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# Joseph Liouville (1809–1882)

In 1809, the French mathematician Jean Fourier discovered that a complex wave is the sum of several simple waves; Napoléon Bonaparte was engaged against the British forces in the Peninsular War; and in the little town of Saint Omer, Pas de Calais, France, Joseph Liouville was born on March 24. His father, Claude Joseph, was a captain in Napoléon's army; his mother, Thérèse Balland, was a maternal cousin of Claude's. They came from a distinguished, upper middle-class Lorraine family. The Liouvilles had an impeccable reputation, many of the family serving France honorably in the military, government and the law. The baby Joseph had a brother, Felix, born six years earlier at the family home in Toul, near Nancy.

Because of their father's profession, the family could not always be together and the two Liouville boys spent some of their childhood with an uncle in Vignot, near Commercy, where they attended their first school. Joseph, according to C.-E. Dumont [97], writing in 1843, was no child prodigy. Indeed, his first teacher predicted he would not go far because he played too much. Though this early assessment proved to be very wrong, young Joseph demonstrated his early inclination for mathematics through games, most notably chess.

When Captain Liouville retired the family moved back to Toul. There, Joseph entered the local *collège* where he studied ancient languages. He continued his studies in mathematics at the *Collège St. Louis* in Paris and began sketching his own mathematical ideas, transcribing them into notebooks, a practice he maintained throughout his life. When he graduated from the *Collège St. Louis* at the age of sixteen, he enrolled next in the famous engineering institute *École Polytechnique*, established in 1794 and considered the premier institution of its kind in Europe. His teachers were of the highest calibre and the course of study was rigorous. Liouville studied applied analysis, geometry, mechanics, physics, chemistry, history, topographic drawing, architecture, geodesy, and literature. He was taught mechanical analysis by André Marie Ampère (1775–1836), a

distinguished professor and mentor of Augustin Cauchy (1789–1857), who also taught at the school and was considered the bright light of the mathematical community in Paris.

In 1827, Liouville transferred to the engineering college, *École des Ponts et Chaussées*. During his three years at that institution he distinguished himself by preparing seven memoirs, mainly on the theory of electricity and heat, which he presented to the *Académie des Sciences* where they were well received. Some were published in the current journals of the day and Liouville enjoyed an early reputation as a rising star in the scientific community. It was at this time (1830) that four important things occurred in his life. He was ordered by the school to train "in the field" as an engineer in Normandy; he became ill – perhaps with rheumatism – and returned to Toul on a special leave of absence; his mother died; and he decided to marry his cousin, Marie Louise Balland. The result was that after many letters to and from the school, several delays, and a brief honeymoon following his June 15 wedding, Liouville did report for engineering duty near Grenoble. His interest in engineering, however, had ended and in October he resigned his post to follow a career in mathematics. He was fortunate to be appointed as *répétiteur* (or substitute) for Claude Louis Mathieu (1783–1875) at the *École Polytechnique* in 1831.

In 1833, through the good graces of his friend Jean Colladon (1802–1893), Liouville was appointed professor of rational mechanics at the *École Centrale des Arts et Manufactures*, a position he held until 1838 in spite of reports that his teaching was too theoretical and strayed from the curriculum. During this period, in order to make ends meet at home, Liouville also held positions at various private schools, at a *collège*, as well as at the *École Polytechnique*, teaching from 35 to 40 hours a week. The salaries paid to instructors were low and there was strenuous competition among aspiring academics to secure work. In Liouville's case, his father continued to support him and Marie Louise until 1833. The young family settled down in Toul where Liouville spent the long summer vacations from June until November. He also maintained an apartment in Paris. In 1836 he earned his doctorate, a necessary step in eventually teaching at the university level.

Although teaching kept him active and involved, Liouville's interest in research required another forum. He therefore joined various discussion groups and in 1832 was elected to the *Société Philomatique*. The *Académie des Sciences*, however, offered the definitive judgement on original work and Liouville continued to submit works to that body, including his comprehensive study on fractional calculus and his theory of integration in finite terms. The latter was favorably reviewed by Siméon-Denis Poisson (1781–1840) of the academy and published in one of its official reports. The paper contained what would later be labeled "Liouville's Theorem."

The state of French learned journals, especially scientific journals, was in flux during the 1830s, but Liouville continued to find outlets for his work. The *Journal de l'École Polytechnique* accepted two of his papers and he turned next to a German publication edited by August Leopold Crelle (1780–1855). This publication was Germany's answer to the respected *Annales de Mathématiques Pures et Appliquées* edited by Joseph Diez Gergonne (1771–1859), which had printed its last number in 1831. Crelle's journal accepted mainly work by German authors, but Liouville was able to join a small group of French scholars who had papers published. This gave Liouville his first contact with, and exposure to, German mathematicians. In particular, some time in 1830 Liouville met and began a correspondence with a famous German mathematician, Peter Gustav Dirichlet (1805–1859), a few years his senior and on a level with such well-known German mathematicians as Carl Jacobi (1804–1851) and even the great Carl Friedrich Gauss (1777–1855).

By 1833, Liouville decided that France needed a new journal of mathematics, one that would be devoted entirely to the study of mathematics and would at last replace the *Annales*, which had finished in 1831. To that end, he set about assembling his first edition. As Jesper Lützen describes in his book [192]:

> Liouville intended to accept both original papers on advanced subjects and new approaches to more elementary mathematics. Liouville also made it clear that he would not allow his Journal to degenerate into a forum for the everlasting and often slanderous quarrels in the competitive Parisian academic circles.
>
> *(Lützen, 1990, p. 38)*

From the first edition in 1836, the journal was of an excellent and enduring quality and attracted the best and brightest mathematical minds in France, as well as from Germany and later, other parts of Europe. Liouville demonstrated a skill at selection and editing not usually seen in a scholar so young and inexperienced. In his first editions he published contributions from the likes of Peter Gustav Dirichlet, Charles Sturm (1803–1855), Ándre Ampère (1775–1836), Michel Chasles (1793–1880), Count Gugliemo Libri-Carrucci (1803–1869), Gabriel Lamé (1795–1870), Victor Lebesgue (1791–1875) and Julius Plücker (1801–1868). The journal was so respected, so popular, and so successful that it soon became known simply as *Journal de Liouville*. In addition to the works of established mathematicians, Liouville accepted papers by such promising young mathematicians as Joseph Serret (1819–1885), Urbain Leverrier (1811–1877) and Charles Hermite (1822–1901). He continued to edit the journal for the next 40 years. It provided him with an international reputation and a forum for his own work.

While continuing his exhaustive routine of teaching, research and editing, Liouville, during the 1830s and 1840s, was unrelenting in his quest for secure

appointments at prestigious institutions, in part to further his career but also to provide an adequate income for his growing family. The Liouvilles produced four children, three daughters and a son. They seem to have been a close-knit group, enjoying the company of uncles, aunts, cousins and grandparents, with long sojourns in Toul tending the family vineyards and gardens and making occasional forays to visit relatives and friends.

Liouville lost a close contest to Jean Marie Constant Duhamel (1797–1872) for a permanent professorship at the *École Polytechnique* in 1835, but was soon in contention again, this time for a seat at the prestigious *Académie des Sciences*. Liouville chose to compete for the seat in geometry and submitted various results to the academy for consideration, among them new work he had completed with Charles Sturm, which later became known as Sturm-Liouville theory. According to Jesper Lützen, at the last moment, he

> devised an ingenious convergence proof for the Fourier series of the second-order Sturm-Liouville problems.
>
> *(Lützen, 1990, p. 45)*

In the end, however, it was not enough and the seat went to the more experienced Sturm. Finally, in 1837 Liouville obtained a position at the *Collège de France* as a substitute for Jean Baptiste Biot (1774–1862), teaching mathematical physics.

In 1838 he was named Professor of Analysis and Mechanics at the *École Polytechnique*. Liouville was an active member of the teaching staff at the *École Polytechnique*, and in addition to those duties he served on numerous administrative committees and was in charge of the library at least until 1847. He also served as editor of the school's journal, the *Journal de l'École Polytechnique* for many years.

In 1839, the astronomer Michel Lefrançois Lalande (1766–1839) died causing a vacancy at the *Académie des Sciences*, and, ever on the alert for well-paying academic positions, Liouville was again in competition for a seat at the Académie. Having earlier done work on celestial mechanics and planetary theory, Liouville presented three papers to the academy within two weeks of Lalande's death. Vigorously opposing his bid was his nemesis, Libri, whom he had accused publicly of plagiarism. Nonetheless, with the strong support of other members of the academy, principally Dominique Arago (1786–1853) and his close friend and colleague, Charles Sturm, Liouville succeeded to the post. The animosity between Libri and Liouville continued at the academy, each man missing no opportunity to snarl at the other. Liouville is quoted as saying in a letter to Dirichlet, that Libri was a man:

> who is beginning to be despised almost as much as he deserves.
>
> *(Lützen, 1990, p. 64)*

(Four years later, in 1843, Liouville resigned his temporary position at the *Collège de France* in protest when Libri was elected to a seat there).

Liouville had many contacts in the international mathematics community, bringing a wide perspective to his participation on the various committees on which he served during his many years at the Académie. He took a special interest in prize competitions and was often asked to select topics as well as to choose and serve on judging committees. In addition, he judged annual competitions at other academic institutions.

In 1840, Liouville was elected to the *Bureau des Longitudes*, a society founded during the French Revolution for the discussion of astronomical ideas. It was the pre-eminent society of its type in France and election to the Bureau was a high honor. It also paid handsomely. Liouville presented various research papers to the Bureau, among them his work on astronomy and geodesy; his discovery of transcendental numbers; and a series of papers on rational mechanics. Liouville served as president of the Bureau three times: in 1843, 1847 and 1872.

By the age of just 31, Liouville had managed to secure a good living from his various teaching positions, his membership at the Bureau and his income from the vineyards in Toul.

Eventually, in 1851, Liouville returned to the *Collège de France*, defeating Augustin Cauchy for a permanent teaching chair in mathematics. In so doing he took over the position his enemy Libri had held at that institution. Libri had fled France for England, in disgrace, having been accused of the theft of valuable books and manuscripts from French libraries. Libri was sentenced to 10 years in prison in absentia for his crime, but he never returned to France.

The *Collège de France* provided an atmosphere for teaching that delighted Liouville as it allowed him to teach topics of his own choosing and present his ideas and research to his students. The astronomer Hervé Faye, recalling Liouville's lectures at this time, had this to say about his former professor:

> Monsieur Liouville was one of the most brilliant professors one has ever heard. His lectures impressed me so strongly in my youth that today I still have a vivid recollection of the startling clarity with which he was endowed.
>
> *(Lützen, 1990, p. 83)*

Faye was not the only student influenced by Liouville. Over the course of his 50-year teaching career, he encouraged, supported and shared ideas with many

fine young mathematicians, among them: Joseph Serret, Jacob Steiner (1796–1863), Charles Hermite, Joseph Bertrand (1822–1900), Paul Laurent (1841–1908) and William Thomson Kelvin (later Lord Kelvin) (1824–1907). He also took on the enormous challenge of trying to explain the work of Evariste Galois (1811–1832) on the theory of equations and elliptic functions. Galois died in 1832 at the age of 20 after fighting a duel and his mathematical ideas had gone largely ignored until Liouville published them in his Journal in 1846. He also gave a series of lectures on Galois' theories and influenced others to study Galois' work.

While it may seem that Liouville led a rather frantic existence – teaching at several institutions, attending meetings and assemblies, conducting his own varied research, editing his *Journal de Liouville* and contributing to others, tending to his family and travelling back and forth from Toul to Paris – nonetheless, as Lützen points out, by the end of the 1840s, mathematics appeared to him as one body of interconnected ideas. Liouville, he says:

> cultivated only limited areas at different places of the mathematical landscape, but there is no doubt that he had a great view over it.
>
> *(Lützen, 1990, p. 147)*

Liouville was an avid republican all his adult life. He had supported his colleague and friend Dominique Arago's proposal for universal suffrage which he made in the Chamber of Deputies in 1840. He took part in so-called *banquets réformistes* to protest government policies, and he acted as chairman (after getting his father's permission) of such an event in Toul in 1840 during which he gathered over 1,000 signatures on an anti-government petition. The result of the unrest of 1840 was a move from the liberal monarchist Louis Adolphe Thiers to the conservative monarchist François Guizot. Although Liouvilles's political ideas were considered radical, he still had no problem swearing an oath of loyalty to King Louis-Phillipe (unlike the Bourbonist Cauchy, who could not do so).

In 1848, a popular uprising, complete with barricades and banners, caused the King to flee Paris and a republican provisional government was formed. Encouraged by Arago, Liouville agreed to stand for election to the Chamber of Deputies on a moderate, republican platform. He was elected in a republican sweep that saw 500 of the 800 seats going to form the "Second Republic." It was a proud moment for the republicans, but it did not last. The workers rebelled against the bourgeoisie and, on December 10, 1848, the French people elected as President Louis-Napoléon Bonaparte, nephew of the former emperor. In the election of 1849, which Liouville again contested as a republican, he lost his seat in a conservative win. In 1850, Louis-Napoléon led a coup against the

assembly, and in a bizarre turn of events, he extended his term as president to 10 years and ultimately was elected Emperor by the French people.

Liouville retreated. He would not meet with friends in Toul, nor would he attend social events in Paris. In a letter to M. Grébus, the principal of the *collège* in Toul he proclaimed:

> I live in peace with my conscience, and I can go with my head held high.
> *(Lützen, 1990, p. 161)*

In fact, the defeat left him feeling bitter and disillusioned. When his beloved father died in 1850, he experienced further misery, proclaiming in a letter to his old friend P. Vogin:

> It is for me an immense and irreparable emptiness and yet another reason for me to sink into solitude.
> *(Lützen, 1990, p. 162)*

The number of mathematics and other science courses declined during the Second Empire due to conservative influences, including the Roman Catholic Church. But there was progress in industry and agriculture and Louis-Napoléon re-built Paris and extended the French railway system. He was also victorious (with England) in the Crimean War. Meanwhile, Liouville stayed out of politics and after a period of inactivity, he again immersed himself in his work, published numerous papers on his new interest of number theory and continued the impressive run of his respected Journal. He was rewarded in 1857 by an appointment as professor of mechanics at the *Faculté des Sciences*. This last achievement assured him of a secure and generous income, for, since his father's death, he had inherited with his brother, Felix, the lucrative remuneration from the vineyards in Toul.

In his lifetime Liouville made many friends, especially among his mathematics colleagues in France and elsewhere. He also maintained the alliances of his youth and befriended young and aspiring scientists. He enjoyed the friendship and respect of some of the great men in public life and was honored in many ways by them. In contrast, there were enemies, notably Libri, but also his former student, Serret, who turned against him after acrimonious arguments over "possession rights" to Galois' work and, in 1857, the fierce competition between them for the chair in mechanics at the *Académie des Sciences* (which Liouville won, as mentioned previously). In 1854, however, the most bitter hostility arose between Liouville and his former student Urbain Jean Joseph Leverrier.

Leverrier was appointed director of the Observatory in Paris following the death of the great scientist and public figure, Dominique Arago (1786–1853).

The appointment of the conservative, Catholic, Leverrier was destined to cause problems, not only for Liouville but also for his son, Ernest, who had been appointed by Arago as an *élève astronome* at the Observatory in 1852. Whether deliberately or not, Liouville renewed old quarrels with Leverrier; he criticized his work, took issue with his research; and strenuously opposed Leverrier's increasing powers and interference with the *Bureau des Longitudes*. Liouville was not alone in opposing Leverrier, however he was probably the only member of the scientific community who had a son working directly for his enemy. Ernest was dismissed along with many others who had worked at the Observatory with Arago. In a letter to Dirichlet, recorded in its entirety in Jesper Lützen's biography, Liouville described the painful circumstances surrounding this episode. He wrote:

> You are probably aware of the painful emotions I have had to face during the six months which have passed since your nice visit. The death of our excellent Arago, the expulsion of his family and his friends from their formerly peaceful stay at the Observatory, the degradation of the Bureau des Longitudes, the suicide of poor Mauvais, whose life was made unbearable by too much trouble, this is but a small part of what we have had to suffer.
>
> *(Lützen, 1990, p. 183)*

Ernest soon found a new apartment, gave up his career as an astronomer and turned to the profession of his Uncle Felix, the law. He continued his interest in astronomy, however, and his father published several of his papers on the subject in his Journal. Ernest was also of great assistance to his father with the editing of the Journal during Liouville's various illnesses.

The one true and enduring friendship in Liouville's life was with Gustav Dirichlet, the most authoritative expert of his day on the theory of numbers. From the time of their first meeting in Paris in 1830, until Dirichlet's death in 1859, the two corresponded, visited each other frequently, involved their families in the friendship, and perhaps, most important, collaborated in the field of number theory. Liouville had been instrumental in getting Dirichlet, who by 1855 was successor to Gauss at Göttingen, appointed "foreign associate" at the *Académie des Sciences*. Dirichlet returned the favor by arranging to have Liouville receive the Gauss Medal and securing his appointment as foreign member of the Academy at Göttingen.

By 1856, Dirichlet's friendship and influence had led Liouville to an examination of analytic number theory. It was a new field for him and he embraced it with enthusiasm. Indeed, he promised to send his ideas on the theory of numbers to Dirichlet, and in 1857 he lectured on the subject at the *Collège de France*. In addition, Liouville presented "notes" to the Académie, wrote numerous papers, published several in his Journal, and corresponded with a

Russian number theorist, Viktor Bouniakowsky (1804–1889). In a letter dated August, 1858, Dirichlet praises his friend's efforts:

> The theorem you prove and the theorem of Mr. Bouniakowsky (whose memoir I have not seen yet) seem to me the more remarkable because they are of an entirely new type.
>
> *(Lützen, 1990, p. 193)*

Jesper Lützen concludes that the years 1856 and 1857 were two of the most productive years in Liouville's career. He states:

> Not only did he publish notable works on rational mechanics, definite integrals and number theory, he also laid the foundation for all his subsequent works in the latter field.
>
> *(Lützen, 1990, p. 193)*

At the *Collège de France* where he was free to choose the subjects of his lectures, he devoted 11 courses over the next 20 years on "Theory of Numbers."

There is much warmth in the personal correspondence between Dirichlet and Liouville, attesting to the respect and affection they held for each other. The Dirichlet family spent time in Toul; the wives were fond of one another and also corresponded; the Liouvilles were invited to visit in Germany (although they were never able to go). Liouville was therefore devastated when he heard, in rapid succession, of Dirichlet's heart attack in Switzerland in 1858, his wife's unexpected demise in 1859, and finally, the death of Dirichlet himself on May 5, 1859. With the passing of Dirichlet, Liouville lost a faithful friend, colleague and supporter, and his most prolific correspondent.

As early as the 1830s Liouville complained of ill health. At that time it was stomach problems and possibly rheumatism. Later, he developed gout, a painful condition affecting the toes and feet and caused by a surfeit of uric acid in the blood. There was no treatment for gout at that time except to elevate the affected limb. Liouville endured many bouts of pain and immobility due to this ailment. In 1858, Liouville wrote wistfully to his brother Felix, who was on holiday in Italy after a bout of cancer:

> I myself, who only like my hole, cannot think without enthusiasm of the pleasure I would enjoy by taking a walk in this beautiful country full of great places. But if God has denied me this pleasure forever, he has at least granted me in the study of mathematics, a great consolation within my reach.
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> *(Lützen, 1990, p. 217)*

Felix died in 1860, missing the opportunity to see his brother Joseph named an Officer of the *Légion d'Honneur*. Liouville then became the head of the family.

From 1860, Liouville immersed himself in number theory, using his Journal to pour out notes (more than 50 in two years) about number-theoretical functions. As Lützen explains:

> . . . he continued to publish rather uninteresting applications of his general ideas instead of revealing the core of these ideas and his methods of proof.
>
> *(Lützen, 1990, p. 229)*

Lützen goes on to speculate that Liouville may have, as Hermite stated in a letter 20 years later:

> . . . kept for himself the entire harvest of all the consequences of his original discovery.
>
> *(Lützen, 1990, p. 229)*

because he never was able to find the time or energy for the task. Although he promised himself and others that there would be published results, not only on number-theoretical functions, but also on celestial mechanics, rational mechanics, quadratic forms and other subjects, none were forthcoming. For one so prolific in his notebooks, it was a strange omission.

Through the 1860s and 1870s, in spite of ill health, Liouville kept to his rigorous teaching schedule, attended meetings of the *Bureau des Longitudes* and the *Académie*, encouraged and promoted young talent, and made copious entries in his personal notebooks. He also remained at the helm of the *Journal de Liouville*, which he referred to fondly as "his child." It was not until 1875 that he finally turned it over to Henri Resal, a member of the *Académie des Sciences*. But it was a bad choice. Resal did not attract top-rank mathematicians and by 1885 the journal was in serious trouble. It was finally rescued by Camille Jordan, who brought it back to its former glory, and the Journal has continued to this day with the title page still bearing the name of its originator.

It must have given Liouville considerable satisfaction to record in his notebook that in 1870, with a new liberal government encouraging the Emperor to adopt reforms, his enemy Leverrier was dismissed from the Observatory. Liouville himself served on a committee to select a replacement, however the choice of the committee, Charles Delaunay, died within two years and Leverrier returned to the position. Four years later Leverrier died. Liouville, who had written of him:

> such a stupid tyrant cannot last
>
> *(Lützen, 1990, p. 240)*

had the satisfaction of gluing the death notice in his notebook.

By 1870, France was plunged into the Franco-Prussian War and the Liouville family home in Toul was under siege. Liouville escaped to Paris, the Emperor