/\!\sim \to S^1$.

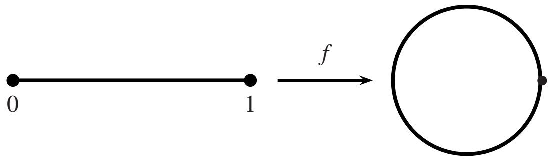


Fig. 7.2. The unit circle as a quotient space of the unit interval.

Proposition 7.3. *The function $\bar{f}\colon I/\!\sim \to S^1$ is a homeomorphism.*

Proof. Since f is continuous, $\bar{f}$ is also continuous by Proposition 7.1. Clearly, $\bar{f}$ is a bijection. As the continuous image of the compact set I, the quotient $I/\!\sim$ is compact. Thus, $\bar{f}$ is a continuous bijection from the compact space $I/\!\sim$ to the Hausdorff space S^1. By Corollary A.36, $\bar{f}$ is a homeomorphism. □

7.4 A Necessary Condition for a Hausdorff Quotient

The quotient construction does not in general preserve the Hausdorff property or second countability. Indeed, since every singleton set in a Hausdorff space is closed, if $\pi\colon S \to S/\!\sim$ is the projection and the quotient $S/\!\sim$ is Hausdorff, then for any $p \in S$, its image $\{\pi(p)\}$ is closed in $S/\!\sim$. By the continuity of π, the inverse image $\pi^{-1}(\{\pi(p)\}) = [p]$ is closed in S. This gives a necessary condition for a quotient space to be Hausdorff.

Proposition 7.4. *If the quotient space $S/\!\sim$ is Hausdorff, then the equivalence class $[p]$ of any point p in S is closed in S.*

Example. Define an equivalence relation $\sim$ on $\mathbb{R}$ by identifying the open interval $]0,\infty[$ to a point. Then the quotient space $\mathbb{R}/\sim$ is not Hausdorff because the equivalence class $]0,\infty[$ of $\sim$ in $\mathbb{R}$ corresponding to the point $]0,\infty[$ in $\mathbb{R}/\sim$ is not a closed subset of $\mathbb{R}$.

7.5 Open Equivalence Relations

In this section we follow the treatment of Boothby [3] and derive conditions under which a quotient space is Hausdorff or second countable. Recall that a map $f\colon X \to Y$ of topological spaces is *open* if the image of any open set under f is open.

Definition 7.5. An equivalence relation $\sim$ on a topological space S is said to be *open* if the projection map $\pi\colon S \to S/\sim$ is open.

In other words, the equivalence relation $\sim$ on S is open if and only if for every open set U in S, the set

$$\pi^{-1}(\pi(U)) = \bigcup_{x\in U} [x]$$

of all points equivalent to some point of U is open.

Example 7.6. The projection map to a quotient space is in general not open. For example, let $\sim$ be the equivalence relation on the real line $\mathbb{R}$ that identifies the two points 1 and -1, and $\pi\colon \mathbb{R} \to \mathbb{R}/\sim$ the projection map.

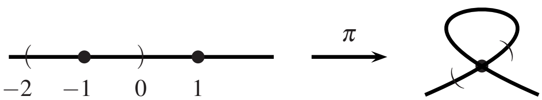


Fig. 7.3. A projection map that is not open.

The map π is open if and only if for every open set V in $\mathbb{R}$, its image $\pi(V)$ is open in $\mathbb{R}/\sim$, which by the definition of the quotient topology means that $\pi^{-1}(\pi(V))$ is open in $\mathbb{R}$. Now let V be the open interval $]-2,0[$ in $\mathbb{R}$. Then

$$\pi^{-1}(\pi(V)) =]-2,0[\cup \{1\},$$

which is not open in $\mathbb{R}$ (Figure 7.3). Therefore, the projection map $\pi\colon \mathbb{R} \to \mathbb{R}/\sim$ is not an open map.

Given an equivalence relation $\sim$ on S, let R be the subset of $S \times S$ that defines the relation

$$R = \{(x,y) \in S \times S \mid x \sim y\}.$$

We call R the *graph* of the equivalence relation $\sim$.

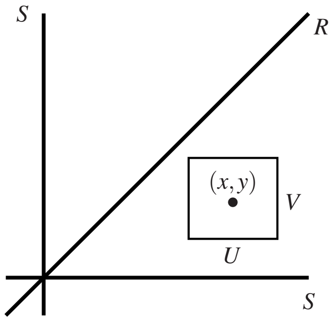


Fig. 7.4. The graph R of an equivalence relation and an open set $U \times V$ disjoint from R.

Theorem 7.7. *Suppose $\sim$ is an open equivalence relation on a topological space S. Then the quotient space $S/\!\sim$ is Hausdorff if and only if the graph R of $\sim$ is closed in $S \times S$.*

Proof. There is a sequence of equivalent statements:

R is closed in $S \times S$
$\iff$ $(S \times S) - R$ is open in $S \times S$
$\iff$ for every $(x,y) \in S \times S - R$, there is a basic open set $U \times V$ containing (x,y) such that $(U \times V) \cap R = \varnothing$ (Figure 7.4)
$\iff$ for every pair $x \nsim y$ in S, there exist neighborhoods U of x and V of y in S such that no element of U is equivalent to an element of V
$\iff$ for any two points $[x] \neq [y]$ in $S/\!\sim$, there exist neighborhoods U of x and V of y in S such that $\pi(U) \cap \pi(V) = \varnothing$ in $S/\!\sim$. $(*)$

We now show that this last statement $(*)$ is equivalent to $S/\!\sim$ being Hausdorff. First assume $(*)$. Since $\sim$ is an open equivalence relation, $\pi(U)$ and $\pi(V)$ are disjoint open sets in $S/\!\sim$ containing $[x]$ and $[y]$ respectively. Therefore, $S/\!\sim$ is Hausdorff.

Conversely, suppose $S/\!\sim$ is Hausdorff. Let $[x] \neq [y]$ in $S/\!\sim$. Then there exist disjoint open sets A and B in $S/\!\sim$ such that $[x] \in A$ and $[y] \in B$. By the surjectivity of π, we have $A = \pi(\pi^{-1}A)$ and $B = \pi(\pi^{-1}B)$ (see Problem 7.1). Let $U = \pi^{-1}A$ and $V = \pi^{-1}B$. Then $x \in U$, $y \in V$, and $A = \pi(U)$ and $B = \pi(V)$ are disjoint open sets in $S/\!\sim$. □

If the equivalence relation $\sim$ is equality, then the quotient space $S/\!\sim$ is S itself and the graph R of $\sim$ is simply the diagonal

$$\Delta = \{(x,x) \in S \times S\}.$$

In this case, Theorem 7.7 becomes the following well-known characterization of a Hausdorff space by its diagonal (cf. Problem A.6).

Corollary 7.8. *A topological space S is Hausdorff if and only if the diagonal Δ in $S \times S$ is closed.*

Theorem 7.9. *Let $\sim$ be an open equivalence relation on a topological space S with projection $\pi: S \to S/\sim$. If $\mathcal{B} = \{B_\alpha\}$ is a basis for S, then its image $\{\pi(B_\alpha)\}$ under π is a basis for $S/\sim$.*

Proof. Since π is an open map, $\{\pi(B_\alpha)\}$ is a collection of open sets in $S/\sim$. Let W be an open set in $S/\sim$ and $[x] \in W$, $x \in S$. Then $x \in \pi^{-1}(W)$. Since $\pi^{-1}(W)$ is open, there is a basic open set $B \in \mathcal{B}$ such that

$$x \in B \subset \pi^{-1}(W).$$

Then

$$[x] = \pi(x) \in \pi(B) \subset W,$$

which proves that $\{\pi(B_\alpha)\}$ is a basis for $S/\sim$. □

Corollary 7.10. *If $\sim$ is an open equivalence relation on a second-countable space S, then the quotient space $S/\sim$ is second countable.*

7.6 Real Projective Space

Define an equivalence relation on $\mathbb{R}^{n+1} - \{0\}$ by

$$x \sim y \iff y = tx \text{ for some nonzero real number } t,$$

where x, $y \in \mathbb{R}^{n+1} - \{0\}$. The *real projective space* $\mathbb{R}P^n$ is the quotient space of $\mathbb{R}^{n+1} - \{0\}$ by this equivalence relation. We denote the equivalence class of a point $(a^0, \ldots, a^n) \in \mathbb{R}^{n+1} - \{0\}$ by $[a^0, \ldots, a^n]$ and let $\pi: \mathbb{R}^{n+1} - \{0\} \to \mathbb{R}P^n$ be the projection. We call $[a^0, \ldots, a^n]$ *homogeneous coordinates* on $\mathbb{R}P^n$.

Geometrically, two nonzero points in $\mathbb{R}^{n+1}$ are equivalent if and only if they lie on the same line through the origin, so $\mathbb{R}P^n$ can be interpreted as the set of all lines through the origin in $\mathbb{R}^{n+1}$. Each line through the origin in $\mathbb{R}^{n+1}$ meets the unit

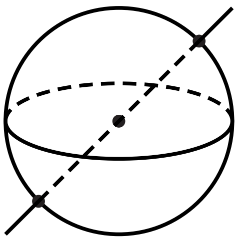


Fig. 7.5. A line through 0 in $\mathbb{R}^3$ corresponds to a pair of antipodal points on S^2.

sphere S^n in a pair of antipodal points, and conversely, a pair of antipodal points on S^n determines a unique line through the origin (Figure 7.5). This suggests that we define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

We then have a bijection $\mathbb{R}P^n \leftrightarrow S^n/\sim$.

Exercise 7.11 (Real projective space as a quotient of a sphere).* For $x = (x^1, \ldots, x^n) \in \mathbb{R}^n$, let $\|x\| = \sqrt{\sum_i (x^i)^2}$ be the modulus of x. Prove that the map $f \colon \mathbb{R}^{n+1} - \{0\} \to S^n$ given by

$$f(x) = \frac{x}{\|x\|}$$

induces a homeomorphism $\bar{f} \colon \mathbb{R}P^n \to S^n/\sim$. (*Hint*: Find an inverse map

$$\bar{g} \colon S^n/\sim \to \mathbb{R}P^n$$

and show that both $\bar{f}$ and $\bar{g}$ are continuous.)

Example 7.12 (*The real projective line* $\mathbb{R}P^1$).

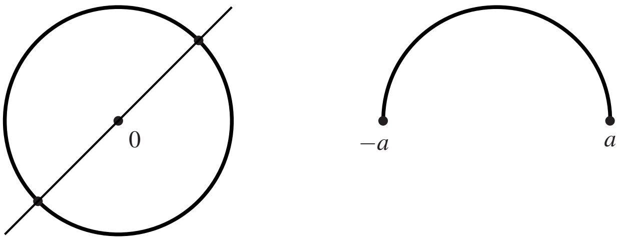


Fig. 7.6. The real projective line $\mathbb{R}P^1$ as the set of lines through 0 in $\mathbb{R}^2$.

Each line through the origin in $\mathbb{R}^2$ meets the unit circle in a pair of antipodal points. By Exercise 7.11, $\mathbb{R}P^1$ is homeomorphic to the quotient $S^1/\sim$, which is in turn homeomorphic to the closed upper semicircle with the two endpoints identified (Figure 7.6). Thus, $\mathbb{R}P^1$ is homeomorphic to S^1.

Example 7.13 (*The real projective plane* $\mathbb{R}P^2$). By Exercise 7.11, there is a homeomorphism

$$\mathbb{R}P^2 \simeq S^2/\{\text{antipodal points}\} = S^2/\sim.$$

For points not on the equator, each pair of antipodal points contains a unique point in the upper hemisphere. Thus, there is a bijection between $S^2/\sim$ and the quotient of the closed upper hemisphere in which each pair of antipodal points on the equator is identified. It is not difficult to show that this bijection is a homeomorphism (see Problem 7.2).

Let H^2 be the closed upper hemisphere

$$H^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1,\ z \geq 0\}$$

and let D^2 be the closed unit disk

$$D^2 = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq 1\}.$$

These two spaces are homeomorphic to each other via the continuous map

$$\varphi\colon H^2 \to D^2,$$
$$\varphi(x,y,z) = (x,y),$$

and its inverse

$$\psi\colon D^2 \to H^2,$$
$$\psi(x,y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right).$$

On H^2, define an equivalence relation $\sim$ by identifying the antipodal points on the equator:

$$(x,y,0) \sim (-x,-y,0), \quad x^2 + y^2 = 1.$$

On D^2, define an equivalence relation $\sim$ by identifying the antipodal points on the boundary circle:

$$(x,y) \sim (-x,-y), \quad x^2 + y^2 = 1.$$

Then φ and ψ induce homeomorphisms

$$\bar{\varphi}\colon H^2/\!\sim \to D^2/\!\sim, \quad \bar{\psi}\colon D^2/\!\sim \to H^2/\!\sim .$$

In summary, there is a sequence of homeomorphisms

$$\mathbb{R}P^2 \overset{\sim}{\to} S^2/\!\sim \overset{\sim}{\to} H^2/\!\sim \overset{\sim}{\to} D^2/\!\sim$$

that identifies the real projective plane as the quotient of the closed disk D^2 with the antipodal points on its boundary identified. This may be the best way to picture $\mathbb{R}P^2$ (Figure 7.7).

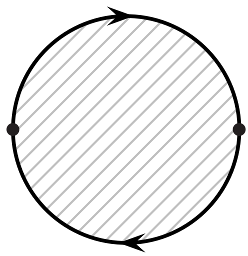


Fig. 7.7. The real projective plane as the quotient of a disk.

The real projective plane $\mathbb{R}P^2$ cannot be embedded as a submanifold of $\mathbb{R}^3$. However, if we allow self-intersection, then we can map $\mathbb{R}P^2$ into $\mathbb{R}^3$ as a cross-cap (Figure 7.8). This map is not one-to-one.

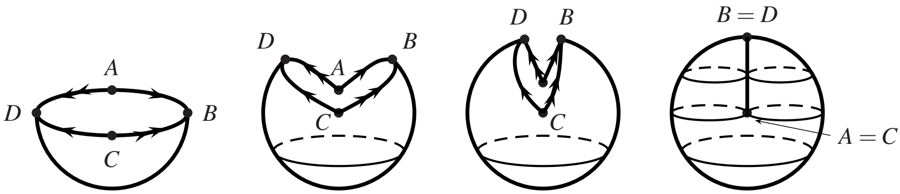


Fig. 7.8. The real projective plane immersed as a cross-cap in $\mathbb{R}^3$.

Proposition 7.14. *The equivalence relation* $\sim$ *on* $\mathbb{R}^{n+1}-\{0\}$ *in the definition of* $\mathbb{R}P^n$ *is an open equivalence relation.*

Proof. For an open set $U \subset \mathbb{R}^{n+1}-\{0\}$, the image $\pi(U)$ is open in $\mathbb{R}P^n$ if and only if $\pi^{-1}(\pi(U))$ is open in $\mathbb{R}^{n+1}-\{0\}$. But $\pi^{-1}(\pi(U))$ consists of all nonzero scalar multiples of points of U; that is,

$$\pi^{-1}(\pi(U)) = \bigcup_{t\in\mathbb{R}^\times} tU = \bigcup_{t\in\mathbb{R}^\times} \{tp \mid p \in U\}.$$

Since multiplication by $t \in \mathbb{R}^\times$ is a homeomorphism of $\mathbb{R}^{n+1}-\{0\}$, the set tU is open for any t. Therefore, their union $\bigcup_{t\in\mathbb{R}^\times} tU = \pi^{-1}(\pi(U))$ is also open. □

Corollary 7.15. *The real projective space* $\mathbb{R}P^n$ *is second countable.*

Proof. Apply Corollary 7.10. □

Proposition 7.16. *The real projective space* $\mathbb{R}P^n$ *is Hausdorff.*

Proof. Let $S = \mathbb{R}^{n+1}-\{0\}$ and consider the set

$$R = \{(x,y) \in S\times S \mid y = tx \text{ for some } t \in \mathbb{R}^\times\}.$$

If we write x and y as column vectors, then $[x\ y]$ is an $(n+1)\times 2$ matrix, and R may be characterized as the set of matrices $[x\ y]$ in $S\times S$ of rank ≤ 1. By a standard fact from linear algebra, $\operatorname{rk}[x\ y] \le 1$ is equivalent to the vanishing of all 2×2 minors of $[x\ y]$ (see Problem B.1). As the zero set of finitely many polynomials, R is a closed subset of $S\times S$. Since $\sim$ is an open equivalence relation on S, and R is closed in $S\times S$, by Theorem 7.7 the quotient $S/\sim\ \simeq \mathbb{R}P^n$ is Hausdorff. □

7.7 The Standard C^∞ Atlas on a Real Projective Space

Let $[a^0,\dots,a^n]$ be homogeneous coordinates on the projective space $\mathbb{R}P^n$. Although a^0 is not a well-defined function on $\mathbb{R}P^n$, the condition $a^0 \ne 0$ is independent of the choice of a representative for $[a^0,\dots,a^n]$. Hence, the condition $a^0 \ne 0$ makes sense on $\mathbb{R}P^n$, and we may define

$$U_0 := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^0 \neq 0\}.$$

Similarly, for each $i = 1, \dots, n$, let

$$U_i := \{[a^0, \dots, a^n] \in \mathbb{R}P^n \mid a^i \neq 0\}.$$

Define

$$\phi_0 \colon U_0 \to \mathbb{R}^n$$

by

$$[a^0, \dots, a^n] \mapsto \left(\frac{a^1}{a^0}, \dots, \frac{a^n}{a^0} \right).$$

This map has a continuous inverse

$$(b^1, \dots, b^n) \mapsto [1, b^1, \dots, b^n]$$

and is therefore a homeomorphism. Similarly, there are homeomorphisms for each $i = 1, \dots, n$:

$$\phi_i \colon U_i \to \mathbb{R}^n,$$
$$[a^0, \dots, a^n] \mapsto \left(\frac{a^0}{a^i}, \dots, \widehat{\frac{a^i}{a^i}}, \dots, \frac{a^n}{a^i} \right),$$

where the caret sign $\widehat{\ }$ over a^i/a^i means that that entry is to be omitted. This proves that $\mathbb{R}P^n$ is locally Euclidean with the (U_i, ϕ_i) as charts.

On the intersection $U_0 \cap U_1$, we have $a^0 \neq 0$ and $a^1 \neq 0$, and there are two coordinate systems

$$[a^0, a^1, a^2, \dots, a^n]$$
$$\phi_0 \swarrow \qquad \searrow \phi_1$$
$$\left(\frac{a^1}{a^0}, \frac{a^2}{a^0}, \dots, \frac{a^n}{a^0} \right) \qquad \qquad \left(\frac{a^0}{a^1}, \frac{a^2}{a^1}, \dots, \frac{a^n}{a^1} \right).$$

We will refer to the coordinate functions on U_0 as $x^1, \dots, x^n$, and the coordinate functions on U_1 as $y^1, \dots, y^n$. On U_0,

$$x^i = \frac{a^i}{a^0}, \quad i = 1, \dots, n,$$

and on U_1,

$$y^1 = \frac{a^0}{a^1}, \quad y^2 = \frac{a^2}{a^1}, \quad \dots, \quad y^n = \frac{a^n}{a^1}.$$

Then on $U_0 \cap U_1$,

$$y^1 = \frac{1}{x^1}, \quad y^2 = \frac{x^2}{x^1}, \quad y^3 = \frac{x^3}{x^1}, \quad \ldots, \quad y^n = \frac{x^n}{x^1},$$

so

$$(\phi_1 \circ \phi_0^{-1})(x) = \left(\frac{1}{x^1}, \frac{x^2}{x^1}, \frac{x^3}{x^1}, \ldots, \frac{x^n}{x^1} \right).$$

This is a C^∞ function because $x^1 \neq 0$ on $\phi_0(U_0 \cap U_1)$. On any other $U_i \cap U_j$ an analogous formula holds. Therefore, the collection $\{(U_i, \phi_i)\}_{i=0,\ldots,n}$ is a C^∞ atlas for $\mathbb{R}P^n$, called the *standard atlas*. This concludes the proof that $\mathbb{R}P^n$ is a C^∞ manifold.

Problems

7.1. Image of the inverse image of a map
Let $f\colon X \to Y$ be a map of sets, and let $B \subset Y$. Prove that $f(f^{-1}(B)) = B \cap f(X)$. Therefore, if f is surjective, then $f(f^{-1}(B)) = B$.

7.2. Real projective plane
Let H^2 be the closed upper hemisphere in the unit sphere S^2, and let $i\colon H^2 \to S^2$ be the inclusion map. In the notation of Example 7.13, prove that the induced map $f\colon H^2/{\sim} \to S^2/{\sim}$ is a homeomorphism. (*Hint*: Imitate Proposition 7.3.)

7.3. Closedness of the diagonal of a Hausdorff space
Deduce Theorem 7.7 from Corollary 7.8. (*Hint*: To prove that if $S/{\sim}$ is Hausdorff, then the graph R of $\sim$ is closed in $S \times S$, use the continuity of the projection map $\pi\colon S \to S/{\sim}$. To prove the reverse implication, use the openness of π.)

7.4.* Quotient of a sphere with antipodal points identified
Let S^n be the unit sphere centered at the origin in $\mathbb{R}^{n+1}$. Define an equivalence relation $\sim$ on S^n by identifying antipodal points:

$$x \sim y \iff x = \pm y, \quad x, y \in S^n.$$

(a) Show that $\sim$ is an open equivalence relation.
(b) Apply Theorem 7.7 and Corollary 7.8 to prove that the quotient space $S^n/{\sim}$ is Hausdorff, without making use of the homeomorphism $\mathbb{R}P^n \simeq S^n/{\sim}$.

7.5.* Orbit space of a continuous group action
Suppose a right action of a topological group G on a topological space S is continuous; this simply means that the map $S \times G \to S$ describing the action is continuous. Define two points x, y of S to be equivalent if they are in the same orbit; i.e., there is an element $g \in G$ such that $y = xg$. Let S/G be the quotient space; it is called the *orbit space* of the action. Prove that the projection map $\pi\colon S \to S/G$ is an open map. (This problem generalizes Proposition 7.14, in which $G = R^\times = \mathbb{R} - \{0\}$ and $S = \mathbb{R}^{n+1} - \{0\}$. Because $\mathbb{R}^\times$ is commutative, a left $\mathbb{R}^\times$-action becomes a right $\mathbb{R}^\times$-action if scalar multiplication is written on the right.)

7.6. Quotient of $\mathbb{R}$ by $2\pi\mathbb{Z}$
Let the additive group $2\pi\mathbb{Z}$ act on $\mathbb{R}$ on the right by $x \cdot 2\pi n = x + 2\pi n$, where n is an integer. Show that the orbit space $R/2\pi\mathbb{Z}$ is a smooth manifold.

7.7. The circle as a quotient space

(a) Let $\{(U_\alpha, \phi_\alpha)\}_{\alpha=1}^2$ be the atlas of the circle S^1 in Example 5.7, and let $\bar{\phi}_\alpha$ be the map ϕ_α followed by the projection $\mathbb{R} \to \mathbb{R}/2\pi\mathbb{Z}$. On $U_1 \cap U_2 = A \amalg B$, since ϕ_1 and ϕ_2 differ by an integer multiple of 2π, $\bar{\phi}_1 = \bar{\phi}_2$. Therefore, $\bar{\phi}_1$ and $\bar{\phi}_2$ piece together to give a well-defined map $\bar{\phi}\colon S^1 \to \mathbb{R}/2\pi\mathbb{Z}$. Prove that $\bar{\phi}$ is C^∞.
(b) The complex exponential $\mathbb{R} \to S^1$, $t \mapsto e^{it}$, is constant on each orbit of the action of $2\pi\mathbb{Z}$ on $\mathbb{R}$. Therefore, there is an induced map $F\colon \mathbb{R}/2\pi\mathbb{Z} \to S^1$, $F([t]) = e^{it}$. Prove that F is C^∞.
(c) Prove that $F\colon \mathbb{R}/2\pi\mathbb{Z} \to S^1$ is a diffeomorphism.

7.8. The Grassmannian $G(k,n)$

The Grassmannian $G(k,n)$ is the set of all k-planes through the origin in $\mathbb{R}^n$. Such a k-plane is a linear subspace of dimension k of $\mathbb{R}^n$ and has a basis consisting of k linearly independent vectors $a_1, \dots, a_k$ in $\mathbb{R}^n$. It is therefore completely specified by an $n \times k$ matrix $A = [a_1 \ \cdots \ a_k]$ of rank k, where the *rank* of a matrix A, denoted by $\operatorname{rk} A$, is defined to be the number of linearly independent columns of A. This matrix is called a *matrix representative* of the k-plane. (For properties of the rank, see the problems in Appendix B.)

Two bases $a_1, \dots, a_k$ and $b_1, \dots, b_k$ determine the same k-plane if there is a change-of-basis matrix $g = [g_{ij}] \in \mathrm{GL}(k, \mathbb{R})$ such that

$$b_j = \sum_i a_i g_{ij}, \quad 1 \le i, j \le k.$$

In matrix notation, $B = Ag$.

Let $F(k,n)$ be the set of all $n \times k$ matrices of rank k, topologized as a subspace of $\mathbb{R}^{n\times k}$, and $\sim$ the equivalence relation

$$A \sim B \quad \text{iff} \quad \text{there is a matrix } g \in \mathrm{GL}(k,\mathbb{R}) \text{ such that } B = Ag.$$

In the notation of Problem B.3, $F(k,n)$ is the set $D_{\max}$ in $\mathbb{R}^{n\times k}$ and is therefore an open subset. There is a bijection between $G(k,n)$ and the quotient space $F(k,n)/\sim$. We give the Grassmannian $G(k,n)$ the quotient topology on $F(k,n)/\sim$.

(a) Show that $\sim$ is an open equivalence relation. (*Hint*: Either mimic the proof of Proposition 7.14 or apply Problem 7.5.)
(b) Prove that the Grassmannian $G(k,n)$ is second countable. (*Hint*: Apply Corollary 7.10.)
(c) Let $S = F(k,n)$. Prove that the graph R in $S \times S$ of the equivalence relation $\sim$ is closed. (*Hint*: Two matrices $A = [a_1 \ \cdots \ a_k]$ and $B = [b_1 \ \cdots \ b_k]$ in $F(k,n)$ are equivalent if and only if every column of B is a linear combination of the columns of A if and only if $\operatorname{rk}[A\ B] \le k$ if and only if all $(k+1) \times (k+1)$ minors of $[A\ B]$ are zero.)
(d) Prove that the Grassmannian $G(k,n)$ is Hausdorff. (*Hint*: Mimic the proof of Proposition 7.16.)

Next we want to find a C^∞ atlas on the Grassmannian $G(k,n)$. For simplicity, we specialize to $G(2,4)$. For any 4×2 matrix A, let A_{ij} be the 2×2 submatrix consisting of its ith row and jth row. Define

$$V_{ij} = \{A \in F(2,4) \mid A_{ij} \text{ is nonsingular}\}.$$

Because the complement of V_{ij} in $F(2,4)$ is defined by the vanishing of $\det A_{ij}$, we conclude that V_{ij} is an open subset of $F(2,4)$.

(e) Prove that if $A \in V_{ij}$, then $Ag \in V_{ij}$ for any nonsingular matrix $g \in \mathrm{GL}(2,\mathbb{R})$.

Define $U_{ij} = V_{ij}/\sim$. Since $\sim$ is an open equivalence relation, $U_{ij} = V_{ij}/\sim$ is an open subset of $G(2,4)$.

For $A \in V_{12}$,

$$A \sim AA_{12}^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ * & * \\ * & * \end{bmatrix} = \begin{bmatrix} I \\ A_{34}A_{12}^{-1} \end{bmatrix}.$$

This shows that the matrix representatives of a 2-plane in U_{12} have a canonical form B in which B_{12} is the identity matrix.

(f) Show that the map $\tilde{\phi}_{12} \colon V_{12} \to \mathbb{R}^{2\times 2}$,

$$\tilde{\phi}_{12}(A) = A_{34}A_{12}^{-1},$$

induces a homeomorphism $\phi_{12} \colon U_{12} \to \mathbb{R}^{2\times 2}$.

(g) Define similarly homeomorphisms $\phi_{ij} \colon U_{ij} \to \mathbb{R}^{2\times 2}$. Compute $\phi_{12} \circ \phi_{23}^{-1}$, and show that it is C^∞.

(h) Show that $\{U_{ij} \mid 1 \le i < j \le 4\}$ is an open cover of $G(2,4)$ and that $G(2,4)$ is a smooth manifold.

Similar consideration shows that $F(k,n)$ has an open cover $\{V_I\}$, where I is a strictly ascending multi-index $1 \le i_1 < \cdots < i_k \le n$. For $A \in F(k,n)$, let A_I be the $k \times k$ submatrix of A consisting of i_1th, ..., i_kth rows of A. Define

$$V_I = \{A \in G(k,n) \mid \det A_I \neq 0\}.$$

Next define $\tilde{\phi}_I \colon V_I \to \mathbb{R}^{(n-k)\times k}$ by

$$\tilde{\phi}_I(A) = (AA_I^{-1})_{I'},$$

where $(\ )_{I'}$ denotes the $(n-k) \times k$ submatrix obtained from the complement I' of the multi-index I. Let $U_I = V_I/\sim$. Then $\tilde{\phi}$ induces a homeomorphism $\phi \colon U_I \to \mathbb{R}^{(n-k)\times k}$. It is not difficult to show that $\{(U_I, \phi_I)\}$ is a C^∞ atlas for $G(k,n)$. Therefore the Grassmannian $G(k,n)$ is a C^∞ manifold of dimension $k(n-k)$.

7.9.* Compactness of real projective space

Show that the real projective space $\mathbb{R}P^n$ is compact. (*Hint*: Use Exercise 7.11.)