Ocean Dynamics

Bearbeitet von
Dirk Olbers, Jürgen Willebrand, Carsten Eden

1. Auflage 2012. Buch. XXIII, 703 S. Hardcover
ISBN 978 3 642 23449 1
Format (B x L): 0 x 0 cm
Gewicht: 1555 g

Weitere Fachgebiete > Geologie, Geographie, Klima, Umwelt > Geologie > Hydrologie, Hydrogeologie

Zu Inhaltsverzeichnis

Conservation Laws for Moving Fluids

2

In this chapter the basic conservation laws for mass, momentum, and energy are presented, including a discussion of appropriate boundary conditions. From the basic laws, budgets of other variables such as temperature and vorticity are derived. The result of the discussion will be a set of equations suitable to describe oceanic motions at all scales.

The equations which govern the evolution of fluid properties and its motion are differential equations, derived from elementary conservation laws of physics. They are usually formulated as *conservation equations* and arise if a property $\chi(\boldsymbol{x}, t)$ (units property per mass) respectively $\rho(\boldsymbol{x}, t)\chi(\boldsymbol{x}, t)$ (units property per volume) of a substance is changed due to sources and sinks, represented by $C(\boldsymbol{x}, t)$ (units property per volume and time), and due to transport by a flux $\boldsymbol{J}(\boldsymbol{x}, t)$ (this is a vector in the direction of transport, with the magnitude giving the units property per area and time). If C is nonzero, there, is actually no conservation of the property, not locally and generally not in the integral sense. Even in case of vanishing sources/sinks there is no local conservation unless the divergence of the flux vector vanishes in addition to C. An important aspect is the condition of fluxes through the boundary of the domain, governed by the normal component $\boldsymbol{J} \cdot \boldsymbol{n}$ where $\boldsymbol{n}$ is the outward normal vector (see the figure on page 1). Hence χ is generally not constant, but still we will call the equation governing $\chi(\boldsymbol{x}, t)$ a conservation equation. Fluid dynamics and thermodynamics largely deal with conservation equations.

2.1 General Form of Conservation Equations

Consider a volume V, which is *fixed* in space and bounded by a surface A, and an arbitrary scalar fluid property χ (in units per mass) or $\rho\chi$ (in units per volume). The total amount of the property χ in the volume, given by $\int_V \rho\chi \mathrm{d}V$, may change in time in one of two ways:

- **Transport through the surface:**
  The volume transport (volume per unit time) through a surface element $\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\mathrm{d}A$ ($\boldsymbol{n}$ outward normal unit vector) is $\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A}$, the outward transport of χ is thus $\rho\chi\boldsymbol{u} \cdot \mathrm{d}\boldsymbol{A}$. Here $\rho\chi\boldsymbol{u}$ is the *advective flux* of χ (unit $[\chi]\,\mathrm{m}^{-2}\,\mathrm{s}^{-1}$). In addition,

D. Olbers, J. Willebrand, C. Eden, *Ocean Dynamics*,
DOI 10.1007/978-3-642-23450-7_2,

a nonadvective flux $\boldsymbol{J}_\chi$ may occur, e. g. by diffusion, heat conduction, radiation etc. The total outward transport across A is hence given by

$$\oint_A \left(\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi\right) \cdot \mathrm{d}\boldsymbol{A} \tag{2.1}$$

- **Interior sources and sinks:**
  Let C_χ denote the net sources/sinks of χ, measured in χ-units per time and volume, through internal processes within the volume. Examples are heat sources, radioactive decay, chemical reaction, consumption of χ material etc. The change within the volume is then given by

$$\int_V C_\chi \mathrm{d}V$$

Both quantities, the flux $\boldsymbol{J}_\chi$ and the sources/sinks C_χ, have to be specified for the corresponding fluid property. They contribute to the total rate of change of the χ-content in the volume according to

$$\frac{\partial}{\partial t}\int_V \rho\chi \mathrm{d}V = -\oint_A \left(\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi\right) \cdot \mathrm{d}\boldsymbol{A} + \int_V C_\chi \mathrm{d}V \tag{2.2}$$

For a fixed volume the time rate of change may be placed inside the integral over the volume, and with GAUSS'[1] theorem (see Appendix A.1) the surface integral in (2.2) may be rewritten as a volume integral according to

$$\oint_A \left(\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi\right) \cdot \mathrm{d}\boldsymbol{A} = \int_V \nabla \cdot \left(\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi\right) \mathrm{d}V$$

so that (2.2) can be written as

$$\int_V \left[\frac{\partial}{\partial t}(\rho\chi) + \nabla \cdot \left(\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi\right) - C_\chi\right] \mathrm{d}V = 0 \tag{2.3}$$

As this holds for an *arbitrary* volume, the integrand has to vanish identically, i. e.

$$\frac{\partial}{\partial t}(\rho\chi) = -\nabla \cdot \left(\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi\right) + C_\chi \tag{2.4}$$

which is the general conservation equation for the χ property in the so-called *flux form*. The right-hand side contains the divergence of the total χ-flux $\rho\chi\boldsymbol{u} + \boldsymbol{J}_\chi$, as well as interior sources/sinks contained in C_χ. It should be mentioned that the advective part of the flux – despite its nonlinear character – describes a conceptually simple process because the fluid velocity $\boldsymbol{u}$ and the property χ belong to the state vector, and thus usually to the resolved part of the system. The challenge is to obtain the nonadvective fluxes and the sources and sinks as a correct physical, mathematical and resolved form. For many properties of the ocean $\boldsymbol{J}_\chi(\boldsymbol{x}, t)$ and $C_\chi(\boldsymbol{x}, t)$ are not well known. In Part IV we describe the most elementary attempts to construct parameterizations of turbulent fluxes in the fluid dynamical and thermodynamics equations.

---

[1] CARL FRIEDRICH GAUSS, *1777 in Braunschweig, †1855 in Göttingen, mathematician.

General Boundary Condition

The air-sea interface is of fundamental importance for ocean dynamics. With the exception of tides, nearly all oceanic motions are forced by the exchange of momentum, energy, and water between atmosphere and ocean across the sea surface. Momentum exchange with the solid earth across the bottom boundary is also important, whereas the water, heat and salt exchanges at the bottom play only a limited role for oceanic motions.

The considerations leading to conservation equations for energy, mass, momentum, etc. within the ocean must also be valid at the boundary between ocean and atmosphere. The general conservation equation for a *fixed* volume in its integral form was given in (2.2). As the air-sea boundary in general will move in time, we now consider an infinitesimal volume instead, which is moving with the velocity $\boldsymbol{u}_I$ of the air-sea interface and which includes the interface at all times. More specific, the volume is bounded by a surface A which consists of a surface I just above the interface between ocean and atmosphere on the one side, and corresponding surface I' inside the fluid and the associated side boundaries (see Figure 2.1). The following analysis applies as well at any interface on the fluid boundaries or inside the fluid. At the bottom where I is the bottom surface, however, $\boldsymbol{u}_I$ is to be taken as zero. At the surface of the test volume, only the difference velocity $\boldsymbol{u} - \boldsymbol{u}_I$ between the fluid and the moving volume is relevant for the budget inside the volume. Notice that $\boldsymbol{u}_I \neq \boldsymbol{u}$ in general since the surface I is not necessarily a material surface. With the same notation as before it follows that we have now

$$\frac{\partial}{\partial t} \int\limits_V (\rho\chi) \mathrm{d}V = -\oint\limits_A \left[\rho\chi\left(\boldsymbol{u} - \boldsymbol{u}_I\right) + \boldsymbol{J}_\chi\right] \cdot \mathrm{d}\boldsymbol{A} + \int\limits_V C_\chi \mathrm{d}V \tag{2.5}$$

where $\mathrm{d}\boldsymbol{A}$ is the surface element pointing in the outward normal direction. The contributions of the volume integrals in (2.5) can be made small compared to those arising from the surface integrals by taking a pillbox of sufficiently small height. Neglecting thus the volume integral for an infinitesimal pillbox height, one has

$$\oint\limits_A \left[\rho\chi\left(\boldsymbol{u} - \boldsymbol{u}_I\right) + \boldsymbol{J}_\chi\right] \cdot \mathrm{d}\boldsymbol{A} = 0 \tag{2.6}$$

The integrand in (2.6) describes the sum of the advective flux of χ (relative to the moving boundary) and the diffusive flux. As the outward normal vector points into opposite directions on both sides of the interface, the condition (2.6) requires in the limit of very small height that the normal component of the flux vector is continuous

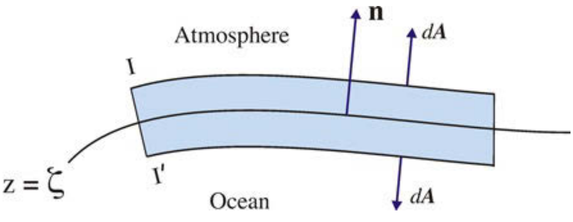

Fig. 2.1 Sketch of the ocean-atmosphere boundary and the test volume described in the text

across the interface, and one obtains the general boundary condition

$$\left[\rho\chi\left(\boldsymbol{u}-\boldsymbol{u}_I\right)+\boldsymbol{J}_\chi\right]\cdot\boldsymbol{n}=\mathcal{J}_\chi^{\text{tot}} \tag{2.7}$$

Here $\boldsymbol{n}$ is the unit vector normal to the boundary, and $\mathcal{J}_\chi^{\text{tot}}$ denotes the normal component of the *total* flux of the substance χ in the other system, e. g. the atmosphere or the solid earth. Note that $\mathcal{J}_\chi^{\text{tot}}$ may also include flux components associated with phase transitions at the boundary, such as evaporation at the sea surface.

The velocity $\boldsymbol{u}-\boldsymbol{u}_I$ in (2.7) can be expressed in terms of the state variables. For the air-sea interface, the location of the boundary can be implicitly represented by $I(\boldsymbol{x}_\text{h},z,t)=z-\zeta(\boldsymbol{x}_\text{h},t)=0$. The normal vector $\boldsymbol{n}$, taken positive upwards from the sea surface, is

$$\boldsymbol{n}=\frac{\nabla I}{|\nabla I|}=\frac{(-\nabla_\text{h}\zeta,1)}{\sqrt{1+(\nabla_\text{h}\zeta)^2}} \tag{2.8}$$

On the surface I, one has $\mathrm{d}I=\partial I/\partial t\,\mathrm{d}t+\nabla I\cdot\mathrm{d}\boldsymbol{x}=0$ and $\mathrm{d}\boldsymbol{x}/\mathrm{d}t=\boldsymbol{u}_I$, and hence $\partial I/\partial t+\boldsymbol{u}_I\cdot\nabla I=0$. Then, the normal velocity of the moving boundary is found as

$$\boldsymbol{u}_I\cdot\boldsymbol{n}=-\frac{\partial I/\partial t}{|\nabla I|}=\frac{\partial\zeta/\partial t}{\sqrt{1+(\nabla_\text{h}\zeta)^2}} \tag{2.9}$$

and the difference to the normal component of the fluid velocity is thus given by

$$(\boldsymbol{u}_I-\boldsymbol{u})\cdot\boldsymbol{n}=\frac{\partial\zeta/\partial t+\boldsymbol{u}_\text{h}\cdot\nabla_\text{h}\zeta-w}{\sqrt{1+(\nabla_\text{h}\zeta)^2}} \tag{2.10}$$

For the ocean bottom, with $I(\boldsymbol{x}_\text{h},z)=z+h(\boldsymbol{x}_\text{h})=0$ where $h(\boldsymbol{x}_\text{h})$ is the ocean depth referred to the mean sea level, the normal vector $\boldsymbol{n}$ defined in (2.8) is pointing from the bottom into the fluid. Equation (2.7) then applies with

$$(\boldsymbol{u}_I-\boldsymbol{u})\cdot\boldsymbol{n}=-\boldsymbol{u}\cdot\boldsymbol{n}=-\frac{w+\boldsymbol{u}_\text{h}\cdot\nabla_\text{h}h}{\sqrt{1+(\nabla_\text{h}h)^2}} \tag{2.11}$$

In most applications, the square root terms in the above relations can be replaced by 1 because the slopes of the interfaces are very small.

2.2 Mass Conservation

As discussed in Section 1.2.1, it is sufficient for almost all dynamical purposes to take the chemical composition of seawater as constant. Thus it suffices to consider the two components salt and pure water, with concentrations s and $w=1-s$ measured in kg salt, respectively kg freshwater, per kg seawater, and densities $\rho_\text{s}=\rho s=\rho S/1{,}000$, $\rho_\text{w}=\rho w$, in kg m^{-3}. The total mass of the respective molecules must be conserved, i. e. the general conservation equation must hold for mass (notice that we ignore here freezing of freshwater and melting of sea ice). Formally, it is possible to simply use $\chi=1$ and $\boldsymbol{J}_\chi=0$ and $C_\chi=0$ in (2.4) (the total mass has neither sources/sinks nor a nonadvective flux) to derive a conservation budget for total mass. However, we consider first budgets for the partial masses of salt and freshwater independently. The combination yields the total mass and the salt conservation equation.

2.2.1 Total Mass and Salt Conservation Equation

Consider the partial mass δM_s of salt in a moving material volume element with volume δV which is defined with respect to the mean motion $\boldsymbol{u}_\mathrm{s}$ of the salt molecules. This volume element is moving with the salt velocity which may differ from the flow velocity $\boldsymbol{u}$, and thus corresponds to a Lagrangian derivative $\mathrm{D_s}/\mathrm{D}t$ formed with $\boldsymbol{u}_\mathrm{s}$. In this Lagrangian frame of reference δM_s must be constant, thus $\mathrm{D_s}(\delta M_\mathrm{s})/\mathrm{D}t = 0$. For the density $\rho_\mathrm{s} = \delta M_\mathrm{s}/\delta V$ of salt we thus have

$$\frac{\mathrm{D_s}}{\mathrm{D}t}\rho_\mathrm{s} = -\frac{\delta M_\mathrm{s}}{(\delta V)^2}\frac{\mathrm{D_s}}{\mathrm{D}t}\delta V$$

According to (1.9), the volume change is given by $\mathrm{D_s}(\delta V)/\mathrm{D}t = \delta V \nabla \cdot \boldsymbol{u}_\mathrm{s}$, and it follows that

$$\frac{\mathrm{D_s}}{\mathrm{D}t}\rho_\mathrm{s} = -\rho_\mathrm{s}\nabla \cdot \boldsymbol{u}_\mathrm{s} \tag{2.12}$$

Both components, salt and freshwater, must independently satisfy a conservation equation of the form (2.12). In the Eulerian form, these become

$$\frac{\partial \rho_\mathrm{s}}{\partial t} + \nabla \cdot \rho_\mathrm{s}\boldsymbol{u}_\mathrm{s} = 0\ , \quad \frac{\partial \rho_\mathrm{w}}{\partial t} + \nabla \cdot \rho_\mathrm{w}\boldsymbol{u}_\mathrm{w} = 0 \tag{2.13}$$

The transport velocities $\boldsymbol{u}_\mathrm{s}$ and $\boldsymbol{u}_\mathrm{w}$ for both components generally differ. The total mass per volume has the density $\rho = \rho_\mathrm{s}+\rho_\mathrm{w}$, and the mass-weighted mean velocity – the flow velocity $\boldsymbol{u}$ – is defined as the barycentric mean

$$\rho\boldsymbol{u} = \rho_\mathrm{s}\boldsymbol{u}_\mathrm{s} + \rho_\mathrm{w}\boldsymbol{u}_\mathrm{w} = \rho(s\boldsymbol{u}_\mathrm{s} + w\boldsymbol{u}_\mathrm{w}) \tag{2.14}$$

Addition of the two equations in (2.13) yields the conservation of total mass,

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \rho\boldsymbol{u} = 0 \tag{2.15}$$

which is also called the *continuity equation*.

The velocities of the partial masses are not easily measurable and hence are not suitable to describe the flow. Using $\rho_\mathrm{s} = \rho S/1{,}000$ and introducing the advective flux of salinity by the flow velocity $\boldsymbol{u}$, one obtains from (2.13) and (2.14) the salt conservation in the form

$$\frac{\partial \rho S}{\partial t} + \nabla \cdot \rho S\boldsymbol{u} = -\nabla \cdot [(\boldsymbol{u}_\mathrm{s} - \boldsymbol{u})\,\rho S] = -\nabla \cdot \boldsymbol{J}_\mathrm{S} \tag{2.16}$$

Here, the *diffusive* salinity flux $\boldsymbol{J}_\mathrm{S} = (\boldsymbol{u}_\mathrm{s} - \boldsymbol{u})\rho S$ is introduced as a consequence of eliminating the salt velocity $\boldsymbol{u}_\mathrm{s}$. It results from the difference of the mean velocity $\boldsymbol{u}_\mathrm{s}$ of salt and the flow velocity $\boldsymbol{u}$. As outlined in Section 2.5 below, the diffusive flux is empirically related to the salinity gradient (and possibly gradients of other fluid properties).

The continuity equation (2.15) allows to rewrite the general conservation equation (2.4) in the form

$$\rho\frac{\mathrm{D}\chi}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{J}_\chi + C_\chi \tag{2.17}$$

which will be referred to as the *parcel form*, in contrast to the *flux form* (2.4). The flux form (2.4) is preferred for integral budgets, whereas the particle form (2.17) permits to follow the particle properties. Mathematically, both formulations are equivalent. Alternative formulations of mass conservation (2.15) are given by

$$\frac{1}{\rho}\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\nabla\cdot\boldsymbol{u}\ , \quad \rho\frac{\mathrm{D}\upsilon}{\mathrm{D}t} = \nabla\cdot\boldsymbol{u} \tag{2.18}$$

with the *specific volume* $\upsilon = 1/\rho$. Note that the equations are in parcel form; thus the velocity divergence is the volume source of the specific volume.

The total (advective plus diffusive) salt flux (kg salt per m^2 and s) is given by $\rho S\boldsymbol{u} + \boldsymbol{J}_\mathrm{S}$. It enters the *flux form* of salt conservation,

$$\frac{\partial\rho S}{\partial t} = -\nabla\cdot(\rho S\boldsymbol{u} + \boldsymbol{J}_\mathrm{S}) \tag{2.19}$$

corresponding to the general form (2.4), without interior sources/sinks. Alternatively, the *parcel form* of salt conservation reads

$$\rho\frac{\mathrm{D}S}{\mathrm{D}t} = -\nabla\cdot\boldsymbol{J}_\mathrm{S} \tag{2.20}$$

2.2.2 Boundary Conditions for the Fluxes of Total Mass and Salt

For $\chi = 1$ and $\boldsymbol{J}_\chi = 0$ the relation (2.7) yields the boundary condition for the total mass at the sea surface,

$$\rho\,(\boldsymbol{u} - \boldsymbol{u}_I)\cdot\boldsymbol{n} = \mathcal{J}_\mathrm{mass} = \mathcal{E} - \mathcal{P} \tag{2.21}$$

The mass flux $\mathcal{J}_\mathrm{mass} = \mathcal{E} - \mathcal{P}$ (in $\mathrm{kg\,m^{-2}\,s^{-1}}$) across the air-sea interface (positive upwards) is given as the difference between evaporation $\mathcal{E}$ and precipitation $\mathcal{P}$ of pure water[2]. With (2.10) one obtains

$$\frac{\partial\zeta}{\partial t} + \boldsymbol{u}_\mathrm{h}\cdot\nabla_\mathrm{h}\zeta - w = -\frac{\mathcal{J}_\mathrm{mass}}{\rho} = \frac{\mathcal{P}-\mathcal{E}}{\rho} \quad \text{at} \quad z = \zeta(\boldsymbol{x}_\mathrm{h}, t) \tag{2.22}$$

which is referred to as *kinematical* boundary condition since no forces are involved. We have assumed $(\nabla_\mathrm{h}\zeta)^2 \ll 1$ to write the condition in the above commonly used form.

At a solid boundary as the bottom, where $\boldsymbol{u}_I \equiv 0$ and $\mathcal{J}_\mathrm{mass} = 0$, (2.7) leads to kinematical boundary condition

$$\boldsymbol{u}\cdot\boldsymbol{n} = \boldsymbol{u}_\mathrm{h}\cdot\nabla h + w = 0 \quad \text{at} \quad z = -h(\boldsymbol{x}_\mathrm{h}) \tag{2.23}$$

so that the normal component of the velocity vector has to vanish.

The processes of evaporation and precipitation exchange pure water with the atmosphere, but there is no salt flux through the interface. With $\chi = S$ and $\boldsymbol{J}_\chi = \boldsymbol{J}_\mathrm{S}$ equation (2.7) yields

$$[\rho S\,(\boldsymbol{u} - \boldsymbol{u}_I) + \boldsymbol{J}_\mathrm{S}]\cdot\boldsymbol{n} = 0 \tag{2.24}$$

---

[2] The mass flux of other substances is very small and negligible for dynamical purposes.

and with (2.21), this can be written as

$$\boldsymbol{J}_{\mathrm{S}} \cdot \boldsymbol{n} = -S\mathcal{J}_{\mathrm{mass}} = S(\mathcal{P} - \mathcal{E}) = \rho \frac{S}{1 - S/1000}(\mathcal{P}_{\mathrm{w}} - \mathcal{E}_{\mathrm{w}}) \tag{2.25}$$

at the sea surface. In the last relation $\mathcal{E}_{\mathrm{w}} = \mathcal{E}/\rho_{\mathrm{w}}, \mathcal{P}_{\mathrm{w}} = \mathcal{P}/\rho_{\mathrm{w}}$ are the corresponding *volume* fluxes (in units of $\mathrm{m^3\,m^{-2}\,s^{-1}} = \mathrm{m\,s^{-1}}$). Notice that although no flux is transported through the surface, (2.25) appears as if a virtual flux of salt enters the ocean from the atmosphere and is carried further there by salt diffusion. Therefore, the right hand side $S(\mathcal{P}-\mathcal{E})$ is called *equivalent* salinity flux. It describes the changes of salinity due to the freshwater flux $\mathcal{P} - \mathcal{E}$. At the bottom, assuming zero exchange of salt with the ocean floor, one has a vanishing diffusive salt flux normal to the bottom, $\boldsymbol{J}_{\mathrm{S}} \cdot \boldsymbol{n} = 0$. Note, however, that the transport of water and salt by rivers can be relevant.

2.3 Conservation of Momentum

With respect to spatial scales, two different kinds of forces acting on any part of a fluid can be distinguished. Long-range forces have a *macroscopic* range and affect all parts of a fluid. They are generally proportional to the fluid volume, and for that reason they are called *volume forces*. The principal volume force influencing motions in ocean and atmosphere is the force of gravity. Other relevant volume forces are the centrifugal and Coriolis forces discussed in Section 2.3.3 below. Short-range forces have a *microscopic* range of the order of the distance between molecules. If a volume is deformed, the relative positions of molecules change and the deformation generates internal forces, so-called *stresses*, that attempt to bring the volume shape back into its equilibrium state. These internal stresses cannot exist in a nondeformed volume; they are the result of molecular forces which have a very short range, of the order of the distance between molecules (e. g. VAN DER WAALS[3] forces). These forces hence must be proportional to the surface area and are independent of the volume. They can affect the motion of a fluid parcel only through contact of the parcel with the surrounding fluid (or the boundary of the fluid), and, therefore, they are termed *surface forces*. Examples for surface forces are molecular friction or surface tension. The mathematical treatment of surface forces is attributed to CAUCHY[4].

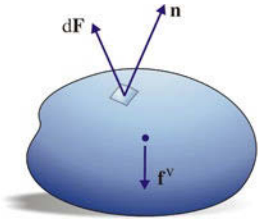

Fig. 2.2 Material fluid parcel experiencing a body (volume) force $\boldsymbol{f}^{\,\mathrm{v}}$ and a surface force $\mathrm{d}\boldsymbol{F}$ acting on on infinitesimal surface element

Following NEWTON's[5] second law of motion, the conservation of momentum is the balance between the mass times acceleration and the sum of all forces applied to a fluid parcel[6],

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \boldsymbol{f} = \boldsymbol{f}^{\,\mathrm{v}} + \boldsymbol{f}^{\,\mathrm{s}} \tag{2.26}$$

---

[3] JOHANNES DIDERIK VAN DER WAALS, *1837 in Leiden, †1923 in Amsterdam, physicist and thermodynamicist.

[4] AUGUSTIN LOUIS CAUCHY, *1789 in Paris, †1857 in Sceaux, mathematician.

[5] SIR ISAAC NEWTON, *1643 in Woolsthorpe-by-Colsterworth/Lincolnshire, †1727 in Kensington, physicist, philosopher, mathematician and astronomer.

[6] The differential form of Newton's second law (2.26) is actually not found in Newton's *Principia* of 1687 or any of his work. Newton still formulated differential equations with geometrical approaches. It was Euler who first wrote the fundamental law of dynamics in the above form (in 1749).

Here $\boldsymbol{f}$ is the force per volume, separated into those forces arising from volume ($\boldsymbol{f}^{\mathrm{v}}$, long range) and from surface ($\boldsymbol{f}^{\mathrm{s}}$, short range; with infinitesimal contribution $\mathrm{d}\boldsymbol{F}$) related origin (see sketch in Figure 2.2). Note that the form of momentum conservation given by (2.26) is valid only in an inertial frame. Rotating frames of reference, more appropriate to the ocean, are considered in Section 2.3.3. In the following section, we will consider the mathematical representation of the forces proportional to the surface, while in Section 2.3.4 we consider the forces proportional to volume. Boundary conditions for momentum are discussed in Section 2.3.2.

2.3.1 Stresses, Pressure and Frictional Forces

A first result is gained by considering (2.26) for a *material* volume $V = V(t)$ where volume forces $\boldsymbol{f}^{\mathrm{v}}$ must appear as an integral over the volume and surface forces $\boldsymbol{f}^{\mathrm{s}}$ as an integral of a vector $\boldsymbol{t}^{\mathrm{s}}$ over the surface

$$\frac{\mathrm{d}}{\mathrm{d}t} \int\limits_{V(t)} \rho \boldsymbol{u} \mathrm{d}V = \int\limits_{V(t)} \boldsymbol{f}^{\mathrm{v}} \mathrm{d}V + \int\limits_{A(t)} \boldsymbol{t}^{\mathrm{s}} \mathrm{d}A \tag{2.27}$$

The vector $\boldsymbol{t}^{\mathrm{s}}$ is thus a force per unit area, called a *stress*. The relation of $\boldsymbol{t}^{\mathrm{s}}$ to a representation in terms of a force $\boldsymbol{f}^{\mathrm{s}}$ per volume is not yet clear, but the integral form allows immediately to point out an important property of surface forces: because the volume terms in (2.27) are proportional to d^3 and the surface integral to d^2 in terms of a characteristic scale d of the volume, the limit of small d cancels the volume contribution faster than the surface contribution. Hence

$$\lim_{d \to 0} \frac{1}{d^2} \int\limits_{A(t)} \boldsymbol{t}^{\mathrm{s}} \mathrm{d}A = 0 \tag{2.28}$$

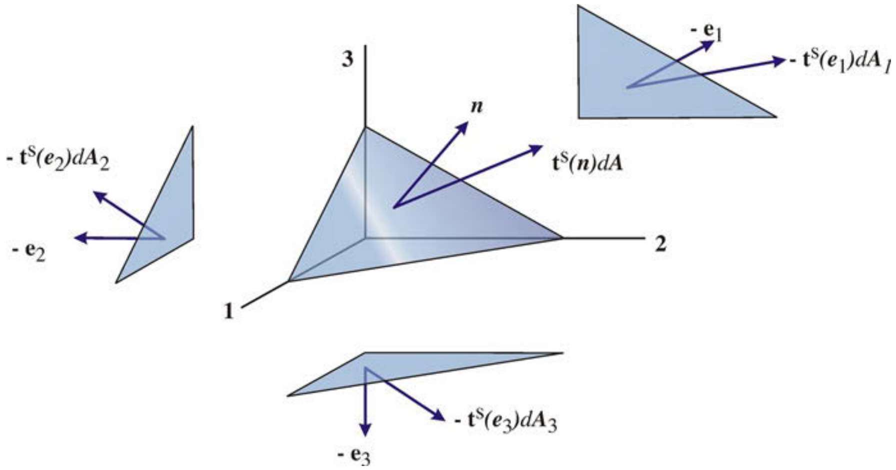

Fig. 2.3 The tetrahedron with side areas, associated units vectors, and stress components. The tetrahedron is bounded by three surfaces which are normal to the coordinate axes and a surface with arbitrary orientation specified by the normal vector $\boldsymbol{n}$. Redrawn from Aris (1989)

stating that the stresses must be locally in equilibrium. This statement allows to express the relation of the surface stress $\boldsymbol{t}^{\mathrm{s}}$ to the volume term $\boldsymbol{f}^{\mathrm{s}}$ in (2.26) in a simple mathematical form. The direction of the resulting force $\mathrm{d}\boldsymbol{F} = \boldsymbol{t}^{\mathrm{s}}\mathrm{d}A$ may have an arbitrary angle with the surface element which is characterized by the normal vector $\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\mathrm{d}A$ (see Figure 2.2). Cauchy postulated that $\mathrm{d}\boldsymbol{F}$ at any given time t and at any position $\boldsymbol{x}$ only depends upon the orientation $\boldsymbol{n}$, i. e. the curvature of the surface is irrelevant for the stress. This postulate is indicated by the notation $\boldsymbol{t}^{\mathrm{s}} = \boldsymbol{t}^{\mathrm{s}}(\boldsymbol{n};\boldsymbol{x},t)$. Expressing now the stress equilibrium (2.28) for a small tetrahedron (see Figure 2.3), having three of its faces parallel to the coordinate planes with unit normals $\boldsymbol{e}_j$ and area elements $\mathrm{d}A_j$, $j = 1, 2, 3$, and the fourth with the normal $\boldsymbol{n}$ and area $\mathrm{d}A$ (see Figure 2.3), we find

$$\boldsymbol{t}^{\mathrm{s}}(\boldsymbol{n})\mathrm{d}A - \boldsymbol{t}^{\mathrm{s}}(\boldsymbol{e}_j)\mathrm{d}A_j = 0$$

using $\boldsymbol{t}^{\mathrm{s}}(-\boldsymbol{e}_j) = -\boldsymbol{t}^{\mathrm{s}}(\boldsymbol{e}_j)$, a consequence of the principle of action and reaction, which also follows from Cauchy's postulate. A geometric consideration yields $\mathrm{d}A_j = n_j\mathrm{d}A$ for $\boldsymbol{n} = (n_1, n_2, n_3)$, and hence

$$\boldsymbol{t}^{\mathrm{s}}(\boldsymbol{n}) = n_j\boldsymbol{t}^{\mathrm{s}}(\boldsymbol{e}_j) \quad \text{or} \quad t_i^{\mathrm{s}}(\boldsymbol{n}) = n_j t_i^{\mathrm{s}}(\boldsymbol{e}_j)$$

stating a *linear proportionality* of the stress with the normal of the respective surface element. The latter relation is component-wise, and denoting the ith component of the stress vector $\boldsymbol{t}^{\mathrm{s}}(\boldsymbol{e}_j)$, acting on the area element normal to the coordinate axis j, by Π_{ji}, we may write

$$\mathrm{d}F_i = \mathrm{d}An_j\Pi_{ji} \quad \text{or} \quad \mathrm{d}\boldsymbol{F} = \mathrm{d}\boldsymbol{A}\cdot\Pi \tag{2.29}$$

for the stress force $\mathrm{d}\boldsymbol{F}$ acting on the infinitesimal area $\mathrm{d}A$. The proportionality factor Π_{ji} stands for the i-component of the force per unit area on the area perpendicular to the j-axis. The Π_{ji} can be combined into a tensor $\Pi = (\Pi_{ij})$ (with 3×3 components), the *stress tensor*. The components with associated surfaces are displayed in Figure 2.4. Hence $n_j\Pi_{ji}$ is the i-component of the force per unit area on the area perpendicular to $\boldsymbol{n}$.

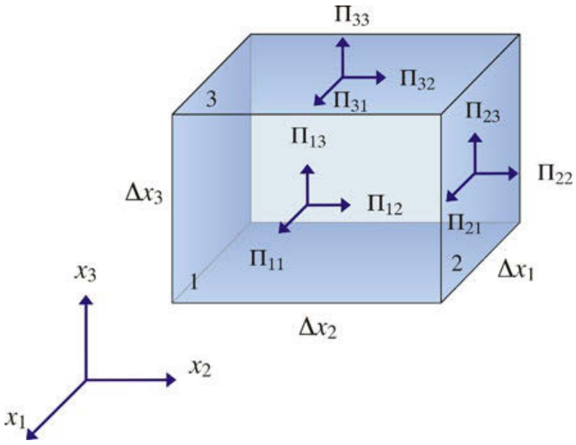

Fig. 2.4 The components of the stress tensor

The resulting force $\boldsymbol{F}$ on a volume V results from all forces $\mathrm{d}\boldsymbol{F}$ on the surface of the volume, and integration yields

$$\boldsymbol{F} = \oint_A \mathrm{d}\boldsymbol{A} \cdot \Pi = \int_V \nabla \cdot \Pi \mathrm{d}V = \int_V \boldsymbol{f}^{\mathrm{s}} \mathrm{d}V \tag{2.30}$$

which clarifies that the volume form $\boldsymbol{f}^{\mathrm{s}}$ of the surface forces is given by the divergence $\nabla \cdot \Pi$ of the stress tensor. The relation follows by application of Gauss' theorem. This mathematical form guarantees that the surface forces, acting between different interior parts of the volume, must sum up to zero as action and reaction must compensate on every interior surface. The i-component $\partial \Pi_{ji}/\partial x_j$ constitutes a force per volume in the i-direction. Likewise, Π_{ji} is the flux of i-momentum in the *negative* j-direction. Notice that this sign convention used for the molecular flux of momentum, given by Π and below also by Σ, differs from the convention of sign used for molecular fluxes of partial masses and heat (compare to the Sections 2.2 and 2.4.3).

It may be somewhat counterintuitive that the last relation expresses the *surface* forces again by a *volume* term, but now the spatial variations of the stress tensor come into play. In fact, if Π is constant, the first expression in (2.30) vanishes as well. Taking (2.30) for an infinitesimal volume V, the force per volume becomes $\nabla \cdot \Pi$, and the momentum balance (2.26) is written in the form

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \Pi + \boldsymbol{f}^{\mathrm{v}} \tag{2.31}$$

This form of momentum conservation is due to Cauchy. It is valid for any continuum, but the stress tensors of different materials can be rather different.

The stress tensor for fluids is symmetric, i.e. $\Pi_{ij} = \Pi_{ji}$, according to BOLTZMANN's[7] postulate. It may be shown that the symmetry of the stress tensor is equivalent to the postulate that the angular momentum of a material fluid element is only changed by torques $\boldsymbol{x} \times \boldsymbol{f}^{\mathrm{v}}$ associated with the volume forces, and torques $\boldsymbol{x} \times \boldsymbol{t}^{\mathrm{s}}$ arising from the stress forces, expressed by

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V(t)} \rho \boldsymbol{x} \times \boldsymbol{u} \mathrm{d}V = \int_{V(t)} \boldsymbol{x} \times \boldsymbol{f}^{\mathrm{v}} \mathrm{d}V + \int_{A(t)} \boldsymbol{x} \times \boldsymbol{t}^{\mathrm{s}} \mathrm{d}A \tag{2.32}$$

Put in mathematical terms, the torque of the volume representation $\boldsymbol{x} \times \nabla \cdot \Pi$ can only be expressed by a divergence $\nabla \cdot \boldsymbol{x} \times \Pi$ if the stress tensor is symmetric. Details are found in the box on p. 35. We like to emphasize that the symmetry of (Π_{ij}) is a postulate which is very well satisfied for fluids: a nonsymmetric stress tensor would generate stress couples as body torques, arising by a net volume contribution from the stress forces. These are not observed in fluid motions.

The diagonal elements of (Π_{ij}) which act normal to the corresponding surface are *normal stresses*. The off-diagonal elements of (Π_{ij}) act tangential; they are the *tangential stresses*. The normal stresses on different surfaces of a volume element need not be identical. The mean normal inward stress is the *pressure*

$$p = -\frac{1}{3}\left(\Pi_{11} + \Pi_{22} + \Pi_{33}\right) = -\frac{1}{3}\Pi_{ii} = -\frac{1}{3}\mathrm{tr}\,\Pi \tag{2.33}$$

---

[7] LUDWIG BOLTZMANN, *1844 in Linz, †1908 in Duino near Triest, physicist.

The relation (2.33) defines the *mechanical pressure* which is now assumed to be identical[8] to the thermodynamical pressure introduced in Section 1.2.

It is useful to decompose the stress tensor according to the isotropic pressure part and the remaining terms,

$$\Pi_{ij} = -p\delta_{ij} + \Sigma_{ij} \tag{2.34}$$

The remainder (Σ_{ij}) is called the *friction tensor* (or viscous stress tensor or deviatoric stress tensor). It does not transport any normal component of momentum in the mean since by definition the sum of the viscous normal stresses must vanish, $\Sigma_{ii} = 0$. For the surface force per volume $\boldsymbol{f}^{\mathrm{s}}$ acting on a volume element we have then

$$\boldsymbol{f}^{\mathrm{s}} = \boldsymbol{\nabla} \cdot \Pi = -\boldsymbol{\nabla} p + \boldsymbol{\nabla} \cdot \Sigma$$

In a fluid at rest or in uniform motion all tangential stresses must vanish; otherwise fluid elements would start to move relative to neighboring elements. Likewise, all normal stresses must be identical, otherwise the volume would be deformed and also induce relative motion. More generally, an *ideal or perfect fluid* is defined by the vanishing of all viscous stresses, $\Sigma \equiv 0$. The surface forces are then given by the gradient of pressure alone, $\boldsymbol{f}^{\mathrm{s}} = -\boldsymbol{\nabla} p$. In real fluids, the viscous stresses do generally not vanish. The behavior of a solid body and a fluid is fundamentally different. In a deformed body, forces occur directly as result of deformations and hence a relation between stress tensor and deformation tensor is expected. This *constitutive law* or *stress–strain relation* is a material property of the medium. With respect to changes

7. Balance of Angular Momentum

The stress tensor for fluids must be symmetric, $\Pi_{ij} = \Pi_{ji}$, as any antisymmetric component would result in net torques internally in the fluid and hence would lead to a rotation of respective volume elements. To prove this property, we write (2.31) in the flux form and proceed to the balance of angular momentum (with respect to the coordinate origin),

$$\frac{\partial}{\partial t}\rho\boldsymbol{x} \times \boldsymbol{u} + \boldsymbol{x} \times \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u}\boldsymbol{u} - \Pi) = \boldsymbol{x} \times \boldsymbol{f}^{\mathrm{v}}$$

obtained by taking the cross-product with the position vector $\boldsymbol{x}$. As $\boldsymbol{u}\boldsymbol{u}$ is a symmetric tensor, we have

$$\boldsymbol{x} \times \boldsymbol{\nabla} \cdot (\rho\boldsymbol{u}\boldsymbol{u}) = \boldsymbol{\nabla} \cdot \rho\boldsymbol{u}(\boldsymbol{x} \times \boldsymbol{u})$$

so that only the divergence of the advective flux of angular momentum is of relevance. Likewise, for the stress tensor we must require

$$\boldsymbol{x} \times \boldsymbol{\nabla} \cdot \Pi = \boldsymbol{\nabla} \cdot \boldsymbol{x} \times \Pi \tag{B7.1}$$

as otherwise torques would arise from the stresses in the interior of any volume. Written in component form, this results in $\epsilon_{ijk}\Pi_{jk} = \boldsymbol{0}$, which directly shows that the tensor $\boldsymbol{\Pi}$ must be symmetric. The relation (B7.1) or directly $\Pi_{ij} = \Pi_{ji}$ is known as Boltzmann's postulate. Then, in the Lagrangian view, the angular momentum balance becomes

$$\rho\frac{\mathrm{D}(\boldsymbol{x} \times \boldsymbol{u})}{\mathrm{D}t} = \boldsymbol{\nabla} \cdot (\boldsymbol{x} \times \Pi) + \boldsymbol{x} \times \boldsymbol{f}^{\mathrm{v}}$$

---

[8] The thermodynamic pressure relies on the concept of thermodynamic equilibrium; the mechanical pressure is defined for a moving fluid which is not in equilibrium. The equivalence of the two pressure concepts is nontrivial; we refer the reader to Batchelor (1977).

in volume, most materials react in a similar elastic way. Differences between materials occur, however, when the deformation causes a strain rather than a volume change.

In a fluid, displacements generally do not result in forces. Only a relative motion of a neighboring parcel exerts a stress. It is phenomenologically well known that for a constant velocity field all tangential stresses vanish. A Newtonian fluid is defined by a linear relation between stress and velocity shear, for instance a shear flow $u(y)$ in the x-direction the tangential stress Σ_{yx} (describing flux of x-momentum in the negative y-direction) is proportional to the gradient $\partial u/\partial y$, i. e.

$$\Sigma_{yx} = \nu \frac{\partial u}{\partial y}$$

with a down-gradient direction of transport (note here again the above-mentioned sign convention). The (dynamical) viscosity ν is a property of the fluid. A general *linear* relation between stress and strain must involve both the stress tensor and the deformation tensor. Both tensors are symmetric, but the stress tensor has a vanishing trace. The deviator $D^{\star}_{ij}$ of the deformation tensor D_{ij}, introduced in (1.10), describes shearing motion without a change in volume. It is symmetric and also has a vanishing trace. The simplest relation between stress and strain hence is given by

$$\Sigma_{ij} = 2\nu \mathrm{D}^{\star}_{ij} = 2\nu \left(\mathrm{D}_{ij} - \frac{1}{3}\mathrm{D}_{\ell\ell}\delta_{ij} \right) \tag{2.35}$$

or explicitly

$$\Sigma_{ij} = \nu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_\ell}{\partial x_\ell}\delta_{ij} \right) \tag{2.36}$$

Relation (2.36) defines a general *Newtonian fluid*, and has been confirmed experimentally for gases and most fluids. However, there are fluids which behave differently, e. g. where a nonlinear or more complicated, (e. g. time-dependent) relation between stress and strain is found. Such fluids are called non-Newtonian fluids; prominent examples are suspensions like paint, which gets less viscous when painted and more viscous when it remains at the wall or paper, or ketchup, which also often needs some time-dependent treatment to get less viscous and leave the bottle. The dynamical viscosity of pure water is about 10^{-3} N s m^{-2} at 20°C and decreases (increases) significantly for higher (lower) temperatures. There are not many fluids which are less viscous than water, e. g. acetone has a viscosity of 3×10^{-4} N s m^{-2} at 25°C, but many which are more viscous, e. g. olive oil with a viscosity of 0.08 N s m^{-2}. Figure 2.5 shows the dynamical viscosity of seawater, which is very similar to pure water, for range of temperature and salinity representative for the ocean. The dependency of ν on salinity is much weaker compared to the one on temperature.

The divergence of the frictional stress tensor, i. e. the frictional force per volume, is given by

$$\frac{\partial \Sigma_{ji}}{\partial x_j} = \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_i}{\partial x_j} \right) + \frac{\partial}{\partial x_j}\left(\nu \frac{\partial u_j}{\partial x_i} \right) - \frac{2}{3}\frac{\partial}{\partial x_i}\left(\nu \frac{\partial u_j}{\partial x_j} \right) \tag{2.37}$$

In most applications it is sufficiently accurate to neglect the spatial variations of ν (which is due to the dependence of ν on thermodynamic variables as shown in

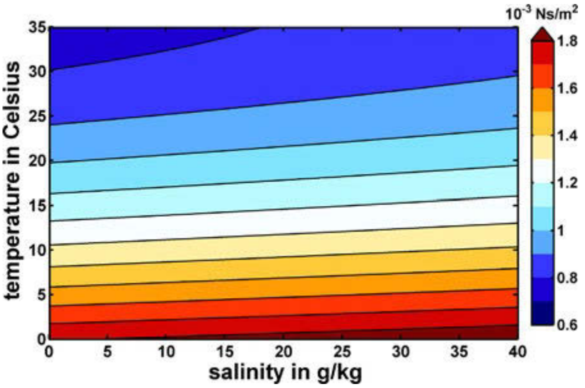

Fig. 2.5 Dynamical viscosity of seawater ν in $10^{-3}\,\mathrm{N\,s\,m^{-2}}$ as a function of temperature and salinity. After Sharqawy et al. (2010)

Figure 2.5) so that

$$\frac{\partial \Sigma_{ji}}{\partial x_j} = \nu \left(\frac{\partial^2 u_i}{\partial x_j^2} + \frac{1}{3} \frac{\partial^2 u_j}{\partial x_i \partial x_j} \right) \tag{2.38}$$

Incorporating the force of the molecular stresses into Newton's law of motion, the balance of momentum (2.26) thus takes the form

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \nabla \cdot \Pi + \boldsymbol{f}^{\mathrm{v}} = -\nabla p + \nabla \cdot \Sigma + \boldsymbol{f}^{\mathrm{v}} \tag{2.39}$$

where $\boldsymbol{f}^{\mathrm{v}}$ denotes the volume force which still remains to be specified (compare Section 2.3.4). Using the continuity equation (2.15), we obtain the equivalent flux form

$$\frac{\partial}{\partial t} \rho \boldsymbol{u} = -\nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} - \Pi) + \boldsymbol{f}^{\mathrm{v}} = -\nabla p - \nabla \cdot (\rho \boldsymbol{u}\boldsymbol{u} - \Sigma) + \boldsymbol{f}^{\mathrm{v}} \tag{2.40}$$

with the *momentum flux tensor* $\rho \boldsymbol{u}\boldsymbol{u} - \Pi$ which includes the momentum flux both through advection and through surface forces. Occasionally, only the advective contribution is called momentum flux.

Expressing the divergence of the frictional tensor by (2.38) (i. e. neglecting the spatial dependency of viscosity) leads to the NAVIER–STOKES[9] equations

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -\nabla p + \nu \nabla^2 \boldsymbol{u} + \frac{\nu}{3} \nabla(\nabla \cdot \boldsymbol{u}) + \boldsymbol{f}^{\mathrm{v}} \tag{2.41}$$

written here in the parcel form. These equations form the basis of most applications of hydrodynamics.

---

[9] CLAUDE-LOUIS NAVIER, *1785 in Dijon, †1836 in Paris, engineer and physicist. He laid down the complete Navier–Stokes equations in 1823 for an incompressible fluid. His work received little contemporary attention, and so the equations were rediscovered at least four times: by Cauchy in 1823, by Poisson in 1829, by Saint-Venant in 1837, and by Stokes in 1845 (Darrigol, 2009). GEORGE GABRIEL STOKES, *1819 in Skreen, †1903 in Cambridge, mathematician and physicist.

2.3.2 Boundary Condition for the Momentum Flux

For the momentum (per mass) $\boldsymbol{u}$ and the momentum flux Π one obtains from (2.7) and (2.21)

$$\boldsymbol{u}\mathcal{J}_{\text{mass}} - \Pi \cdot \boldsymbol{n} = -\Pi^a \cdot \boldsymbol{n} \tag{2.42}$$

where $\mathcal{J}_{\text{mass}} = \mathcal{E} - \mathcal{P}$ as before, and Π^a is stress tensor of the neighboring medium, considering the sea surface. The first term on the left-hand side represents the transfer of momentum associated with vapor leaving or rain drops entering the ocean. It is generally very small compared to molecular stresses exchanging momentum (at the sea surface: $\mathcal{J}_{\text{mass}}/\rho = (\mathcal{E} - \mathcal{P})\rho$ is a very small velocity of order $1\,\text{m/y} \sim 10^{-7}\text{m}\,\text{s}^{-1}$, and the second term is of order $\nu/(\rho\Delta) \sim 10^{-6}/\Delta\,\text{m}^2\,\text{s}^{-1}$ where Δ is the scale of the velocity gradient. The first term in (2.42) is, therefore, of negligible magnitude.

Decomposing (2.42) into normal and tangential components, we obtain the *dynamic* boundary conditions

$$-p + \Sigma_{33} = -p^a + \Sigma^a_{33} \tag{2.43}$$

$$\Sigma_{j3} = \Sigma^a_{j3} = \tau^a_j\,, \quad j = 1, 2 \tag{2.44}$$

where $\boldsymbol{\tau}^a$ is the tangential force (windstress) and p^a the pressure at the lower boundary of the atmosphere, and $\boldsymbol{n}$ has been taken vertically for simplicity. Remember, however, that the stresses considered here are of molecular character, and the implied molecular boundary layers at the interfaces contain extremely large velocity shears to carry e. g. the flux of momentum imparted by the surface wind field. Usually the viscous part in (2.43) is small, so that the pressure field is very nearly continuous at the surface, $p \approx p^a$.

At the bottom of the ocean, the pressure within the solid earth is not constrained by the oceanic pressure, and the relation (2.43) does not apply. In fluid mechanics two possibilities for the dynamic boundary conditions are popular at solid interfaces: $\Sigma \cdot \boldsymbol{n} = 0$, representing the case of no stress or *free slip*, and $\boldsymbol{u} = 0$, requiring a vanishing flow speed, termed *no slip*. Figure 2.6 displays the structure of the near-bottom velocity for the two conditions. These conditions then require a certain stress at the bottom which is determined from the solution of the respective hydrodynamical problem. A parameterization which is intermediate between the limiting forms

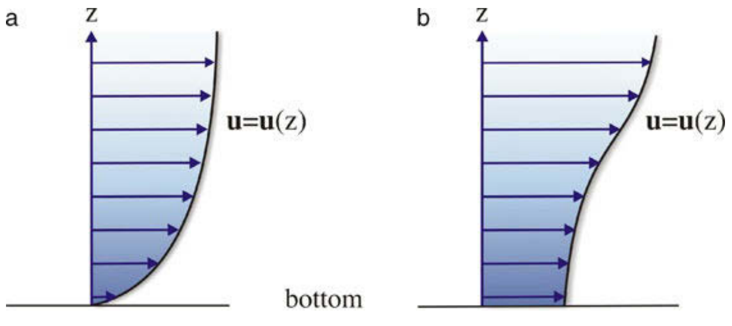

Fig. 2.6 Sketch of no-slip (**a**) and free-slip (**b**) conditions at the ocean bottom

of free slip and no slip is given by

$$\Sigma \cdot \boldsymbol{n} = r\boldsymbol{u} \cdot \boldsymbol{t} \tag{2.45}$$

where $\boldsymbol{t}$ is the tangential unit vector in the bottom surface. The coefficient r is empirical, frequently expressed as $r = r^*|\boldsymbol{u} \cdot \boldsymbol{t}|$ which makes the parameterization nonlinear. Clearly, for small r we regain the free-slip condition and for large r the no-slip condition, because additionally to $\boldsymbol{u} \cdot \boldsymbol{t} = 0$ we also have $\boldsymbol{u} \cdot \boldsymbol{n} = 0$ from the the kinematic condition at the sea floor.

2.3.3 Conservation Equations on the Rotating Earth

So far we have used the field variables without any reference to a particular coordinate system. We have introduced scalar fields χ and vector fields $\boldsymbol{v}$ (not necessarily velocity) and implicitly used the notion of a Cartesian coordinate system. However, whereas two observers with their orientation in two different coordinate frames would certainly agree on the value of a scalar property $\chi(t)$ of a fluid parcel at all times and its temporal change $\mathrm{D}\chi/\mathrm{D}t$, they would not necessarily agree on vector properties $\boldsymbol{v}(t)$. A vector is characterized by a length and direction, and both depend on a predefined frame of reference. In particular, the two observers might get different values for $\boldsymbol{v}(t)$ and its material rate of change $\mathrm{D}\boldsymbol{v}/\mathrm{D}t$.

Rotating Coordinate System

The Earth is turning itself but also rotating around the Sun. It is, therefore, convenient to consider a rotating coordinate system. An *inertial* frame is characterized by constant unit vectors $\boldsymbol{e}_i{}^\star$ in the coordinate direction $i = 1, 2, 3$, while for a *rotating* frame of reference the unit vectors $\boldsymbol{e}_i(t)$ are time dependent and rotate with the angular velocity $\boldsymbol{\Omega}$. Figure 2.7 outlines the situation in a two-dimensional sketch. The magnitude of $\boldsymbol{\Omega}$ denotes the angular rotation speed of $\Omega = \mathrm{d}\lambda/\mathrm{d}t$, where $\lambda(t)$ denotes the angle between the individual unit vectors $\boldsymbol{e}_i(t)$ and $\boldsymbol{e}_i{}^\star$ of the two coordinate systems. The direction of $\boldsymbol{\Omega}$ defines the rotation axis, which points out of the plane in the scheme of Figure 2.7. While the rotating frame has time-dependent unit vectors $\boldsymbol{e}_i(t)$ when viewed from the inertial frame, an observer in the rotating system would, of course, see 'his' $\boldsymbol{e}_i$ as vectors which are constant in time. In the following we assume that the coordinate systems have the same origin; the situation of the Earth is a bit more complicated and discussed in the box on p. 41.

It is worth mentioning that though $\mathrm{D}\chi/\mathrm{D}t$ is the same for the two observers, they obtain different local rates of change and different advective rates of change, i. e. expanding $\mathrm{D}/\mathrm{D}t$ in the inertial and the rotating system into the Eulerian form,

$$\frac{\mathrm{D}\chi}{\mathrm{D}t} = \frac{\partial\chi}{\partial t} + \boldsymbol{u}^\star \cdot \nabla^\star\chi \quad \text{and} \quad \frac{\mathrm{D}\chi}{\mathrm{D}t} = \frac{\partial\chi}{\partial t} + \boldsymbol{u} \cdot \nabla\chi$$

the $\partial\chi/\partial t$ are different in the two expressions because different Eulerian spatial coordinates are held fixed. These are $x_i^\star = \boldsymbol{e}_i^\star \cdot \boldsymbol{x}$ and $x_i = \boldsymbol{e}_i \cdot \boldsymbol{x}$, respectively.

We will now compare the material rate of change $\mathrm{D}\boldsymbol{v}/\mathrm{D}t$ of a vector, measured in the inertial frame, to the rate of change which an observer in the rotating frame

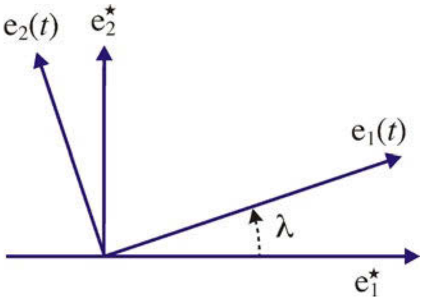

Fig. 2.7 A sketch outlining the rotating coordinate system. The inertial frame is given by $\boldsymbol{e}_1{}^\star$ and $\boldsymbol{e}_2{}^\star$. The frame given by $\boldsymbol{e}_1$ and $\boldsymbol{e}_2$ is rotating with an angular velocity $\boldsymbol{\Omega} = \mathrm{d}\lambda/\mathrm{d}t$

obtains. We will first consider the arbitrary vector $\boldsymbol{v}(t)$, afterward the result will be evaluated for the position vector and the velocity of a fluid parcel. Because $\boldsymbol{e}_1(t + \Delta t) = \boldsymbol{e}_1(t) + \Delta\lambda\boldsymbol{e}_2(t)$ for a unit base vector which is normal to rotation axis (see Figure 2.7), we obtain $\mathrm{d}\boldsymbol{e}_1/\mathrm{d}t = \Omega\boldsymbol{e}_2$ and similarly $\mathrm{d}\boldsymbol{e}_2/\mathrm{d}t = -\Omega\boldsymbol{e}_1$ where $\Omega = \mathrm{d}\lambda/\mathrm{d}t$. Hence, in general

$$\frac{\mathrm{d}\boldsymbol{e}_i}{\mathrm{d}t} = \boldsymbol{\Omega} \times \boldsymbol{e}_i$$

is observed in the inertial frame. Consider now an arbitrary time-dependent vector $\boldsymbol{v}(t)$ with components $v_i^*(t) = \boldsymbol{v}(t) \cdot \boldsymbol{e}_i{}^*$ in the inertial frame and components $v_i(t) = \boldsymbol{v}(t) \cdot \boldsymbol{e}_i(t)$ in the rotating frame. The temporal rate of change of $\boldsymbol{v}(t)$ is thus

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \frac{\mathrm{d}v_i^*}{\mathrm{d}t}\boldsymbol{e}_i{}^* = \frac{\mathrm{d}v_i}{\mathrm{d}t}\boldsymbol{e}_i + v_i\frac{\mathrm{d}\boldsymbol{e}_i}{\mathrm{d}t}$$

The first term in the second expression is the change which a corotating observer measures. We use the notation $(\mathrm{d}/\mathrm{d}t)^{\mathrm{rot}}\boldsymbol{v}$ to write this term in vector form. The second term is just $\boldsymbol{\Omega} \times \boldsymbol{v}$, hence

$$\frac{\mathrm{d}\boldsymbol{v}}{\mathrm{d}t} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathrm{rot}} \boldsymbol{v} + \boldsymbol{\Omega} \times \boldsymbol{v}$$

Application to the position vector $\boldsymbol{X}(t)$ of a fluid parcel, now a material property, yields

$$\frac{\mathrm{D}\boldsymbol{X}}{\mathrm{D}t} = \boldsymbol{u} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathrm{rot}} \boldsymbol{X} + \boldsymbol{\Omega} \times \boldsymbol{X} = \boldsymbol{u}^{\mathrm{rot}} + \boldsymbol{\Omega} \times \boldsymbol{X}$$

The first term, $\boldsymbol{u}^{\mathrm{rot}} = (\mathrm{d}/\mathrm{d}t)^{\mathrm{rot}}\boldsymbol{X}$, is the velocity vector measured by the corotating observer. Further, for the acceleration, we find

$$\begin{aligned}
\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathrm{rot}} \boldsymbol{X} + \boldsymbol{\Omega} \times \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} \\
&= \left(\frac{\mathrm{d}^2}{\mathrm{d}t^2}\right)^{\mathrm{rot}} \boldsymbol{X} + \boldsymbol{\Omega} \times \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathrm{rot}} \boldsymbol{X} + \boldsymbol{\Omega} \times \left[\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathrm{rot}} \boldsymbol{X} + \boldsymbol{\Omega} \times \boldsymbol{X}\right] \\
&= \left(\frac{\mathrm{d}}{\mathrm{d}t}\right)^{\mathrm{rot}} \boldsymbol{u}^{\mathrm{rot}} + 2\boldsymbol{\Omega} \times \boldsymbol{u}^{\mathrm{rot}} + \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})
\end{aligned} \tag{2.46}$$

8. A Rotating Coordinate System for the Earth

A natural coordinate system attached to the Earth is the geocentric Cartesian frame with its origin in the Earth's center. However, the Earth rotates around the Sun and also around itself. Further, the solar system and the whole galaxy might rotate, which, however, should add only small contributions. We first consider the rotation of the Earth around the Sun using a Cartesian system in the center of the Sun, rotating with a period of 365.24 days. We assume a constant angular velocity of $\Omega_{\text{sun}} = 2\pi/(365.24 \times 86400\,\text{s}) = 0.02 \times 10^{-5}\,\text{s}^{-1}$ and a circular path of the Earth around the Sun, such that the Earth remains now at a fixed position in the rotating frame. In this coordinate system a corresponding Coriolis and centrifugal force in the momentum budget (2.47) will appear.

Next, we consider the rotation of the Earth itself. First we move the center of the coordinate system into the center of the Earth and we also rotate the system (once) from the ecliptic into the equatorial plane of the Earth (which has an angle with the ecliptic of about $23°$). Note that this simple transformation will have no consequences in the momentum budget, except that $\boldsymbol{\Omega}_{\text{sun}}$ will be rotated by about $-23°$ with respect to the equatorial plane. Since the Earth is still rotating, we finally switch to a rotating coordinate system with rotation vector $\boldsymbol{\Omega}_{\text{earth}}$ pointing towards the North Pole with magnitude $\Omega_{\text{earth}} = 2\pi 86400\,\text{s} = 7.272 \times 10^{-5}\,\text{s}^{-1}$. This last transformation will add another Coriolis and centrifugal force, such that the momentum budget becomes

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -2\rho\boldsymbol{\Omega} \times \boldsymbol{u} - \nabla p + \nabla \cdot \Sigma - \rho\boldsymbol{\Omega}_{\text{earth}} \times (\boldsymbol{\Omega}_{\text{earth}} \times \boldsymbol{X}) \\ - \rho\boldsymbol{\Omega}_{\text{sun}} \times (\boldsymbol{\Omega}_{\text{sun}} \times \boldsymbol{X}) + \boldsymbol{f}^{\text{v}}$$

with $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\text{earth}} + \boldsymbol{\Omega}_{\text{sun}}$ with magnitude $\Omega \approx 7.292 \times 10^{-5}\,\text{s}^{-1}$ and direction almost parallel to $\boldsymbol{\Omega}_{\text{earth}}$. Note that the centrifugal forces can be written as potentials which then both add to the geopotential discussed in Section 2.3.4. The contribution of the centrifugal potential of the Sun will be much smaller than the one of the Earth.

The left-hand side is the acceleration of the fluid element as seen in the absolute frame, and the first term on the right-hand side is the acceleration which the corotating observer measures for the same parcel. Note that if $\boldsymbol{\Omega}$ is time-dependent, a term $\mathrm{d}\boldsymbol{\Omega}/\mathrm{d}t \times \boldsymbol{X}$ will appear on the right-hand side of (2.46).

Momentum Budget in a Rotating System

The equations of motion in the preceding sections hold for an absolute coordinate system (*inertial system*). In a *rotating* frame, the left hand side of the momentum equation (2.39) must hence be replaced by the right hand side of (2.46). Dropping the index rot, this yields

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -2\rho\boldsymbol{\Omega} \times \boldsymbol{u} - \nabla p + \nabla \cdot \Sigma - \rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X}) + \boldsymbol{f}^{\text{v}} \tag{2.47}$$

where $\boldsymbol{u}$ now denotes the velocity seen by the corotating observer. Two additional forces appear in the rotating frame which are entirely due to the coordinate transformation and, therefore, have a virtual character:

- The CORIOLIS[10]–force
  $-2\rho\boldsymbol{\Omega} \times \boldsymbol{u}$ (or Coriolis acceleration $2\boldsymbol{\Omega} \times \boldsymbol{u}$) depends on the flow velocity, always trying to bend the parcel's path normal to the direction of $\boldsymbol{\Omega}$ and to $\boldsymbol{u}$. In other

---

[10] GASPARD GUSTAVE DE CORIOLIS, *1792 in Paris, †1843 in Paris, engineer and mathematician. Coriolis published his work in 1835, but almost 60 years earlier, in 1768, PIERRE-SIMON LAPLACE correctly posed the problem of forces on the rotating Earth in his work on tides.

words: looking down with the North Pole in view, we see a counterclockwise rotation of the Earth and thus a deviation of the motion on the northern hemisphere to the right. With the South Pole in view, we see a clockwise rotation, and the motion on the southern hemisphere is seen to deviate to the left. The Coriolis force is fundamental for all large-scale motions in ocean and atmosphere.

- The *centrifugal force* $-\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X})$ is independent of the flow variables; it will be discussed in the next section.

Both forces are also referred to as *apparent forces* as they are apparent only to a corotating observer.

2.3.4 The Force of Gravity on the Earth

According to Newton's law of gravitation a point mass M attracts another point mass m with the force

$$\mathcal{F}(\boldsymbol{r}) = -G_{\mathrm{N}} \frac{mM}{r^2} \frac{\boldsymbol{r}}{r} \tag{2.48}$$

Here $\boldsymbol{r}$ denotes the distance vector from M to m, and $G_{\mathrm{N}} = 6.673 \times 10^{-11}\,\mathrm{m^3\,kg^{-1}\,s^{-2}}$ is the universal constant of gravitation. The force $\mathcal{F}/m$ per unit mass can be derived from the gravity potential according to

$$\frac{\mathcal{F}}{m} = -\nabla\Phi^{\star} = \nabla G_{\mathrm{N}} \frac{M}{r} \tag{2.49}$$

The force per mass is $-\nabla\Phi^{\star}$, and the force per volume is $-\rho\nabla\Phi^{\star}$. Since forces are additive, the potentials from different masses add as well, and the attraction of a fluid parcel by different masses can hence be treated separately. The volume force exerted by gravity can thus be represented by a potential $\Phi^{\star}$. In oceanography and meteorology the gravity potentials of the Earth, Moon, and Sun are relevant, i. e. $\Phi^{\star} = \Phi_{\mathrm{E}} + \Phi_{\mathrm{m}} + \Phi_{\mathrm{s}}$. We discuss how to treat the contributions from the Earth and the Moon. The contribution of the Sun is analogous to the Moon.

Gravity Potential of the Earth

Newton's law of attraction holds if both m and M can be considered as *mass points* with negligible spatial extent. A result of potential theory is that Newton's law also holds for a body of finite extent, provided that its mass distribution is *spherically symmetric*, i. e. only dependent on the distance from the center of mass. The Earth's surface has, however, no spherical symmetry but is more accurately described as oblate spheroidal ellipsoid. Moreover, the mass distribution in the interior is also not spherically symmetric. Outside the Earth, the gravity potential is, therefore, more accurately given by an expansion of the potential in spherical harmonics with the leading terms

$$\Phi_{\mathrm{E}}(r,\lambda,\varphi) = -G_{\mathrm{N}} \frac{M_{\mathrm{E}}}{r} \left[1 - \varepsilon_1 \frac{a_{\mathrm{e}}^2}{r^2} \left(3\sin^2\varphi - 1\right) + \varepsilon_{\star}(r,\lambda,\varphi)\right] \tag{2.50}$$

(see e.g. Stacey, 1992). Here r denotes the distance from the mass center of the Earth, λ is the geographical longitude and φ the geographical latitude. The mass of the Earth is $M_{\mathrm{E}} = 5.97 \times 10^{24}$ kg, the equatorial radius of the Earth is $a_{\mathrm{e}} = 6378.1$ km, and $\varepsilon_1 = 0.541 \cdot 10^{-3}$ is a dimensionless coefficient. The residual term $\varepsilon_\star(r, \lambda, \varphi)$ (see Figure 2.9 below) has a magnitude of $\approx 10^{-5}$ and can be neglected for all dynamical purposes. Likewise, corrections to (2.50) resulting from the atmospheric mass distribution (see e. g. Rummel and Rapp, 1976) can be neglected.

Geopotential

Similar to the gravity force, the centrifugal force in the momentum budget (2.47) can be rewritten in terms of a potential as shown in this section. Note that we only consider here the centrifugal force resulting from the Earth rotation since this is the most important (compare the box on p. 41). Only the component $\boldsymbol{X}_\perp$ of the position vector which is perpendicular to the axis of rotation contributes to the centrifugal force. This is because of $\boldsymbol{\Omega} \times \boldsymbol{X} = \boldsymbol{\Omega} \times \boldsymbol{X}_\perp$. With the vector identity $\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c}$ one finds $\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X}_\perp) = -\Omega^2 \boldsymbol{X}_\perp$. Assuming that the axis of rotation is the z–direction, we have $\boldsymbol{X}_\perp = (x, y, 0) = \frac{1}{2}\nabla(x^2 + y^2) = \frac{1}{2}\nabla r^2 \cos^2\varphi$ where φ is the latitude coordinate and r the length of the vector $\boldsymbol{X}$. The centrifugal acceleration can thus be derived from a potential according to

$$\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X}) = -\nabla\left(\frac{1}{2}\Omega^2 r^2 \cos^2\varphi\right)$$

and hence be combined with the gravity, arising for the Earth's attraction, to the *geopotential* $\Phi = \Phi_{\mathrm{E}} - \frac{1}{2}\Omega^2 r^2 \cos^2\varphi$ (frequently called 'apparent' gravity potential) so that

$$\Phi(r, \lambda, \varphi) = -G_{\mathrm{N}}\frac{M_{\mathrm{E}}}{r}\left[1 - \varepsilon_1 \frac{a_{\mathrm{e}}^2}{r^2}\left(3\sin^2\varphi - 1\right) + \frac{1}{2}\varepsilon_2\left(\frac{r}{a_{\mathrm{e}}}\right)^3 \cos^2\varphi + \varepsilon_\star(r, \lambda, \varphi)\right] \tag{2.51}$$

where $\varepsilon_2 = \Omega^2 a_{\mathrm{e}}^3/(G_{\mathrm{N}} M_{\mathrm{E}}) \approx 3.47 \times 10^{-3}$ is a dimensionless constant. The combined force in the equation of motion is thus $-\rho\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{X}) - \rho\nabla\Phi_{\mathrm{E}} = -\rho\nabla\Phi$. The geometric condition of the gravitational and geopotential surfaces with associated accelerations is depicted in Figure 2.8.

Surfaces of constant geopotential $r_{\mathrm{g}} = r_{\mathrm{g}}(\lambda, \varphi)$ are defined by $\Phi(r_{\mathrm{g}}, \lambda, \varphi) =$ const. With (2.51), and neglecting the residual $\varepsilon_\star(r, \lambda, \varphi)$, one has

$$-G_{\mathrm{N}}\frac{M_{\mathrm{E}}}{r_{\mathrm{g}}}\left[1 - \varepsilon_1\left(\frac{a_{\mathrm{e}}}{r_{\mathrm{g}}}\right)^2\left(3\sin^2\varphi - 1\right) + \frac{1}{2}\varepsilon_2\left(\frac{r_{\mathrm{g}}}{a_{\mathrm{e}}}\right)^3\cos^2\varphi\right] = \mathrm{const} \tag{2.52}$$

The geopotential surface corresponding to the mean sea level is called the *geoid*. It can be derived from (2.52) to a first approximation by replacing $r_{\mathrm{g}} \approx a_{\mathrm{e}}$ in the small terms, with the result

$$r_{\mathrm{g}} \approx a_{\mathrm{e}}\left(1 - \varepsilon_3 \sin^2\varphi\right) \tag{2.53}$$

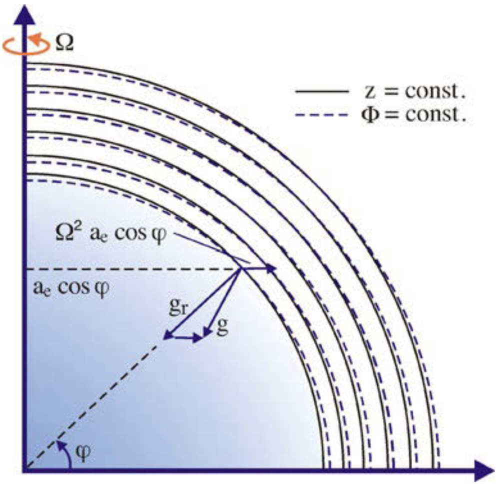

Fig. 2.8 Gravity as sum of the Earth attraction and the centrifugal acceleration. Surfaces of equal height (solid lines) and equal geopotential (dashed lines) are schematically displayed, i. e. the difference between both is exaggerated

with $\varepsilon_3 \approx 3\varepsilon_1 + \varepsilon_2/2 = 3.36 \times 10^{-3}$. In this approximation, the geoid hence has the shape of an oblate ellipsoid: its height increases from the pole (polar radius $a_\mathrm{p} = 6356.8\,\mathrm{km}$) to the equator by $\varepsilon_3 a_\mathrm{e} \approx 21.4\,\mathrm{km}$ as demonstrated in Figure 2.8.

Deviations of the geoid from the reference ellipsoid (shown in Figure 2.9) are related to the residual term $\varepsilon_\star(r, \lambda, \varphi)$ in (2.51). These deviations reflect the non-spherical mass distribution in the Earth, and have the form of irregular undulations with a maximum amplitude of about 100 m. The exact form of the geoid is of great importance in various contexts, in particular in connection with satellite altimetry.

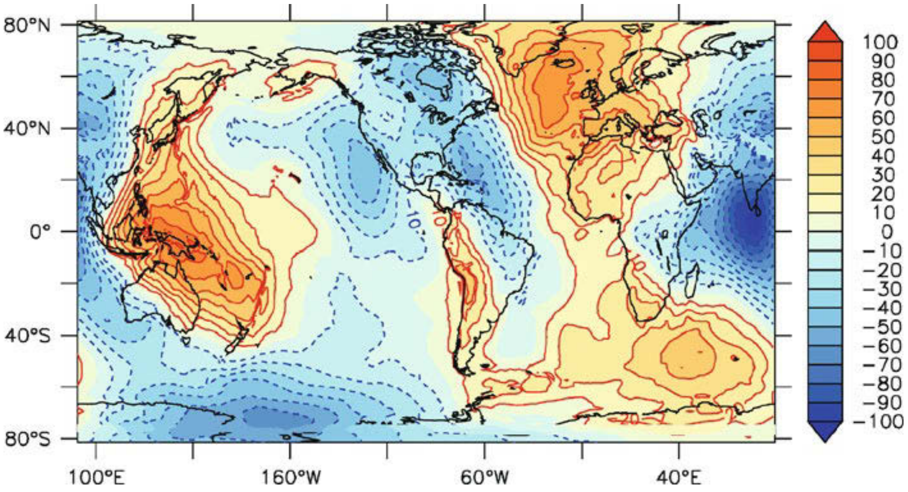

Fig. 2.9 Deviation of the geoid from the reference ellipsoid, related to the residual term $\varepsilon_\star(r, \lambda, \varphi)$ in (2.50) (in m). Data are taken from the official Earth Gravitational Model EGM2008 released by the US National Geospatial-Intelligence Agency. Black lines denote the continental shape

The gravitational acceleration (in short *gravity*) $\boldsymbol{g} = -\nabla\Phi$ is perpendicular to the geopotential surfaces. It follows from (2.52) that $\boldsymbol{g}$ is very nearly parallel to the radius vector $\boldsymbol{r}$ (the ratio of horizontal to vertical component is at most ε_3). With (2.51) and (2.53), the magnitude of gravity $g = |\nabla\Phi| \approx \partial\Phi/\partial r$ at the Earth's surface $r = r_\mathrm{g}(\varphi)$ is hence approximately given by

$$g = \frac{G_\mathrm{N} M_\mathrm{E}}{a_\mathrm{e}^2\left(1 - \varepsilon_3 \sin^2\varphi\right)}\left[1 - 3\varepsilon_1\left(3\sin^2\varphi - 1\right) - \varepsilon_2\cos^2\varphi\right] \approx g_\mathrm{e}\left(1 + \varepsilon_4\sin^2\varphi\right) \tag{2.54}$$

with $g_\mathrm{e} = (G_\mathrm{N}M_\mathrm{E}/a_\mathrm{e}^2)(1 + 3\varepsilon_1 - \varepsilon_2)$ and $\varepsilon_4 = 2\varepsilon_2 - 3\varepsilon_1 \approx 5.3 \times 10^{-3}$. The gravitational acceleration is thus not constant on geopotential surfaces but increases from the equatorial value $g_\mathrm{e} \approx 9.78\,\mathrm{m\,s^{-2}}$ to $g_\mathrm{e}(1 + \varepsilon_4) \approx 9.83\,\mathrm{m\,s^{-2}}$ at the poles. It is customary to ignore these changes and to apply a constant value of $g = 9.81\,\mathrm{m\,s^{-2}}$, causing a relative error of 3×10^{-3}.

The component of gravity in the horizontal directions (normal to $\boldsymbol{r}$) is at most around $\varepsilon_3 g \approx 0.03\,\mathrm{m\,s^{-2}}$. While this is very small compared to the vertical component, it is not small in comparison to other acceleration terms in the same (horizontal) direction. It is hence very convenient and useful to use a coordinate system which has $\Phi = \text{const}$ as one coordinate surface. For orthogonal coordinates, gravity must thus coincide exactly with one coordinate direction, i. e. $\boldsymbol{g} = (0, 0, -g)$. The geopotential is then dependent on the vertical coordinate z. Referring the potential to the mean surface, i. e. $\Phi(z = 0) = 0$, we have

$$\Phi(z) = \int_0^z \mathrm{d}z' g(z') \approx gz \tag{2.55}$$

The geopotential is thus the work which must be applied to lift a unit mass from $z = 0$ to the height z.

Tidal Potential

By far the largest part of the attraction from the Moon and the Sun is compensated by the centrifugal force arising from the orbital motion. The compensation is not complete, however, and a residual *tidal force* remains. As example let us consider the tidal potential of the Moon. The distance between the centers of mass of Earth and Moon is $r_\mathrm{m} = 384{,}000\,\mathrm{km}$, approximately 60 times the Earth's radius, and the mass ratio is $M_\mathrm{E}/M_\mathrm{m} \approx 81.5$. To a good approximation, both bodies rotate on circular orbits around their joint center of mass which is located at a distance $s = r_\mathrm{m}M_\mathrm{m}/(M_\mathrm{E} + M_\mathrm{m}) \approx 4{,}600\,\mathrm{km}$ from the Earth's center of mass, still within the Earth. It is sufficient to consider the Earth as a sphere with radius $a = 6{,}371\,\mathrm{km}$.

The motion of the Earth around the joint center of mass is a revolution without rotation rather than a rigid rotation, i. e. each mass point within the Earth follows a circle with the same radius s but a different origin, as shown in Figure 2.10. Hence the centrifugal force $\omega^2\boldsymbol{s}$ is identical for all points. In order for the Earth–Moon system to remain in equilibrium, mass attraction by the Moon and centrifugal force must compensate exactly in the Earth's center of mass,

$$\omega^2\boldsymbol{s} - G_\mathrm{N}\frac{M_\mathrm{m}}{r_\mathrm{m}^2}\frac{\boldsymbol{r}_\mathrm{m}}{r_\mathrm{m}} = 0 \tag{2.56}$$

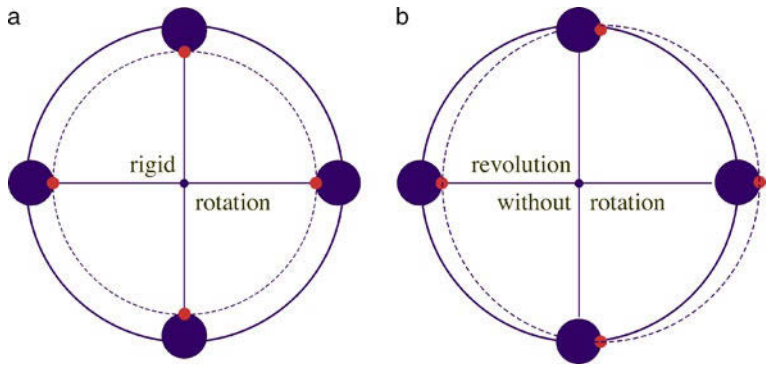

Fig. 2.10 **a** shows a rigid rotation of the large blue body, where all points (red dot) on the body move on circles with a common center but different radii. For a revolution without rotation, shown in **b**, the points rotate on circles with identical radii but different origin

At any point $\boldsymbol{r}$ which is in a distance $r_\star$ from the Moon so that $\boldsymbol{r}_\mathrm{m} + \boldsymbol{r} = \boldsymbol{r}_\star$ (see Figure 2.11), the total force per mass (i. e. the acceleration) is thus

$$\boldsymbol{f} = -G_\mathrm{N} \frac{M_\mathrm{m}}{r_\star^2} \frac{\boldsymbol{r}_\star}{r_\star} + \omega^2 \boldsymbol{s} = -G_\mathrm{N} M_\mathrm{m} \left(\frac{\boldsymbol{r}_\star}{r_\star^3} - \frac{\boldsymbol{r}_\mathrm{m}}{r_\mathrm{m}^3} \right)$$

where (2.56) has been used for the second form. The acceleration can be expressed as gradient of a potential, i. e. $\boldsymbol{f} = -\nabla \Phi_\mathrm{m}$, with

$$\Phi_\mathrm{m} = -G_\mathrm{N} M_\mathrm{m} \left(\frac{1}{r_\star} - \frac{1}{r_\mathrm{m}} - \frac{r}{r_\mathrm{m}^2} \cos\vartheta \right) \tag{2.57}$$

For convenience, the constant $G_\mathrm{N} M_\mathrm{m}/r_\mathrm{m}$ has been added here. For the distance $r_\star$ it follows geometrically (law of cosines, see Figure 2.11) that $r_\star^2 = r_\mathrm{m}^2 + r^2 - 2 r r_\mathrm{m} \cos\vartheta$ (here ϑ is the angle between $\boldsymbol{r}$ and $\boldsymbol{r}_\mathrm{m}$). As $r/r_\mathrm{m} \approx 0.017$ at the Earth's surface, a Taylor expansion for $r/r_\mathrm{m} \ll 1$ yields

$$\frac{r_\mathrm{m}}{r_\star} = \left(1 - 2\frac{r}{r_\mathrm{m}} \cos\vartheta + \frac{r^2}{r_\mathrm{m}^2} \right)^{-\frac{1}{2}} = 1 + \frac{r}{r_\mathrm{m}} \cos\vartheta - \frac{1}{2} \frac{r^2}{r_\mathrm{m}^2} + \frac{3}{2} \frac{r^2}{r_\mathrm{m}^2} \cos^2\vartheta + o\left(\frac{r^3}{r_\mathrm{m}^3} \right)$$

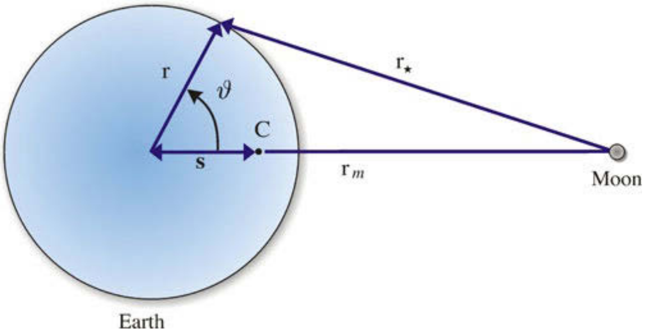

Fig. 2.11 Configuration of the Earth–Moon system. $\boldsymbol{C}$ is the center of mass of the two bodies

Hence the Moon's potential follows from (2.57), up to second order in r/r_m, as

$$\Phi_\mathrm{m}(r,\vartheta) = -G_\mathrm{N} M_\mathrm{m} \frac{r^2}{r_\mathrm{m}^3} \frac{1}{2} \left(3\cos^2\vartheta - 1\right) \tag{2.58}$$

The magnitude of the tidal force $F^\mathrm{r} = \partial\Phi_\mathrm{m}/\partial r$ in the radial direction compared to the gravity force at the Earth's surface is very small,

$$\frac{F^\mathrm{r}}{g^\star} = \frac{G_\mathrm{N} M_\mathrm{m} a / r_\mathrm{m}^3}{G_\mathrm{N} M_\mathrm{E} / a^2} = \frac{M_\mathrm{m}}{M_\mathrm{E}} \left(\frac{a}{r_\mathrm{m}}\right)^3 \approx 10^{-7}$$

The vertical component of the tidal force hence is always negligible. For the horizontal component it suffices to set $r = a$ in (2.58).

The angle ϑ depends on the astronomical variables describing the Moon's orbit relative to the position $\boldsymbol{r}$, and can be expressed as a function of geographic longitude and latitude as well as time, $\vartheta = \vartheta(\lambda, \varphi, t)$. The details are not given here, see e. g. Pugh (1987). However, it is easy to see that because of the Earth rotation and the rotation of the Moon, ϑ will contain a near-daily and a near-monthly period, and that the potential will be dominated by approximately half-daily and half-monthly periods (note that due to the relation $\cos^2\vartheta = (1 + \cos 2\vartheta)/2$ the forcing period in (2.58) is halved). Such periods can in fact be seen in many tidal records.

For the Sun the situation is completely analogous. Both potentials are of nearly comparable magnitude, $\Phi_\mathrm{s}/\Phi_\mathrm{m} \sim (M_\mathrm{s}/M_\mathrm{m}) \cdot (r_\mathrm{m}/r_\mathrm{s})^3 \approx 0.45$. The sum of the potentials of the Moon and the Sun, $\Phi_\mathrm{tide}(\lambda, \varphi, t) = \Phi_\mathrm{m} + \Phi_\mathrm{s}$, is the full tidal potential, and $\Phi + \Phi_\mathrm{tide}$ combines the attraction by the Earth, Moon, and Sun and includes the effect of the centrifugal force. With this, the momentum budget finally takes the form

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -2\rho\boldsymbol{\Omega} \times \boldsymbol{u} - \nabla p + \nabla \cdot \Sigma - \rho \nabla \left(\Phi + \Phi_\mathrm{tide}\right) \tag{2.59}$$

Remember that the geopotential Φ can be considered time-independent whereas the tidal potential Φ_tide is time-varying.

In the remainder of this book, tidal phenomena will not be considered, except for the energy budgets in Section 2.4 where tidal forcing will be included by redefining $\Phi \to \Phi + \Phi_\mathrm{tide}$.

2.4 Energy Conservation

In this section, we look at various forms of energy in a moving fluid and the corresponding energy budgets. In Section 1.2, the internal energy E has been introduced which contains, among other contributions, the kinetic energy of molecular motion. In a moving fluid, one has to include the kinetic energy of the *macroscopic* hydrodynamic motion, described in terms of the flow velocity $\boldsymbol{u}$, as there may be exchanges between both forms of energy. Motion in the presence of gravity readily leads to the concept of potential energy. Finally, a budget combining all relevant forms of energy is presented. All energy budgets are based on the first law of thermodynamics discussed in Section 1.2 and on the equations discussed in the previous sections of this chapter.

2.4.1 Contributions to the Change of Energy in a Material Volume

The kinetic energy of a parcel with mass m moving with the flow velocity $\boldsymbol{u}$ is $m\boldsymbol{u}^2/2$. Hence the sum of both internal and macroscopic kinetic energies per mass is $E + \boldsymbol{u}^2/2$. In analogy to the first law of thermodynamics (1.22) we formulate the conservation for the energy $\int_V \rho(E + \boldsymbol{u}^2/2)\mathrm{d}V$ in a *material* volume moving with the flow velocity. Note that we could also consider a volume fixed in space as done for the general conservation law in Section 2.1 (for which we would have to include an advective flux of energy in addition to the contributions considered below in this section). We will also derive in the following Section 2.4.2 an individual budget for the macroscopic kinetic energy $\boldsymbol{u}^2/2$ from the momentum budget, which we then subtract from the budget for internal and kinetic energy, which finally yields a budget for the internal energy only. Note that we have to take this route, since it appears cumbersome to directly derive a budget for the internal energy from the first law of thermodynamics given by (1.23).

Changes of energy within the material volume can occur through external fluxes, work on the volume which has been discussed in Section 2.3, and change of the mass composition due to the different enthalpy of the constituents. The individual contributions to the energy budget, i. e. the respective change of energy per volume and time, are as follows:

- **External fluxes**
  Energy can be brought into or out of a volume through molecular heat conduction, resulting in a flux $\boldsymbol{J}_\mathrm{T}$ (in W m^{-2}), and also through radiation by the flux $\boldsymbol{J}_\mathrm{rad}$. The combined flux $\boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad}$ will be referred to as heat flux. The energy flux per unit time through $\mathrm{d}\boldsymbol{A}$ is $(\boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad}) \cdot \mathrm{d}\boldsymbol{A}$, and the transport out of the volume is

  $$\oint_A (\boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad}) \cdot \mathrm{d}\boldsymbol{A} \tag{2.60}$$

  where the surface element $\mathrm{d}\boldsymbol{A}$ is directed outward.
- **Work on the volume I: surface forces**
  As shown in Section 2.3.1, the net surface force on a surface element is $\Pi \cdot \mathrm{d}\boldsymbol{A}$. Within a time δt, the boundary of a material fluid element is displaced by $\boldsymbol{u}\,\delta t$. Hence the work on the volume per unit time is given by

  $$\oint_A \boldsymbol{u} \cdot \Pi \cdot \mathrm{d}\boldsymbol{A} \tag{2.61}$$

- **Work on the volume II: volume forces**
  The force of gravity (per volume) is $-\rho\nabla\Phi$, the work per unit time hence is

  $$-\int_V \rho\boldsymbol{u} \cdot \nabla\Phi \mathrm{d}V \tag{2.62}$$

- **Energy exchange through diffusion of partial masses**
  According to (1.23), a change of $\mathrm{d}S$ in salt concentration causes an energy change of $(\partial H/\partial S)\mathrm{d}S$. The diffusive flux $\boldsymbol{J}_\mathrm{S}$ transports the amount $\boldsymbol{J}_\mathrm{S} \cdot \mathrm{d}\boldsymbol{A}$ and hence the energy $(\partial H/\partial S)\boldsymbol{J}_\mathrm{S} \cdot \mathrm{d}\boldsymbol{A}$ through a surface element. The total energy

transport out of the volume is thus

$$\oint_A \frac{\partial H}{\partial S} \boldsymbol{J}_\mathrm{S} \cdot \mathrm{d}\boldsymbol{A} = \oint_A \boldsymbol{J}_\mathrm{chem} \cdot \mathrm{d}\boldsymbol{A} \tag{2.63}$$

where the flux of 'chemical' energy (also in $\mathrm{W\,m^{-2}}$) is given by

$$\boldsymbol{J}_\mathrm{chem} = \frac{\partial H}{\partial S} \boldsymbol{J}_\mathrm{S} \tag{2.64}$$

Adding all terms of the energy budget and applying the Gaussian theorem, we obtain the local energy conservation in the form

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} (E + E_\mathrm{k}) = -\nabla \cdot \boldsymbol{J}^\mathrm{tot} - \rho \boldsymbol{u} \cdot \nabla \Phi \tag{2.65}$$

with the macroscopic kinetic energy (per mass) $E_\mathrm{k} = \boldsymbol{u}^2/2$. Here $\boldsymbol{J}^\mathrm{tot}$ is given by

$$\boldsymbol{J}^\mathrm{tot} = \boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad} + \boldsymbol{J}_\mathrm{chem} - \boldsymbol{u} \cdot \Pi \tag{2.66}$$

and denotes the sum of all nonadvective energy fluxes. The last contribution $-\boldsymbol{u} \cdot \Pi$ will be identified below as flux of mechanical energy. The flux form of (2.66) is

$$\frac{\partial}{\partial t} \rho (E + E_\mathrm{k}) = -\nabla \cdot \left[\rho (E + E_\mathrm{k}) \boldsymbol{u} + \boldsymbol{J}^\mathrm{tot} \right] - \rho \boldsymbol{u} \cdot \nabla \Phi \tag{2.67}$$

The sum of internal and kinetic energy hence can be changed locally not only through a divergent flux of total energy, but also through the gravity force. In the ocean phase changes occur mainly at the surface (by freezing and melting of sea ice) and, therefore, are generally considered in boundary conditions only. Otherwise a local volume term, describing the conversion, would enter (2.67).

2.4.2 Mechanical Energy

On the rotating Earth we have found the momentum budget (2.59), written as

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\rho \boldsymbol{\Omega} \times \boldsymbol{u} = \nabla \cdot \Pi - \rho \nabla \Phi \tag{2.68}$$

Scalar multiplication with $\boldsymbol{u}$ yields

$$\begin{aligned} \rho \frac{\mathrm{D}E_\mathrm{k}}{\mathrm{D}t} &= \boldsymbol{u} \cdot (\nabla \cdot \Pi) - \rho \boldsymbol{u} \cdot \nabla \Phi \\ &= -\boldsymbol{u} \cdot \nabla p + \boldsymbol{u} \cdot (\nabla \cdot \Sigma) - \rho \boldsymbol{u} \cdot \nabla \Phi \end{aligned} \tag{2.69}$$

as budget for the macroscopic kinetic energy (per mass) $E_\mathrm{k} = \boldsymbol{u}^2/2$. Note that the Coriolis term does not contribute. In the second relation we have used the splitting of the stress according to (2.34), revealing $-\boldsymbol{u} \cdot \nabla p$ as the amount of kinetic energy produced by the work of the pressure force, and $\boldsymbol{u} \cdot \nabla \cdot \Sigma$ as the amount of kinetic energy produced by viscous effects.

The geopotential Φ satisfies the (trivial) conservation equation

$$\rho \frac{\mathrm{D}\Phi}{\mathrm{D}t} = \rho \frac{\partial \Phi}{\partial t} + \rho \boldsymbol{u} \cdot \nabla \Phi \tag{2.70}$$

Here local changes can only be caused by tidal forces (cf. Section 2.3.4), i. e. $\partial\Phi/\partial t = \partial\Phi_{\mathrm{tide}}/\partial t$. The advection term in (2.70) occurs, with opposite sign, also in (2.69). We identify the geopotential Φ with a *potential* energy; hence the advection term describes a conversion between potential and kinetic energy. It is common to use a coordinate system with z normal to geopotential surfaces. Then, with $\boldsymbol{u} = (u, v, w)$, we obtain $\rho\boldsymbol{u} \cdot \nabla\Phi \equiv \rho g w$.

The first term in (2.69) can be split as $\boldsymbol{u} \cdot (\nabla \cdot \Pi) = \nabla \cdot (\boldsymbol{u} \cdot \Pi) - (\Pi \cdot \nabla) \cdot \boldsymbol{u}$ and the last term in this relation can be rewritten as

$$(\Pi \cdot \nabla) \cdot \boldsymbol{u} = \Pi_{ij} \frac{\partial u_i}{\partial x_j} = \Pi_{ij} D_{ij} = \left(-p\delta_{ij} + \Sigma_{ij}\right) D_{ij}$$

with the deformation tensor D_{ij} as defined by (1.6). Separating $D_{ij} = D_{\ell\ell}\delta_{ij}/3 + D_{ij}^{\star}$ into the deviator D_{ij}^{*} and the remainder, we find $\Sigma_{ij} D_{ij} = \Sigma_{ij} D_{ij}^{*}$ because the frictional stress tensor has zero trace, $\Sigma_{ii} = 0$. With the definition

$$\rho\epsilon = \Sigma_{ij} D_{ij}^{*} \tag{2.71}$$

we then obtain $\boldsymbol{u} \cdot (\nabla \cdot \Pi) = \nabla \cdot (\boldsymbol{u} \cdot \Pi) + p\nabla \cdot \boldsymbol{u} - \rho\epsilon$. Inserting this result into (2.69) yields the kinetic energy conservation as

$$\rho \frac{\mathrm{D}E_{\mathrm{k}}}{\mathrm{D}t} = -\nabla \cdot (-\boldsymbol{u} \cdot \Pi) + p\nabla \cdot \boldsymbol{u} - \rho\epsilon - \rho\boldsymbol{u} \cdot \nabla\Phi \tag{2.72}$$

Addition of (2.69) and (2.70) yields the conservation of *mechanical* energy, i. e. the sum of kinetic and potential energies, as

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(E_{\mathrm{k}} + \Phi\right) = -\nabla \cdot (-\boldsymbol{u} \cdot \Pi) + p\nabla \cdot \boldsymbol{u} - \rho\epsilon + \rho \frac{\partial\Phi_{\mathrm{tide}}}{\partial t} \tag{2.73}$$

The work terms for pressure and viscous forces is separated into a flux divergence and a remainder. The flux of mechanical energy is $-\boldsymbol{u} \cdot \Pi = p\boldsymbol{u} - \boldsymbol{u} \cdot \Sigma$. The role of the remaining terms will be discussed in the next section.

2.4.3 Internal Energy and Enthalpy

Subtracting the kinetic energy budget (2.72) from (2.65) leads to the conservation of internal energy,

$$\rho \frac{\mathrm{D}E}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{J}_{\mathrm{H}} - p\nabla \cdot \boldsymbol{u} + \rho\epsilon \tag{2.74}$$

Here the flux $\boldsymbol{J}_{\mathrm{H}}$ of internal energy (and also of enthalpy, see below) is given as

$$\boldsymbol{J}_{\mathrm{H}} = \boldsymbol{J}_{\mathrm{T}} + \boldsymbol{J}_{\mathrm{rad}} + \boldsymbol{J}_{\mathrm{chem}} \tag{2.75}$$

Two source/sink terms in (2.74) appear which constitute a conversion between internal and mechanical energy because they appear with opposite sign in the mechanical energy budget (2.73). The term $\rho\epsilon$ is the rate of *energy dissipation* (also known as Joule heating) (per volume, in $\mathrm{kg\,m^{-1}\,s^{-3}}$). As will be shown in Section 2.5.1, ϵ is always positive, and hence constitutes an *irreversible* transformation of mechanical into internal energy caused by friction. The term $p\nabla \cdot \boldsymbol{u}$ is the amount of internal energy converted to mechanical energy, which occurs also in reversible systems, and is usually denoted the *reversible exchange*. This interpretation is, however, strictly correct only in the absence of heat conduction and diffusion.

It is often convenient to formulate (2.74) in terms of the enthalpy $H = E + pv$. Using the mass conservation (2.18) one obtains

$$\rho \frac{\mathrm{D}H}{\mathrm{D}t} = \frac{\mathrm{D}p}{\mathrm{D}t} - \nabla \cdot \boldsymbol{J}_{\mathrm{H}} + \rho\epsilon \tag{2.76}$$

We have discussed $\rho \boldsymbol{u} \cdot \nabla \Phi \equiv \rho g w$ as exchange term between kinetic and potential energy. Note that the reversible term in (2.76) is approximately of the same form, $\mathrm{D}p/\mathrm{D}t \approx w \partial p / \partial z \approx -\rho w g$, for hydrostatic conditions.

2.4.4 Total Energy and Total Enthalpy

The sum of internal, kinetic and potential energy

$$E^{\mathrm{tot}} = E + E_{\mathrm{k}} + \Phi \tag{2.77}$$

is referred to as *total energy*. Adding (2.65) and (2.70) yields

$$\rho \frac{\mathrm{D}E^{\mathrm{tot}}}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{J}^{\mathrm{tot}} + \rho \frac{\partial \Phi_{\mathrm{tide}}}{\partial t} \tag{2.78}$$

where the total energy flux is given by (2.66). The flux form of (2.78) is

$$\frac{\partial}{\partial t} \rho E^{\mathrm{tot}} = -\nabla \cdot \left(\rho E^{\mathrm{tot}} \boldsymbol{u} + \boldsymbol{J}^{\mathrm{tot}} \right) + \rho \frac{\partial \Phi_{\mathrm{tide}}}{\partial t} \tag{2.79}$$

Only the fluxes $\boldsymbol{J}^{\mathrm{tot}}$ at the boundaries change the integrated total energy, otherwise the fluxes can only redistribute this quantity. In the diagram of Figure 2.12 and the box on p. 52 we summarize the energy compartments with the exchange terms and external fluxes.

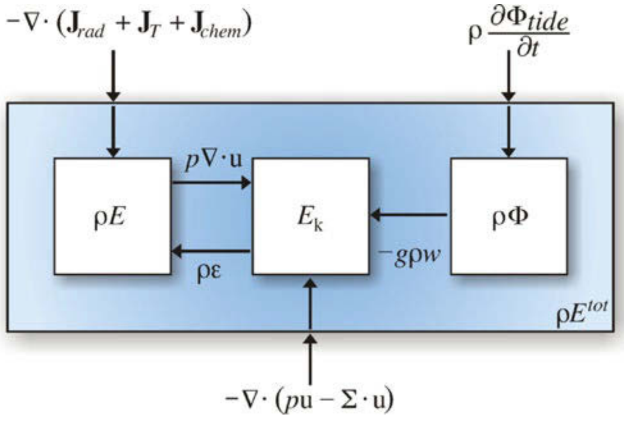

Fig. 2.12 Local source terms for the various energy compartments

9. Fluxes and Source Terms in Energy Budgets

Variable	*Symbol*	Flux	Source/sink (per volume)
Pot. energy	Φ	0	$g\rho w + \rho\dfrac{\partial \Phi_{\text{tide}}}{\partial t}$
Kin. energy	E_{k}	$\boldsymbol{J}_{\text{mech}} = p\boldsymbol{u} - \Sigma\cdot\boldsymbol{u}$	$-g\rho w - \rho\epsilon + p\boldsymbol{\nabla}\cdot\boldsymbol{u}$
Mech. energy	$E_{\text{k}} + \Phi$	$\boldsymbol{J}_{\text{mech}} = p\boldsymbol{u} - \Sigma\cdot\boldsymbol{u}$	$-\rho\epsilon + p\boldsymbol{\nabla}\cdot\boldsymbol{u} + \dfrac{\partial \Phi_{\text{tide}}}{\partial t}$
Int. energy	E	$\boldsymbol{J}_{\text{H}} = \boldsymbol{J}_{\text{rad}} + \boldsymbol{J}_{\text{T}} + \dfrac{\partial H}{\partial S}\boldsymbol{J}_{\text{S}}$	$\rho\epsilon - p\boldsymbol{\nabla}\cdot\boldsymbol{u}$
Enthalpy	H	$\boldsymbol{J}_{\text{H}} = \boldsymbol{J}_{\text{rad}} + \boldsymbol{J}_{\text{T}} + \dfrac{\partial H}{\partial S}\boldsymbol{J}_{\text{S}}$	$\rho\epsilon + \dfrac{\mathrm{D}p}{\mathrm{D}t}$
Total energy	$E^{\text{tot}} = E + E_{\text{k}} + \Phi$	$\boldsymbol{J}^{\text{tot}} = \boldsymbol{J}_{\text{H}} + \boldsymbol{J}_{\text{mech}}$	$\rho\dfrac{\partial \Phi_{\text{tide}}}{\partial t}$
Total enthalpy	$H + E_{\text{k}} + \Phi$	$\boldsymbol{J}^{\text{tot}} = \boldsymbol{J}_{\text{H}} - \Sigma\cdot\boldsymbol{u}$	$\dfrac{\partial p}{\partial t} + \rho\dfrac{\partial \Phi_{\text{tide}}}{\partial t}$

Decomposing the mechanical energy flux in (2.66) according to $\boldsymbol{u}\cdot\Pi = -p\boldsymbol{u} + \boldsymbol{u}\cdot\Sigma$, the total energy flux may be represented as

$$\boldsymbol{J}^{\text{tot}} = p\boldsymbol{u} - \boldsymbol{u}\cdot\Sigma + \boldsymbol{J}_{\text{H}} \tag{2.80}$$

Invoking the identity

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}(pv) = \frac{\partial p}{\partial t} + \nabla\cdot p\boldsymbol{u}$$

which can be derived by means of the continuity equation, one obtains from (2.78)

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left(H + E_{\text{k}} + \Phi\right) = -\nabla\cdot\boldsymbol{J}_{*}^{\text{tot}} + \frac{\partial p}{\partial t} + \rho\frac{\partial \Phi_{\text{tide}}}{\partial t} \tag{2.81}$$

where the reduced energy flux $\boldsymbol{J}_{*}^{\text{tot}} = \boldsymbol{J}^{\text{tot}} - p\boldsymbol{u} = -\boldsymbol{u}\cdot\Sigma + \boldsymbol{J}_{\text{H}}$ contains the fluxes due to molecular friction, diffusion, heat conduction and radiation. In many cases, especially when considering large-scale motions, the pressure is mainly determined by depth, and its local rate of change is negligible. If, in addition, the conditions are adiabatic so that the reduced energy flux $\boldsymbol{J}_{*}^{\text{tot}}$ can be neglected, we obtain the BERNOULLI[11]-equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(H + E_{\text{k}} + \Phi\right) \approx 0 \tag{2.82}$$

The quantity

$$B = H + E_{\text{k}} + \Phi = E + E_{\text{k}} + \Phi + pv \tag{2.83}$$

---

[11] JOHANN BERNOULLI, *1667 in Basel, †1748 in Basel, mathematician, and his son DANIEL BERNOULLI, *1700 in Groningen, †1782 in Basel, are the most prominent members of the scientist family Bernoulli. The above theorem is due to Daniel B. whose most famous work 'Hydrodynamics' appeared in 1738.

is the Bernoulli function in its most general form. Under the above-mentioned conditions, the Bernoulli function of a fluid element is approximately constant. In such situations (2.82) can be more convenient than energy conservation. The Bernoulli function can also be viewed as total enthalpy.

2.4.5 Boundary Condition for the Enthalpy Flux

A boundary condition is conveniently formulated in terms of the enthalpy H of seawater because boundary processes take place at (approximately) constant pressure. From the conservation equation (2.76) it follows that the appropriate flux of enthalpy is $\boldsymbol{J}_{\mathrm{H}}$ which is defined in (2.75) as sum of (sensible) heat flux, radiative heat flux and a flux related to the chemical potential (the latter will later be shown to be small, cf. Section 2.6). Application of (2.7), with $\chi = H$ and $\boldsymbol{J}_\chi = \boldsymbol{J}_{\mathrm{H}}$, yields for the ocean-atmosphere boundary

$$H \mathcal{J}_{\mathrm{mass}} + \boldsymbol{J}_{\mathrm{H}} \cdot \boldsymbol{n} = \mathcal{J}_{\mathrm{H}}^{\mathrm{atm}} \tag{2.84}$$

10. Alternative Derivation of the Budget for Internal Energy

An alternative derivation of the budget for internal energy can be found using the Gibbs relation (1.47), for which the differentials $\mathrm{d}E$, $\mathrm{d}\eta$, etc are written as material derivatives $\mathrm{D}E/\mathrm{D}t$, $\mathrm{D}\eta/\mathrm{D}t$, etc.

$$\frac{\mathrm{D}E}{\mathrm{D}t} = \mu \frac{\mathrm{D}S}{\mathrm{D}t} + T \frac{\mathrm{D}\eta}{\mathrm{D}t} + p/\rho^2 \frac{\mathrm{D}\rho}{\mathrm{D}t}$$

where $-p\mathrm{d}v = p/\rho^2 \mathrm{d}\rho$ was used. Combing this particle form of the Gibbs relation with the budgets for salt and total mass and assuming the standard form for the entropy budget with a nonadvective flux of entropy $\boldsymbol{J}_\eta$ and a source or sink of entropy C_η given by $\rho \mathrm{D}\eta/\mathrm{D}t = -\boldsymbol{\nabla} \cdot \boldsymbol{J}_\eta + C_\eta$ yields

$$\rho \frac{\mathrm{D}E}{\mathrm{D}t} = -\mu \boldsymbol{\nabla} \cdot \boldsymbol{J}_{\mathrm{S}} - T \boldsymbol{\nabla} \cdot \boldsymbol{J}_\eta + T C_\eta - p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

We rewrite this budget for the internal energy again in the standard form including a nonadvective flux and a source or sink term of internal energy, as

$$\rho \frac{\mathrm{D}E}{\mathrm{D}t} = -\boldsymbol{\nabla} \cdot (\mu \boldsymbol{J}_{\mathrm{S}} + T \boldsymbol{J}_\eta) + \boldsymbol{J}_{\mathrm{S}} \cdot \boldsymbol{\nabla} \mu + \boldsymbol{J}_\eta \cdot \boldsymbol{\nabla} T + T C_\eta - p \boldsymbol{\nabla} \cdot \boldsymbol{u}$$

Combining this budget for internal energy with the kinetic and potential energy budgets yields as budget for total energy

$$\begin{aligned} \rho \frac{\partial}{\partial t}(E + E_{\mathrm{k}} + \Phi) &= -\boldsymbol{\nabla} \cdot \left(\mu \boldsymbol{J}_{\mathrm{S}} + T \boldsymbol{J}_\eta - \boldsymbol{u} \cdot \Pi\right) + \boldsymbol{J}_{\mathrm{S}} \cdot \boldsymbol{\nabla} \mu \\ &\quad + \boldsymbol{J}_\eta \cdot \boldsymbol{\nabla} T + T C_\eta - \rho \epsilon + \rho \frac{\partial \Phi}{\partial t} \end{aligned}$$

Now we assume that the total energy is conserved (except for tidal forcing). It follows for the source term of entropy that

$$T C_\eta = -\boldsymbol{J}_{\mathrm{S}} \cdot \boldsymbol{\nabla} \mu - \boldsymbol{J}_\eta \cdot \boldsymbol{\nabla} T + \rho \epsilon$$

must hold, which is identical to (2.91). From the internal energy budget, we also see that the enthalpy flux is given by $\boldsymbol{J}_{\mathrm{H}} = \mu \boldsymbol{J}_{\mathrm{S}} + T \boldsymbol{J}_\eta$. With $\partial H/\partial S = \mu + T \partial\eta/\partial S$ and the entropy flux from (2.90) we recover the form (2.75) for the enthalpy flux.

Here $\mathcal{J}_{\mathrm{H}}^{\mathrm{atm}}$ is the net enthalpy flux and $\mathcal{J}_{\mathrm{mass}} = \mathcal{E} - \mathcal{P}$ the net mass flux, both normal to the surface and pointing into the atmosphere. On the atmospheric side of the boundary, water is transported as water vapor (with enthalpy H_{q}) due to evaporation, and as liquid water (with enthalpy H_{p}) due to precipitation. Therefore, (2.84) can be written as $H(\mathcal{E} - \mathcal{P}) + \boldsymbol{J}_{\mathrm{H}} \cdot \boldsymbol{n} = H_{\mathrm{q}}\mathcal{E} - H_{\mathrm{p}}\mathcal{P} + \boldsymbol{J}_{\mathrm{H}}^{a} \cdot \boldsymbol{n}$ where $\boldsymbol{J}_{\mathrm{H}}^{a}$ is the atmospheric sensible plus radiative heat flux. The *latent heat* of condensation L_{q} is defined as

$$L_{\mathrm{q}} = H_{\mathrm{q}} - H_{\mathrm{f}} \tag{2.85}$$

and has the value of $L_{\mathrm{q}} \approx 2.5 \times 10^{6}\,\mathrm{J\,kg^{-1}}$ at 0°C and atmospheric standard pressure. Here H_{f} is the enthalpy of pure water. Since with (2.85) one has $H_{\mathrm{q}} - H = L_{\mathrm{q}} + H_{\mathrm{f}} - H$, the boundary condition finally takes the form

$$\boldsymbol{J}_{\mathrm{H}} \cdot \boldsymbol{n} = \boldsymbol{J}_{\mathrm{H}}^{a} \cdot \boldsymbol{n} + L_{\mathrm{q}}\mathcal{E} + (H_{\mathrm{f}} - H)\mathcal{E} - (H_{\mathrm{p}} - H)\mathcal{P} \tag{2.86}$$

Sensible and radiative heat fluxes on both sides of the boundary hence differ by three terms associated with the air-sea mass flux, of which the first term (latent heat flux) is by far the most important. The enthalpy differences between pure water and sea water, and between precipitating water and sea water, are much smaller so that the last two terms in (2.86) can usually be ignored.

The exchange of enthalpy between ocean and atmosphere is an important driving mechanism for the ocean circulation. We continue with a detailed discussion in Chapter 13. The exchange at the sea floor is comparatively small, and usually ignored, i. e. $\boldsymbol{J}_{\mathrm{H}} \cdot \boldsymbol{n} = 0$ is assumed. Locally it may however be necessary to include geothermal heating at places such as hydrothermal vents. Furthermore, geothermal heating may play a certain role for the large-scale circulation over long time scales (see e.g. Adcroft et al., 2001; Emile-Geay and Madec, 2009).

2.5 Entropy Budget

The entropy $\eta(S, T, p)$ can be computed numerically as a derivative of the known Gibbs function according to (1.49). Therefore, a consideration of the entropy budget is not in itself necessary to obtain a closed system of equations which allows determination of all state variables. The second law, however, allows conclusions on the direction of frictional/diffusive molecular fluxes.

2.5.1 Entropy Sources and Flux–Gradient Relations

Starting from the Gibbs relation (1.28) in the form

$$T\frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{\mathrm{D}H}{\mathrm{D}t} - \upsilon\frac{\mathrm{D}p}{\mathrm{D}t} - \mu\frac{\mathrm{D}S}{\mathrm{D}t} \tag{2.87}$$

one obtains with (2.76) and (2.20)

$$\rho\frac{\mathrm{D}\eta}{\mathrm{D}t} = \frac{1}{T}\left[-\boldsymbol{\nabla}\cdot\left(\boldsymbol{J}_{\mathrm{rad}} + \boldsymbol{J}_{\mathrm{T}} + \frac{\partial H}{\partial S}\boldsymbol{J}_{\mathrm{S}}\right) + \rho\epsilon + \mu\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\mathrm{S}}\right] \tag{2.88}$$

After some algebra (expansion of $\nabla\mu$, and use of (1.50)) one obtains the entropy conservation in the standard form

$$\rho\frac{\mathrm{D}\eta}{\mathrm{D}t} = -\nabla\cdot\boldsymbol{J}_\eta + C_\eta \tag{2.89}$$

where the entropy flux is defined by

$$\boldsymbol{J}_\eta = \frac{\boldsymbol{J}_{\mathrm{rad}} + \boldsymbol{J}_{\mathrm{T}}}{T} + \frac{\partial\eta}{\partial S}\boldsymbol{J}_{\mathrm{S}} \tag{2.90}$$

The entropy source in (2.89) consists of external and internal contributions, $C_\eta = C_\eta^{\mathrm{ext}} + C_\eta^{\mathrm{int}}$. The external source

$$C_\eta^{\mathrm{ext}} = -\frac{\boldsymbol{J}_{\mathrm{rad}}}{T^2}\cdot\nabla T$$

is due to radiation, and the internal sources, summarized by

$$C_\eta^{\mathrm{int}} = \frac{1}{T}\rho\epsilon - \frac{\boldsymbol{J}_{\mathrm{T}}}{T^2}\cdot\nabla T - \frac{\boldsymbol{J}_{\mathrm{S}}}{T}\cdot\left(\frac{\partial\mu}{\partial S}\nabla S + \frac{\partial\mu}{\partial p}\nabla p\right) \tag{2.91}$$

are due to a suite of irreversible processes within the fluid. The second law of thermodynamics requires the internal entropy source to be positive, $C_\eta^{\mathrm{int}} > 0$. This requirement is certainly satisfied when *all* individual contributions in (2.91) are positive, i. e.

$$\epsilon > 0\,, \quad -\boldsymbol{J}_{\mathrm{T}}\cdot\nabla T > 0\,, \quad -\boldsymbol{J}_{\mathrm{S}}\cdot\left(\frac{\partial\mu}{\partial S}\nabla S + \frac{\partial\mu}{\partial p}\nabla p\right) > 0 \tag{2.92}$$

These inequalities lead to the following restrictions for the direction of molecular transports, as discussed in the following.

- **Mechanical dissipation**
  From (2.71), the definition of the viscous stress tensor (2.36), and the stress–strain relation (2.35) one finds

$$\rho\epsilon = \Sigma_{ij}D_{ij}^\star = 2\nu D_{ij}^\star D_{ij}^\star = \frac{1}{2}\nu\sum_{i,j}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} - \frac{2}{3}\frac{\partial u_\ell}{\partial x_\ell}\delta_{ij}\right)^2 \tag{2.93}$$

  Hence the *energy dissipation rate* ϵ (per mass, in $\mathrm{m}^2\,\mathrm{s}^{-3}$) is positive if the molecular viscosity ν is positive. Instead of ν one frequently uses the *kinematical* viscosity $\kappa_{\mathrm{m}} = \nu/\rho$. For water at 10°C temperature, the kinematic viscosity is $\kappa_{\mathrm{m}} \approx 1.3\times10^{-6}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. It decreases with temperature and varies by ±25% under oceanic conditions.
- **Heat Flux**
  For molecular heat conduction FOURIER'S[12] law states that the diffusive heat flux is opposite to the gradient of temperature,

$$\boldsymbol{J}_{\mathrm{T}} = -\lambda\nabla T \tag{2.94}$$

  with the heat conductivity λ. Hence we have $-\boldsymbol{J}_{\mathrm{T}}\cdot\nabla T > 0$ provided that λ is positive, i. e. that the heat flux is directed from warm to cold regions. Normally, instead of λ, the temperature conductivity $\kappa_{\mathrm{T}} = \lambda/\rho c_p$ is preferred. In the ocean, typical values are $\kappa_{\mathrm{T}} \approx 1.4\times10^{-7}\,\mathrm{m}^2\,\mathrm{s}^{-1}$.

---

[12] JEAN BAPTIST JOSEPH FOURIER, *1768 near Auxerre, †1830 in Paris, mathematician and physicist.

- **Salinity Flux**
  The FICKIAN[13] law relates the salinity flux to the gradient by

  $$\boldsymbol{J}_{\mathrm{S}} \approx -\rho\kappa_{\mathrm{S}}\nabla S \tag{2.95}$$

  However, from the last expression in (2.92) we see that (2.95) is *not* sufficient to guarantee positivity of the entropy production. Including the salinity flux due to the pressure gradient according to

  $$\boldsymbol{J}_{\mathrm{S}} = -\rho\kappa_{\mathrm{S}}\left(\nabla S + \frac{\partial\mu/\partial p}{\partial\mu/\partial S}\nabla p\right) \tag{2.96}$$

  leads, however, to a positive entropy production term provided that the salt diffusivity κ_{S} is positive. Typical values for seawater are $\kappa_{\mathrm{S}} \approx 1.2 \times 10^{-9}\,\mathrm{m}^2\,\mathrm{s}^{-1}$. A consequence of (2.96) is carried out in the box on p. 57.

2.5.2 Onsager Relations

A generalization of the phenomenological relations (2.94) and (2.96) follows from the irreversible thermodynamics developed by ONSAGER[14] in 1931 (Onsager, 1931a,b). The positivity of entropy production (2.91) can also be satisfied by the weaker condition that only the sum, rather than the individual contributions, be positive. We consider here only heat and salt fluxes, and obtain the requirement that

$$C_\eta^\star = -\frac{\boldsymbol{J}_{\mathrm{T}}}{T^2}\cdot\nabla T - \frac{\mu_{\mathrm{S}}\boldsymbol{J}_{\mathrm{S}}}{T}\cdot\left(\nabla S + \frac{\partial\mu/\partial p}{\partial\mu/\partial S}\nabla p\right) > 0$$

A more general linear and isotropic relation between the fluxes $\boldsymbol{J}_{\mathrm{T}}$ and $\boldsymbol{J}_{\mathrm{S}}$ and the vectors on the right-hand side of (2.94) and (2.96) can be written as

$$\boldsymbol{J}_{\mathrm{T}} = -\rho c_p\kappa_{\mathrm{T}}\nabla T - \rho\kappa_{\mathrm{TS}}\left(\nabla S + \frac{\partial\mu/\partial p}{\partial\mu/\partial S}\nabla p\right) \tag{2.97}$$

$$\boldsymbol{J}_{\mathrm{S}} = -\rho\kappa_{\mathrm{S}}\left(\nabla S + \frac{\partial\mu/\partial p}{\partial\mu/\partial S}\nabla p\right) - \rho\kappa_{\mathrm{ST}}\nabla T \tag{2.98}$$

The last term in (2.97) leads to a heat flux in the presence of a combined salinity/pressure gradient (DUFOUR[15]-effect), the last term in (2.98) to a salinity flux due to a temperature gradient (SORET[16]-effect). The entropy production due to heat and salt fluxes then follows as

$$\begin{aligned} C_\eta^\star/\rho = {} & \frac{c_p\kappa_{\mathrm{T}}}{T^2}(\nabla T)^2 + \frac{\kappa_{\mathrm{S}}}{T}\frac{\partial\mu}{\partial S}\left(\nabla S + \frac{\partial\mu/\partial p}{\partial\mu/\partial S}\nabla p\right)^2 \\ & + \left(\frac{\kappa_{\mathrm{TS}}}{T^2} + \frac{\kappa_{\mathrm{ST}}}{T}\frac{\partial\mu}{\partial S}\right)\nabla T\cdot\left(\nabla S + \frac{\partial\mu/\partial p}{\partial\mu/\partial S}\nabla p\right) \end{aligned} \tag{2.99}$$

---

[13] ADOLF FICK, *1829 in Kassel, †1901 in Blankenberge/Flanders. Interestingly, he was not a physicist but a physiologist. He formulated in 1855 the three Fickian diffusion laws.

[14] LARS ONSAGER, *1903 in Oslo, †1976 in Coral Gables/Florida, chemist and physicist.

[15] LOUIS DUFOUR, *1832 in Veytaux, †1892 in Lausanne, physicist.

[16] CHARLES SORET, *1854 in Geneva, †1904 in Geneva, physicist and chemist.

From arguments based on statistical mechanics, Onsager has deduced that the two off-diagonal coefficients must be related by *reciprocity relations* which for κ_{TS} and κ_{ST} read

$$\frac{\partial \mu}{\partial S}\kappa_{\mathrm{ST}} = \frac{\kappa_{\mathrm{TS}}}{T} \tag{2.100}$$

As a consequence, both additional terms contribute equally to entropy production. Positivity of (2.99) then requires that

$$\kappa_{\mathrm{T}} > 0\,, \quad \kappa_{\mathrm{S}} > 0\,, \quad \text{and} \quad |\kappa_{\mathrm{TS}}|^2 < \kappa_{\mathrm{T}}\kappa_{\mathrm{S}}c_p T\frac{\partial \mu}{\partial S}$$

More details are given in Kamenkovich (1977) where also the general case with inclusion of the mechanical dissipation is discussed. However, since experiments show that both κ_{ST}, the Soret-coefficient, and κ_{TS}, the Dufour coefficient, are very small for seawater, they are usually neglected.

2.6 Temperature Equations

With the conservation of internal energy (2.74) and with the relevant thermodynamic relations considered in Section 1.2, we now have all basic conservation laws which in principle are sufficient to determine the state variables $\boldsymbol{u}$, ρ, S, p, and E. In the following we use the enthalpy H instead of E. The temperature T which is e. g. needed to evaluate the state equation, could in principle be determined through inversion of the known thermodynamic relationship $H = H(S, T, p)$. The more familiar procedure is, however, to use that relationship to eliminate H from the system of conservation laws, and to derive an equation directly for the temperature. This elimination is useful because (1) temperature is an *independent* state variable and more convenient than H to evaluate thermodynamic expressions, (2) in contrast to internal energy or enthalpy, temperature can be easily observed, and (3) in this way one can achieve a reduction in the number of variables.

In a second step, the temperature will be replaced by a *conservative temperature* variable which is closer to conservation than *in-situ* temperature. The price to be paid is an increase in algebraic complexity in the respective conservation equations. The conservative temperature replaces the historically used *potential temperature*, since

11. Diffusive Equilibrium

A pressure gradient can cause a diffusive salt flux. Note that $\partial\mu/\partial p = \partial v/\partial S = -\gamma v < 0$ while $\partial\mu/\partial S > 0$. Therefore, the pressure-related component of salt flux is directed towards higher pressure, i. e. downward in the ocean. As an example, consider a state of rest which is in hydrostatic equilibrium (cf. Section 2.9 below) and assume a vanishing diffusive salt flux $\boldsymbol{J}_{\mathrm{S}} = 0$, i. e. a diffusive equilibrium. For the vertical component of (2.96) one finds

$$\frac{\partial S}{\partial z} = -\frac{\partial\mu/\partial p}{\partial\mu/\partial S}\frac{\partial p}{\partial z} \approx \frac{\partial\mu/\partial p}{\partial\mu/\partial S}g\rho \approx -3\,(\mathrm{g\,kg^{-1}})/\mathrm{km}$$

a value which is far above observed salinity gradients in the ocean. It follows that the observed oceanic state is far from its diffusive equilibrium state. In fact, small-scale turbulent mixing as discussed in Chapter 11 and advection by the large-scale ocean circulation prevents a diffusive equilibrium of the ocean.

the former has the best conservation properties of all variables measuring ocean heat content. It is hence the preferred temperature variable. The potential temperature budget is derived here for comparison and for historical reasons.

2.6.1 In-situ Temperature

The enthalpy conservation (2.76) can be written, by expanding the total derivative of $H = H(S, T, p)$, as

$$\rho \left(\frac{\partial H}{\partial S} \frac{\mathrm{D}S}{\mathrm{D}t} + \frac{\partial H}{\partial T} \frac{\mathrm{D}T}{\mathrm{D}t} + \frac{\partial H}{\partial p} \frac{\mathrm{D}p}{\mathrm{D}t} \right) = \frac{\mathrm{D}p}{\mathrm{D}t} - \nabla \cdot \boldsymbol{J}_{\mathrm{H}} + \rho \epsilon$$

The derivatives of the enthalpy, appearing in this relation, are evaluated as

$$\frac{\partial H}{\partial T} = c_p \quad \text{and} \quad \frac{\partial H}{\partial p} = \upsilon + T \frac{\partial \eta}{\partial p} = \upsilon - T \frac{\partial \upsilon}{\partial T} = \upsilon(1 - \alpha T)$$

with the help of (1.50) and (1.57). Use of the salt conservation (2.20) then results in the *in-situ* temperature equation

$$\rho c_p \frac{\mathrm{D}T}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{J}_{\mathrm{H}} + \alpha T \frac{\mathrm{D}p}{\mathrm{D}t} + Q_{\mathrm{T}} \tag{2.101}$$

Hence temperature can be changed not only by heat-flux convergence, but also by a change of pressure or by the temperature source Q_{T} which is given by

$$Q_{\mathrm{T}} = \rho \epsilon + \frac{\partial H}{\partial S} \nabla \cdot \boldsymbol{J}_{\mathrm{S}} \tag{2.102}$$

Note that the contributions from the dissipation $\rho\epsilon$ and from the salt fluxes $\boldsymbol{J}_{\mathrm{S}}$ for the temperature source term Q_{T} are small and usually neglected, as discussed below. The *in-situ* temperature budget (2.101) can also be written as

$$\rho c_p \left(\frac{\mathrm{D}T}{\mathrm{D}t} - \Gamma \frac{\mathrm{D}p}{\mathrm{D}t} \right) = -\nabla \cdot \boldsymbol{J}_{\mathrm{H}} + Q_{\mathrm{T}} \quad \text{with} \quad \Gamma(S, T, p) = \frac{\alpha T}{\rho c_p} = -\frac{\partial^2 G / \partial T \partial p}{\partial^2 G / \partial T^2} \tag{2.103}$$

using the relations in Table 1.1. Under adiabatic conditions (and without mass exchange) we have $\boldsymbol{J}_{\mathrm{H}} = 0$ and $Q_{\mathrm{T}} = 0$, and then changes in *in-situ* temperature and pressure are related by $\mathrm{d}T = \Gamma \mathrm{d}p$. The quantity Γ is, therefore, called *adiabatic temperature gradient* or *lapse rate*. The gradient is meant here with respect to pressure, in fact the adiabatic lapse rate can also be written as $\Gamma = (\partial T / \partial p)_{\mathrm{ad}}$, defining the adiabatic temperature gradient. A typical value for the ocean is $\Gamma \approx 10^{-8}$ K/Pa $= 10^{-4}$ K/dbar which corresponds about to ~ 0.1 K/km.

2.6.2 Conservative Temperature

The *in-situ* temperature equation (2.101) with the source terms (2.102) for the ocean is preferred to the balances of internal energy or enthalpy because T is measurable.

However, the *in-situ* temperature is not conserved even under adiabatic conditions because changes of pressure always lead to changes in temperature of a fluid parcel, manifested by the nonvanishing lapse rate Γ. Therefore, the budget (2.101) is in practice not very useful. The concept of the *conservative temperature* introduces a new temperature variable which takes care of the pressure effects and which is conserved under adiabatic conditions.

Consider the enthalpy H of a fluid parcel in terms of its canonical independent variables (see Figure 1.4) salinity, entropy, and pressure, $H = H(S, \eta, p)$. The enthalpy conservation (2.76) takes the form

$$\rho \left[\left(\frac{\partial H}{\partial S} \right)_{\eta \mathrm{p}} \frac{\mathrm{D}S}{\mathrm{D}t} + \left(\frac{\partial H}{\partial \eta} \right)_{\mathrm{Sp}} \frac{\mathrm{D}\eta}{\mathrm{D}t} + \left(\frac{\partial H}{\partial p} \right)_{\mathrm{S}\eta} \frac{\mathrm{D}p}{\mathrm{D}t} \right] = \frac{\mathrm{D}p}{\mathrm{D}t} - \nabla \cdot \boldsymbol{J}_{\mathrm{H}} + \rho\epsilon \tag{2.104}$$

From Maxwell's relations in Section 1.2.5 one has

$$\left(\frac{\partial H}{\partial \eta} \right)_{S,p} = T \ , \quad \left(\frac{\partial H}{\partial p} \right)_{S,\eta} = \upsilon \tag{2.105}$$

and with $\rho\upsilon = 1$ it is found that both terms with $\mathrm{D}p/\mathrm{D}t$ in (2.104) cancel exactly (in contrast to the *in-situ* temperature budget (2.103)). The *potential enthalpy* H^0 is defined by

$$H(S, \eta, p_0) \equiv H^0(S, \eta) \tag{2.106}$$

and is the heat content of a parcel which is brought adiabatically from the local pressure p to a reference pressure p_0, i. e. without mass (constant S) and energy (constant η) exchange. The reference pressure is usually chosen as the surface pressure, which means that the water parcel is brought adiabatically from its local depth to the surface. Note that the *in-situ* temperature of the water parcel will change because of the change in pressure according to (2.103). Note also that the index 0 means that the quantity is evaluated at the pressure p_0. From the definition of potential enthalpy (2.106) we obtain

$$\rho \frac{\mathrm{D}H^0}{\mathrm{D}t} = \rho \left[\left(\frac{\partial H^0}{\partial S} \right)_{\eta,p} \frac{\mathrm{D}S}{\mathrm{D}t} + \left(\frac{\partial H^0}{\partial \eta} \right)_{S,p} \frac{\mathrm{D}\eta}{\mathrm{D}t} \right] \tag{2.107}$$

From the Maxwell relations (2.105) it follows that the quantity

$$\theta = \left(\frac{\partial H^0}{\partial \eta} \right)_{S,p} \tag{2.108}$$

is the temperature of the water parcel brought adiabatically to the reference pressure p_0, therefore, θ is called the *potential* temperature. Using the salt budget $\rho \mathrm{D}S/\mathrm{D}t = -\nabla \cdot \boldsymbol{J}_{\mathrm{S}}$ to replace $\mathrm{D}S/\mathrm{D}t$ and the entropy budget in the form (2.88) to replace $\mathrm{D}\eta/\mathrm{D}t$ in (2.107) we obtain

$$\rho \frac{\mathrm{D}H^0}{\mathrm{D}t} = \left(\frac{\theta}{T} \frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S} \right) \nabla \cdot \boldsymbol{J}_{\mathrm{S}} + \frac{\theta}{T} \left(-\nabla \cdot \boldsymbol{J}_{\mathrm{H}} + \rho\epsilon \right) \tag{2.109}$$

Following McDougall (2003), the *conservative* temperature $\theta^\star$ is introduced as

$$\theta^\star = H^0 / c_p^\star \tag{2.110}$$

where the constant reference value of the specific heat $c_p^\star = 3991.868\,\mathrm{J\,kg^{-1}\,K^{-1}}$ was chosen such that the numerical value of conservative temperature $\theta^\star$ and potential temperature θ become identical at $S = 35\,\mathrm{g\,kg^{-1}}$. This definition for $\theta^\star$ allows to write the heat balance in the form

$$\rho c_p^\star \frac{\mathrm{D}\theta^\star}{\mathrm{D}t} = \left(\frac{\theta}{T}\frac{\partial H}{\partial S} - \frac{\partial H^0}{\partial S}\right)\nabla \cdot \boldsymbol{J}_\mathrm{S} + \frac{\theta}{T}\left(-\nabla \cdot \boldsymbol{J}_\mathrm{H} + \rho\epsilon\right) \tag{2.111}$$

For adiabatic conditions, i. e. for $\boldsymbol{J}_\mathrm{H} = 0$, $\boldsymbol{J}_\mathrm{S} = 0$ and $\epsilon = 0$, the conservative temperature is a conserved quantity, in contrast to the *in-situ* temperature which is not conserved under adiabatic conditions. Conservative temperature is thus more convenient as a state variable than *in-situ* temperature, since adiabatic processes can be characterized by $S = \text{const}$ and $\theta^\star = \text{const}$, instead of $S = \text{const}$ and $\eta = \text{const}$. We will, therefore, use $\theta^\star$ instead of T in the following as state variable (see also Section 2.6.4). The difference between $\theta^\star$ and T is, however, small (see Figure 2.13).

The equation (2.111) for the conservative temperature is exact. The local sources and sinks arise not only from the physical heat sources $\rho\epsilon - \nabla \cdot \boldsymbol{J}_\mathrm{H}$ (modified by a factor θ/T) but also from an additional term related to the salinity flux divergence. A substantial simplification is, however, possible by considering the magnitude of various source terms in (2.111):

- A first approximation involves the first term on the right-hand side of (2.111) which describes the change in conservative temperature due to a salinity flux divergence. Near the surface where $\theta/T \to 1$ and $H^0 \to H$ this term vanishes. At depth its magnitude remains small, and a salinity flux that changes the salinity of a fluid parcel by as much as $1\,\mathrm{g\,kg^{-1}}$ leads to a change in conservative temperature of less than 0.05 K. For all practical purposes, the direct influence of the salt flux divergence on conservative temperature can, therefore, be neglected (see McDougall, 2003, for a detailed analysis). Note that the contribution of the flux $\boldsymbol{J}_\mathrm{chem}$ to the heat flux $\boldsymbol{J}_\mathrm{H} = \boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad} + \boldsymbol{J}_\mathrm{chem}$ defined in (2.75) is also small and can be neglected.
- A second approximation involves the magnitude of θ/T. In the ocean, the difference between conservative and *in-situ* temperature rarely exceeds 0.5°C. Both

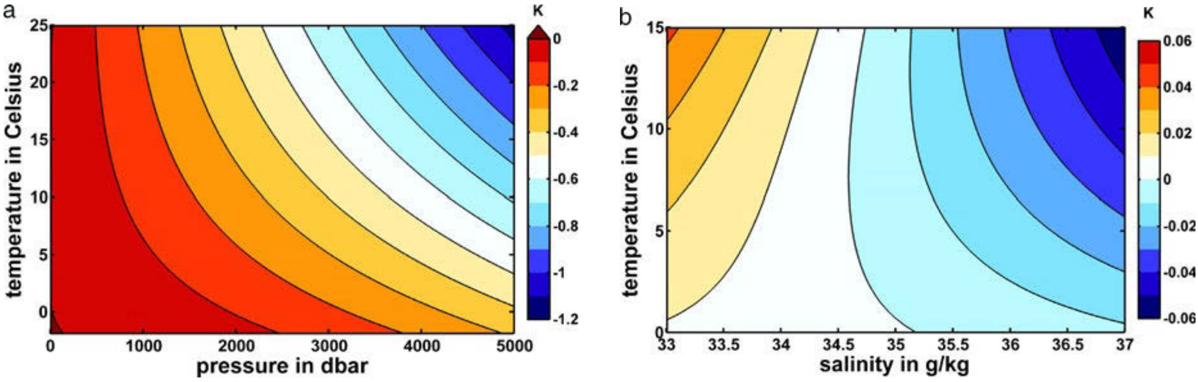

Fig. 2.13 **a** Differences of conservative temperature $\theta^\star$ and in-situ temperature T as function of T and p at $S = 35\,\mathrm{g\,kg^{-1}}$. **b** The difference of $\theta^\star$ and θ is shown as function of S and T for $p = 1{,}000\,\mathrm{dbar}$

temperatures are defined in K, and hence the approximation $\theta/T \approx 1$ in (2.111) has a relative error of no more than 2×10^{-3}, and is even better near the surface where the flux divergence term is most important.

- A third approximation concerns the mechanical dissipation ϵ. As seen from (2.71), the dissipation rate depends on the deformation, respectively shear, of the flow. For a shear of $\partial u/\partial x = 0.1\,\mathrm{s}^{-1}$ (which is already unrealistically high for ocean conditions) one would have $\rho\epsilon \approx 10^{-6}\,\mathrm{m^2 s^{-1}} \times 10^3\,\mathrm{kg\,m^{-3}} \times (0.1)^2\,\mathrm{s^{-2}} \approx 10^{-4}\,\mathrm{W\,m^{-3}}$. According to (2.101), such a value corresponds to a temperature increase of $\sim 10^{-6}\,\mathrm{K/d}$. In the deep ocean, typical values of the dissipation are around $10^{-9}\,\mathrm{W\,m^{-3}}$, so that the corresponding dissipative heating is five orders of magnitude smaller. Note, however, that while mechanical dissipation is negligible for the heat budget, its existence is fundamental for the *mechanical* energy budget.

With these approximation, it follows that

$$\rho c_p^\star \frac{\mathrm{D}\theta^\star}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{J}_\mathrm{H} \tag{2.112}$$

Note that this equation is used in most ocean models. Boundary conditions follow from (2.86). For a more detailed discussion of the approximations involved in (2.112) see IOC, SCOR and IAPSO (2010). We show in Figure 2.13 the difference of the conservative temperature $\theta^\star$ with respect to the *in-situ* temperature T (left panel) and to the potential temperature θ as function of T, S and p. Significant deviations only occur for pressures exceeding 1,000 dbar.

2.6.3 Potential Temperature

We derive in this section a budget for the potential temperature θ. This is done because potential temperature has long been in use in physical oceanography as the 'best' temperature variable, until it became clear that conservative temperature is the superior choice. It is convenient to start from the entropy $\eta = \eta(S, T, p)$ which is a known function of its arguments, as discussed in Section 1.2.5. The potential temperature θ, referred to a fixed reference pressure p_0, is implicitly defined through the relation

$$\eta(S, T, p) = \eta(S, \theta, p_0) \equiv \eta^0(S, \theta) \tag{2.113}$$

which states that the potential temperature θ is the temperature which a particle of temperature T and pressure p will assume when it is brought *adiabatically* (i. e. at constant η and S) to a location with pressure p_0 (usually the surface pressure). Note that the definition (2.113) agrees with (2.108), and also with the more intuitive definition $\theta(S, T, p) = T - \int_{p_0}^{p} \Gamma(S, T(S, \eta, p'), p')\mathrm{d}p'$ as can be seen from (2.118) below. Here S and η remain constant during the integration, and the function $T = T(S, \eta, p)$ can be obtained by solving (2.113) for the *in-situ* temperature.

Solving (2.113) for θ results in

$$\theta = \theta(S, T, p) \tag{2.114}$$

and a prognostic equation for θ can be derived in terms of the time rates of change of S, T, and p,

$$\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\partial\theta}{\partial S}\frac{\mathrm{D}S}{\mathrm{D}t} + \frac{\partial\theta}{\partial T}\frac{\mathrm{D}T}{\mathrm{D}t} + \frac{\partial\theta}{\partial p}\frac{\mathrm{D}p}{\mathrm{D}t} \tag{2.115}$$

The functional form of (2.114) is not explicitly known, but the partial derivatives can be inferred by considering the total differential of (2.113), i. e.

$$\frac{\partial\eta}{\partial S}\mathrm{d}S + \frac{\partial\eta}{\partial T}\mathrm{d}T + \frac{\partial\eta}{\partial p}\mathrm{d}p = \frac{\partial\eta^0}{\partial S}\mathrm{d}S + \frac{\partial\eta^0}{\partial\theta}\mathrm{d}\theta \tag{2.116}$$

Here the index ^{0} indicates that the function of (S, T, p) is evaluated at temperature θ and pressure p_0. From (1.57) it follows that $\partial\eta/\partial T = c_p(S, T, p)/T$ and $\partial\eta^0/\partial\theta = c_p(S, \theta, p_0)/\theta \equiv c_p^0/\theta$. Using (2.116), the derivatives in (2.115) may be expressed as

$$\frac{\partial\theta}{\partial S} = \frac{\theta}{c_p^0}\left(\frac{\partial\eta}{\partial S} - \frac{\partial\eta^0}{\partial S}\right), \quad \frac{\partial\theta}{\partial T} = \frac{c_p}{c_p^0}\frac{\theta}{T}, \quad \frac{\partial\theta}{\partial p} = \frac{\theta}{c_p^0}\frac{\partial\eta}{\partial p} \tag{2.117}$$

In particular one finds, using (1.50) and (1.54),

$$\frac{\partial\theta/\partial p}{\partial\theta/\partial T} = \frac{T}{c_p}\frac{\partial\eta}{\partial p} = -\frac{\alpha T}{\rho c_p} \equiv -\Gamma(S, T, p) \tag{2.118}$$

From (2.118) we see that the term containing $\mathrm{D}p/\mathrm{D}t$ in (2.115) exactly cancels with the corresponding term from (2.103) when eliminating $\mathrm{D}T/\mathrm{D}t$. Invoking (2.20) to eliminate $\mathrm{D}S/\mathrm{D}t$ from (2.115) as well as (2.102) and the first relation in (1.49), a prognostic equation for potential temperature is obtained as (Bacon and Fofonoff, 1996)

$$\rho c_p^0\frac{\mathrm{D}\theta}{\mathrm{D}t} = \frac{\theta}{T}\left(\rho\epsilon - \nabla\cdot\boldsymbol{J}_\mathrm{H}\right) + \frac{\theta}{T}\left(\mu - T\frac{\partial\mu^0}{\partial T}\right)\nabla\cdot\boldsymbol{J}_\mathrm{S} \tag{2.119}$$

Similar to the conservative temperature $\theta^\star$, given by (2.111), the potential temperature θ is also conserved for adiabatic processes, i. e. for $\rho\epsilon = 0$, $\boldsymbol{J}_\mathrm{H} = 0$, and $\boldsymbol{J}_\mathrm{S} = 0$. With the same approximations which lead from (2.111) to (2.112), we obtain for the potential temperature

$$\rho c_p^0\frac{\mathrm{D}\theta}{\mathrm{D}t} = -\nabla\cdot(\boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad}) \quad \text{or} \quad \rho\frac{\mathrm{D}\theta}{\mathrm{D}t} = -\nabla\cdot\boldsymbol{J}_\theta + c_p^0\boldsymbol{J}_\theta\cdot\nabla\frac{1}{c_p^0} \tag{2.120}$$

with $\boldsymbol{J}_\theta = (\boldsymbol{J}_\mathrm{T} + \boldsymbol{J}_\mathrm{rad})/c_p^0$. It becomes obvious that it is not a reference value $c_p^\star$ for specific heat as for the budget for conservative temperature (2.109), but the specific heat c_p^0 at the reference pressure p_0 which enters the balance for potential temperature (2.120). Variations of c_p^0 hence lead to a source or sink of potential temperature. In the relevant oceanic interval of salinities 34–36 g kg^{-1}, values of c_p^0 range from 3850 to 4050 J/kg, i. e. within 1% at most (see Figure 1.7). As an (admittedly extreme) example for this effect, the mixture of two water masses with 0°C temperature, 0 g kg^{-1} salinity and 30 °C, 40 g kg^{-1} at surface pressure results in a potential temperature of 14.6°C, and not 15 °C. Neglecting the source term in (2.120), therefore, causes a small error, which is, however, much larger than the approximations discussed above for the conservative temperature. Using conservative temperature completely avoids this error and conservation properties of $\theta^\star$ are, therefore, better by two orders of magnitude.

On the other hand, the differences between $\theta^\star$ and θ are in general small (see right panel of Figure 2.13). They can reach maximum values of 0.15 K at the surface comparing one ocean basin with another, but there are also comparable differences in regions of strong fronts at the surface, like the Gulf Stream. Differences at depth are much smaller and can in general be neglected, in particular in the light of the various approximations which will be introduced in Part II. Note that we will use conservative temperature as variable, but we will drop the index * for convenience in the remainder of the book (except for the next section). Note also that when referring to potential temperature, either $\theta^\star$ is meant, or differences between $\theta^\star$ and θ are simply neglected.

2.6.4 Conservative Temperature as a State Variable

We now introduce the conservative temperature $\theta^\star$ instead of the *in-situ* temperature T as state variable, since $\theta^\star$ is conserved under adiabatic conditions, while T is not conserved. Using $T = T(S, \theta^\star, p)$ for the state equation $\rho = \rho(S, T, p)$, we obtain the density as function of the new set of state variables S, $\theta^\star$, and p, i. e. by

$$\rho = \rho(S, T(S, \theta^\star, p), p) = F^\star(S, \theta^\star, p) \tag{2.121}$$

An explicit numerical approximation of the function $F^\star(S, \theta^\star, p)$ can be found in IOC, SCOR and IAPSO (2010). For practical applications of the equations of motion (2.121) is the most convenient state equation, whereas for thermodynamic considerations the original form (1.52) of the state equation is preferred.

We will need below the expansion coefficients for density with respect to the new state variables. The coefficients are given by the first derivatives of the relation (2.121) and denoted by

$$\alpha^\star = -\frac{1}{\rho}\left(\frac{\partial\rho}{\partial\theta^\star}\right)_{S,p} \quad \text{modified thermal expansion} \tag{2.122}$$

$$\gamma^\star = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{\theta^\star,p} \quad \text{modified haline contraction} \tag{2.123}$$

$$\kappa^\star = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial p}\right)_{S,\theta^\star} \quad \text{adiabatic compressibility} \tag{2.124}$$

Note that the coefficients $\alpha^\star$ and $\gamma^\star$ are similar in magnitude but not identical to the corresponding coefficients in (1.54)–(1.56) using *in-situ* temperature T as state variable. To express the modified thermal expansion coefficient $\alpha^\star = -(1/\rho)(\partial\rho/\partial\theta^\star)_{S,p} = \alpha(\partial T/\partial\theta^\star)_{S,p}$ in terms of known quantities, we need to know $\partial T/\partial\theta^\star$. Using $\eta(S, T, p) = \eta^0(S, \theta^\star, p_0)$, thus

$$\frac{\partial\eta}{\partial T} = \frac{\partial\eta^0}{\partial\theta^\star}\frac{\partial\theta^\star}{\partial T}$$

and combining with $\partial\eta/\partial T = c_p/T$ and $\partial\theta^\star/\partial\eta^0 = \theta/c_p^\star$ and with the definition (2.110) for $\theta^\star$, for the modified thermal expansion coefficient

$$\alpha^\star = \alpha\frac{c_p^\star T}{c_p\theta} \tag{2.125}$$

The modified haline contraction coefficient $\gamma^\star$ is related to γ by

$$\begin{aligned}\gamma = \frac{1}{\rho}\left(\frac{\partial\rho}{\partial S}\right)_{T,p} &= \frac{1}{\rho}\left[\left(\frac{\partial\rho}{\partial S}\right)_{\theta^\star,p} + \left(\frac{\partial\rho^\star}{\partial\theta}\right)_{S,p}\left(\frac{\partial\theta^\star}{\partial S}\right)_{S,p}\right]\\ &= \gamma^\star - \alpha^\star\left(\frac{\partial\theta^\star}{\partial S}\right)_{S,p}\end{aligned}$$

To find $\partial\theta^\star/\partial S$, use again $\eta(S,T,p) = \eta^0(S,\theta^\star,p_0)$, thus

$$\left.\frac{\partial\eta}{\partial S}\right|_{T,p} = \left.\frac{\partial\eta^0}{\partial S}\right|_{\theta^\star} + \left.\frac{\partial\eta^0}{\partial\theta^\star}\right|_S\frac{\partial\theta^\star}{\partial S} \quad\text{or}\quad \frac{\partial\theta^\star}{\partial S} = \left(\left.\frac{\partial\eta}{\partial S}\right|_{T,p} - \left.\frac{\partial\eta^0}{\partial S}\right|_{\theta^\star}\right)\frac{\theta}{c_p^\star}$$

since $\partial\eta^0/\partial\theta^\star = c_p^\star/\theta$. Thus with (2.125) we find

$$\gamma^\star = \gamma + \alpha^\star\left(\left.\frac{\partial\eta}{\partial S}\right|_{T,p} - \left.\frac{\partial\eta^0}{\partial S}\right|_{\theta^\star}\right)\frac{\theta}{c_p^\star} = \gamma + \alpha\frac{T}{c_p}\left(\left.\frac{\partial\eta}{\partial S}\right|_{T,p} - \left.\frac{\partial\eta^0}{\partial S}\right|_{\theta^\star}\right) \tag{2.126}$$

For the adiabatic compressibility $\kappa^\star$ analogous arguments lead to

$$\kappa^\star = \kappa + \alpha^\star\left(\frac{\partial\theta^\star}{\partial p}\right)_{S,T} = \kappa - \frac{\alpha^2 T}{c_p\rho} \tag{2.127}$$

Since it has a clear physical meaning, we will use in the following the speed of sound c_s instead of the modified adiabatic compressibility $\kappa^\star$. The sound velocity is related to $\kappa^\star$ by

$$\frac{1}{c_\mathrm{s}^2} = \kappa^\star\rho = \left(\frac{\partial\rho}{\partial p}\right)_{S,\theta^\star} \tag{2.128}$$

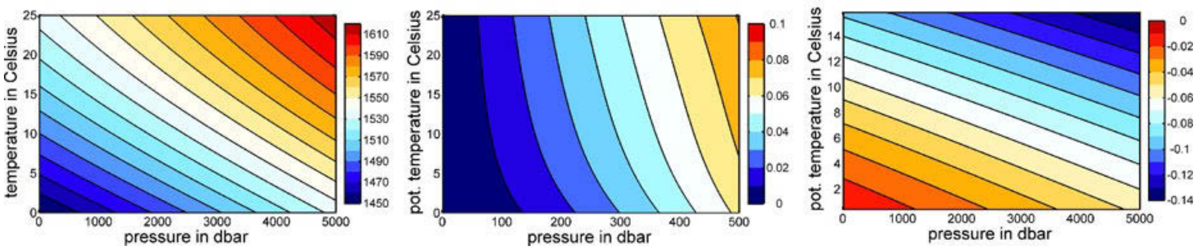

Fig. 2.14 Velocity of sound in seawater, c_s (**a**, in m s^{-1}), difference of the modified thermal expansion coefficient to Figure 1.6, $\alpha^\star - \alpha$ (**b**, in 10^{-4} 1/K) as well as difference of the modified haline contraction coefficient to Figure 1.6, $\gamma^\star - \gamma$ (**c**, in 10^{-4} (g kg^{-1})$^{-1}$), all as a function of T and p at $S = 35\,\mathrm{g\,kg^{-1}}$

The sound speed c_s and the differences in expansion coefficients $\alpha^\star - \alpha$ and $\gamma^\star - \gamma$ for the different state variables are shown in Figure 2.14. Relative differences are for the thermal expansion coefficient significant on the order of 5%, but for the haline contraction coefficient smaller.

Note that from now on, we shall drop the * for the coefficients and use the notation α and γ for the coefficients defined in (2.122) and (2.123). As mentioned above, we shall also drop the * for the conservative temperature (which will be also called occasionally potential temperature, when differences between both are unimportant). Likewise, the notation $F(S, \theta, p)$ will denote the state equation (2.121) whenever conservative temperature is used as independent variable.

2.7 Density Variables

The *in-situ* density ρ is not conserved for constant S and θ, i. e. for adiabatic conditions. However, since molecular and turbulent transports divergences (see also next section) acting on the salinity and conservative temperature budgets are often rather small in the interior of the ocean, a "conserved" density is a useful quantity in particular to trace the spreading of water masses in the ocean or the construction of isopycnal models. Similar to the definition of a "conserved" temperature, we introduce the concept of a potential density in this section. On the other hand, it will turn out that water masses spread along potential density surfaces only under certain conditions, which will be clarified by the concept of neutral surface elements.

2.7.1 Potential Density

In analogy to the concept of conservative temperature a density variable can be introduced which is conserved for adiabatic conditions. The *potential density* is a useful concept more alike the *in-situ* density than either temperature or salinity alone. It is defined by

$$\rho_p(S, \theta) = F(S, \theta, p_\star) \tag{2.129}$$

with the *fixed* reference pressure $p_\star = \text{const}$. For practical use, 1,000 kg m^{-3} is subtracted from the values of potential density. The potential density ρ_p equals the *in-situ* density of a particle (except for the constant removed in (2.129)), when the particle is moved adiabatically from its *in-situ* pressure p to the reference pressure $p_\star$ as sketched in Figure 2.15. Note that the reference pressure $p_\star$ is in principle unrelated to the reference pressure p_0 used in the definition of conservative temperature (2.106). It is, in fact, common to define potential densities not only with reference to the surface but also to deeper levels in the ocean. The reason is that isosurfaces of ρ_p are close to surfaces of adiabatic spreading of water masses when the *in-situ* pressure is close to $p_\star$, as discussed below. The common notation is as follows: $\sigma_0(\sigma_\theta), \sigma_1, \sigma_4$ for $p_\star = 0$, 1,000, 4,000 dbar etc. The conservation of potential density is described by

$$\frac{\mathrm{D}\rho_p}{\mathrm{D}t} = \rho_p \left(\gamma_\star \frac{\mathrm{D}S}{\mathrm{D}t} - \alpha_\star \frac{\mathrm{D}\theta}{\mathrm{D}t} \right) \tag{2.130}$$

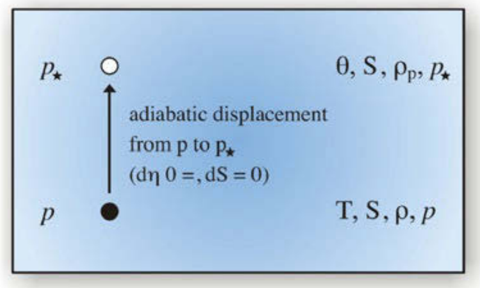

Fig. 2.15 Sketch of adiabatic displacement of a fluid parcel from p to $p_\star$ with respective temperatures, salinities, and densities

where $\gamma_\star, \alpha_\star$ denote the expansion coefficients given by (2.123) and (2.122), respectively, evaluated for the reference pressure $p_\star$. Hence potential density is always conserved for adiabatic motions, for any choice of the reference pressure. Surfaces of constant potential density are called *isopycnal surfaces*. Note, however, that it is the *in-situ* density which appears in the equations of motion. Only near the reference pressure, the potential density is similar to the (dynamically relevant) *in-situ* density. Later in Section 4.1.3 we will derive the balance of *in-situ* density ρ and find that any suitable density variable containing the compressibility is not conserved, even for adiabatic motions.

2.7.2 Neutral Surface Elements

Under adiabatic conditions the advection of water masses occurs along isopycnals, however, only if the pressure changes are not substantial, as shown in this section. To derive a more general concept, that of a so-called *neutral surface*, we start with the equation of state (2.121) and consider a small change of density over the distance $\mathrm{d}\boldsymbol{x}$. It is given by

$$\mathrm{d}\rho = \gamma\rho\nabla S \cdot \mathrm{d}\boldsymbol{x} - \alpha\rho\nabla\theta \cdot \mathrm{d}\boldsymbol{x} + \frac{1}{c_\mathrm{s}^2}\nabla p \cdot \mathrm{d}\boldsymbol{x}$$

Likewise, the change due to an adiabatic motion of a parcel (i. e. with constant S, θ) over distance $\mathrm{d}\boldsymbol{x}$ is

$$(\mathrm{d}\rho)_\mathrm{ad} = \frac{1}{c_\mathrm{s}^2}\mathrm{d}p = \frac{1}{c_\mathrm{s}^2}\nabla p \cdot \mathrm{d}\boldsymbol{x} \tag{2.131}$$

and we thus obtain

$$\mathrm{d}\rho - (\mathrm{d}\rho)_\mathrm{ad} = \gamma\rho\nabla S \cdot \mathrm{d}\boldsymbol{x} - \alpha\rho\nabla\theta \cdot \mathrm{d}\boldsymbol{x} = \boldsymbol{e} \cdot \mathrm{d}\boldsymbol{x}$$

for the difference of density between the background and the parcel. Here the vector

$$\boldsymbol{e} = \gamma\rho\nabla S - \alpha\rho\nabla\theta \equiv \nabla\rho - \frac{1}{c_\mathrm{s}^2}\nabla p \tag{2.132}$$

is introduced. The difference in density between the parcel and the environment vanishes if

$$\boldsymbol{e} \cdot \mathrm{d}\boldsymbol{x} = 0 \tag{2.133}$$

Considering all possible $\mathrm{d}\boldsymbol{x}$ that satisfy this condition, a *neutral* surface element is constructed. The vector $\boldsymbol{e}$ is normal to the element and also normal to the isopycnal $\rho_{\mathrm{p}} = \text{const}$ defined for the local pressure $p = p_\star$, i. e. at the reference point we have $\boldsymbol{e} = \nabla \rho_{\mathrm{p}}$ (see Figure 2.16). Note that since the pressure gradient is almost vertical, the vectors $\boldsymbol{e}$ and $\nabla \rho$ differ by an almost vertical vector.

The condition (2.133), however, defines a surface only locally. A continuous surface of global extent is given by a condition $\chi(\boldsymbol{x}) = \text{const}$ with some scalar function $\chi(\boldsymbol{x})$ and (2.133) is then replaced by $\nabla\chi \cdot \mathrm{d}\boldsymbol{x} = 0$. This implies that only if $\boldsymbol{e} = \varphi\nabla\chi$ would hold with some arbitrary function φ, an entire neutral surface exists. A sufficient condition is the vanishing of the helicity $\mathcal{H} = \boldsymbol{e} \cdot \nabla \times \boldsymbol{e}$. This can be seen by using the condition $\boldsymbol{e} = \varphi\nabla\chi$ for the helicity, which yields $\mathcal{H} = \varphi\nabla\chi \cdot (\varphi\nabla \times \nabla\chi + \nabla\varphi \times \nabla\chi) = 0$. Using the equation of state, the helicity is given by

$$\begin{aligned} \mathcal{H} &= -\alpha\rho\nabla(\gamma\rho) \times \nabla S \cdot \nabla\theta - \gamma\rho\nabla(\alpha\rho) \times \nabla\theta \cdot \nabla S \\ &= \rho^2 \left(\alpha\nabla\gamma - \gamma\nabla\alpha\right) \cdot \nabla\theta \times \nabla S = -\rho^2\gamma^2\nabla\frac{\alpha}{\gamma} \cdot \nabla\theta \times \nabla S \end{aligned}$$

Since the function α/γ depends not only on θ and S but also on pressure, the helicity $\mathcal{H}$ does not vanish in general. Note that the θ or S dependencies of α/γ yield corresponding components of $\nabla\alpha/\gamma$ which are orthogonal to $\nabla\theta \times \nabla S$. Since $\mathcal{H} \neq 0$ there are no neutral surfaces in general. Only for an equation of state in which $\partial(\alpha/\gamma)/\partial p = 0$ would hold (which is not the case), neutral surfaces exists in general. On the other hand, for regions of the ocean with constant θ or S, or for a fixed functional relation $\theta(S)$ or $S(\theta)$, i. e. for a fixed water mass relationship which is often found for rather large regions of the ocean, neutral surfaces can be approximately defined.

In any case, however, there are *neutral trajectories*, i. e. pathways $\boldsymbol{s}$ in the ocean for which $\boldsymbol{e} \cdot \boldsymbol{s} = 0$ holds, such that no buoyant forces would act on water parcels moved along those pathways. McDougall (1987) considers a neutral trajectory in a large circle around the subtropical gyre in the North Atlantic, and finds that the

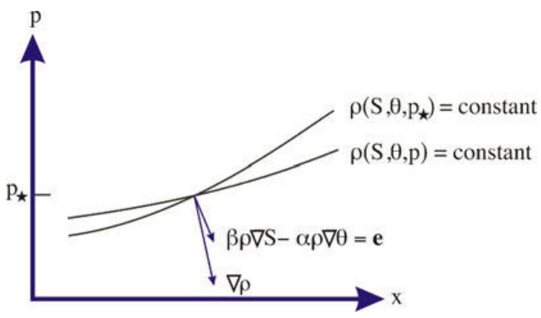

Fig. 2.16 Sketch showing the neutral surface element, the density surface $\rho(S, \theta, p) = \text{const}$, and the local isopycnal $\rho(S, \theta, p_0) = \text{const}$

neutral trajectory of the water parcel would form a large helix, with a change in height in one circle to the other of about 5 m.

12. Approximate Neutral Density

A more restrictive constraint on the vector $\boldsymbol{e}$ is given by $\boldsymbol{e} = \boldsymbol{\nabla}\chi$, for which the quantity χ is called the *neutral density*, denoted by γ_n. Surfaces of $\chi =$ const would then form neutral surfaces. It is obvious that there is no neutral density in general, but approximate forms can be given, for instance by the condition

$$\nabla^2\chi = \boldsymbol{\nabla}\cdot\boldsymbol{e}$$

which was used by Eden and Willebrand (1999) to estimate χ for the North Atlantic. A different method for construction of a neutral density variable γ_{n} has been suggested by Jackett and McDougall (1997), however with an equation of state relating γ_{n} to S, θ, p which includes an explicit spatial dependence.

2.8 Molecular and Turbulent Transports

The conservation equations for mass (2.18), momentum (2.59), salt (2.20), and heat (2.120), in combination with the equation of state (2.121), form a complete system of differential equations for the variables ρ, S, θ, p, and $\boldsymbol{u}$ which we use to represent the oceanic motion. In principle, they can be solved if initial conditions and boundary fluxes of mass, momentum, freshwater, and heat are prescribed.

To understand large-scale properties of the ocean and its interactions with the atmosphere, it is necessary to consider averages over suitable temporal and/or spatial scales. A possible filtering method is discussed below. However, a fundamental property of oceanic dynamics is that the motions at smaller scales influence those at larger scales. That influence occurs through *turbulent* transports which arise from the nonlinear nature of the equations of motion, as shown below. In contrast to molecular transports, which are generally very small, turbulent transports of momentum, heat, and salt are at the heart of ocean dynamics. In this section we review the size of the molecular transports and then derive the mathematical form of the turbulent transports by the commonly used averaging methods. Further treatment of these eddy-driven transports is postponed to Part IV.

2.8.1 Magnitude of Molecular Transports

The nonadvective transports of momentum, salt, and heat are the result of molecular processes. They are usually described by the empirical laws discussed in Section 2.5.1. Ignoring the cross-transports of heat and salt discussed in Section 2.5.2, these laws are (2.36) for the stress in Newtonian fluids, Fourier's law (2.94) for the heat conduction, and the modified Fickian law (2.96) for the salt diffusion.

In order to estimate the role of the molecular fluxes, we assume that typical velocities are of magnitude U, and typical length scales of magnitude L. Specifically, we consider two limiting cases: (1) small-scale motions with $U = 1\,\mathrm{m\,s^{-1}}$ and

$L = 10\,\mathrm{m}$ which are e. g. typical of ocean surface waves, and (2) large-scale motions with $U = 0.1\,\mathrm{m\,s^{-1}}$ and $L = 1{,}000\,\mathrm{km}$ typical for the ocean circulation. In the equations for momentum, salt, and heat we then have the following balances:

- **Friction**
  The order of magnitude of the momentum advection, the Coriolis and the friction terms in (2.59) are given by
  $$\rho \boldsymbol{u} \cdot \nabla \boldsymbol{u} \sim \rho U \frac{U}{L} \tag{2.134}$$
  $$2\rho \boldsymbol{\Omega} \times \boldsymbol{u} \sim 2\rho \Omega U \tag{2.135}$$
  $$\kappa_{\mathrm{M}} \rho \nabla^2 \boldsymbol{u} \sim \rho \kappa_{\mathrm{M}} \frac{U}{L^2} \tag{2.136}$$
  The ratio of the momentum advection (2.134) to the friction term (2.136) is the REYNOLDS[17]-number $\mathrm{Re} = UL/\kappa_{\mathrm{M}}$. With a kinematic viscosity of $\kappa_{\mathrm{M}} \approx 1.3 \times 10^{-6}\,\mathrm{m^2\,s^{-1}}$, we obtain for small-scale motions $\mathrm{Re} \approx 10^7$ and for large-scale motions even $\mathrm{Re} \approx 10^{11}$. In both cases it follows that the molecular momentum transport is negligible compared to the momentum advection. The same conclusion arises from the consideration of the ratio of the friction term and the Coriolis term which is the EKMAN[18]-number $\mathrm{Ek} = \kappa_{\mathrm{M}}/(2\Omega L^2)$. With $\Omega \approx 7 \times 10^{-5}\,\mathrm{s^{-1}}$, we obtain $\mathrm{Ek} \approx 10^{-14}$ for $L = 1{,}000\,\mathrm{km}$.
- **Salt diffusion**
  The order of magnitude of the salt advection and diffusion terms in (2.20) is given by
  $$\rho \boldsymbol{u} \cdot \nabla S \sim \rho U \frac{\Delta S}{L} \tag{2.137}$$
  $$\rho \kappa_{\mathrm{S}} \nabla^2 S \sim \kappa_{\mathrm{S}} \rho \frac{\Delta S}{L^2} \tag{2.138}$$
  The ratio of salt advection (2.137) to salt diffusion (2.138) is the PECLET[19]-number Pe which has a magnitude of $\mathrm{Pe} = UL/\kappa_{\mathrm{S}}$. With $\kappa_{\mathrm{S}} \approx 1.2 \times 10^{-9}\,\mathrm{m^2\,s^{-1}}$, the Peclet-number varies between $\mathrm{Pe} \approx 10^{10}$ (small scales) and $\mathrm{Pe} \approx 10^{14}$ (large scales) for the above examples, and salt diffusion is negligible compared to advection.
- **Heat conduction**
  The situation is analogous to the salt budget. The ratio of salt and temperature coefficients for molecular diffusion is $\kappa_{\mathrm{T}}/\kappa_{\mathrm{S}} \approx 10^2$. It follows that the heat conduction term in (2.120) is of relative order 10^{-8}–10^{-12} compared to temperature advection and hence negligible.

For almost all oceanographic problems the *direct* influence of molecular transports is thus very small. However, the very existence of molecular transports is of fundamental importance, in particular for turbulent energy cascades but also for specific processes such as double diffusion. Moreover, in boundary layers molecular fluxes are always important. Those boundary layers are, however, very thin and hence usually neglected or – more precisely – included in parameterizations for turbulent fluxes in thicker turbulent boundary layers.

---

[17] OSBORNE REYNOLDS, *1842 in Belfast, †1912 in Watchet/Somerset, mechanical engineer and physicist.

[18] VAGN WALFRID EKMAN, *1874 in Stockholm, †1954 in Gostad, oceanographer and physicist.

[19] JEAN CLAUDE EUGÈNE PÈCLET, *1793 in Besançon, †1857 in Paris, physicist.

2.8.2 Reynolds and Hesselberg Averaging

The equations of motion are valid for all scales larger than, say, 1 μm. But we are normally interested only in the oceanic state averaged over much larger temporal and/or spatial scales. Denoting such an average by $\overline{(..)}$, the most simple temporal mean of a variable ψ could e. g. be defined by

$$\overline{\psi}(t) = \frac{1}{\tau} \int\limits_{t-\tau/2}^{t+\tau/2} \psi(t')\mathrm{d}t' \tag{2.139}$$

with an averaging interval of length τ. The average (2.139) is sometimes also called a running mean. Likewise, a spatial or a combined spatial-temporal average could be considered. A disadvantage of any average as (2.139) is, however, that it does *not* satisfy

$$\overline{\overline{\psi}} = \overline{\psi} \tag{2.140}$$

which we will assume nevertheless because it is a rather desirable property, as we see below. Some other averaging operations which satisfy (2.140), but which are more complicated, are discussed in the box on p. 70. For a Boussinesq fluid (which we will introduce in Section 4.1.2), the averaging attributed to Osborn Reynolds is most commonly performed, which is sometimes called *Reynolds averaging*. This procedure will be extensively used and discussed in Chapters 11 and 12. With a non-constant density entering the conservation laws, however, another procedure is more appropriate: the HESSELBERG[20] average (Hesselberg, 1926), which we discuss now.

13. Weighted and Statistical Averages

In principle, one could define an average by

$$\overline{\psi} = \int\limits_{-\infty}^{\infty} W(t,t')\psi(t')\mathrm{d}t' \tag{B13.1}$$

which satisfies (2.140) provided that $\int W(t,t')\mathrm{d}t' = 1$ and $\int W(t,t'')W(t'',t')\mathrm{d}t'' = W(t,t')$. The kernel $W(t,t')$, of course, must contain the appropriate scales corresponding to temporal/spatial averaging. If the motions at smaller scales are described statistically, a statistical average arises which would likewise satisfy (2.140).
To simplify the work arising from the noncommutativity of averaging and differential operators, the averaging procedure is frequently formulated in terms of an ensemble of states $\psi(\boldsymbol{x},t;\lambda)$ such that $\overline{\psi}(\boldsymbol{x},t) = \int \mathrm{d}P(\lambda)\psi(\boldsymbol{x},t;\lambda)$ is the expectation with respect to a probability measure $\mathrm{d}P(\lambda)$ and $\psi' = \psi - \overline{\psi}$ is the deviation of a particular realization. We will assume that the averaging operation obeys (2.140) although in practice this may be difficult to achieve.

In an Eulerian framework, it is most straightforward to consider averages of the state variables *per volume*, not per mass. We thus define the mean state variables by mass-weighted averages according to

$$\rho_\mathrm{m} = \overline{\rho}\ , \quad S_\mathrm{m} = \frac{\overline{\rho S}}{\rho_\mathrm{m}}\ , \quad \theta_\mathrm{m} = \frac{\overline{\rho\theta}}{\rho_\mathrm{m}}\ , \quad \boldsymbol{u}_\mathrm{m} = \frac{\overline{\rho\boldsymbol{u}}}{\rho_\mathrm{m}} \tag{2.141}$$

---

[20] THEODOR HESSELBERG, *1885 in Lierne, †1966 in Oslo, meteorologist.

In this way the mean concentrations per volume are preserved since we have e. g. $\rho_\mathrm{m} S_\mathrm{m} = \overline{\rho S}$ etc. As an example, we will explicitly discuss the salt conservation (2.19), the average of which is

$$\frac{\partial \overline{\rho S}}{\partial t} + \nabla \cdot \overline{\rho S \boldsymbol{u}} = -\nabla \cdot \overline{\boldsymbol{J}}_\mathrm{S} \tag{2.142}$$

Decomposing the variables according to

$$\rho = \rho_\mathrm{m} + \rho' \,, \quad S = S_\mathrm{m} + S' \,, \quad \boldsymbol{u} = \boldsymbol{u}_\mathrm{m} + \boldsymbol{u}' \quad \text{etc.} \tag{2.143}$$

we obtain

$$\frac{\partial \rho_\mathrm{m} S_\mathrm{m}}{\partial t} + \nabla \cdot \rho_\mathrm{m} S_\mathrm{m} \boldsymbol{u}_\mathrm{m} = -\nabla \cdot \left(\overline{\boldsymbol{J}}_\mathrm{S} + \boldsymbol{J}_\mathrm{S}^\mathrm{turb} \right) \tag{2.144}$$

The mean salinity hence satisfies a conservation equation completely identical to (2.19), except for the addition of a *turbulent* salinity flux $\boldsymbol{J}_\mathrm{S}^\mathrm{turb}$ which is given by

$$\begin{aligned} \boldsymbol{J}_\mathrm{S}^\mathrm{turb} &= \overline{\rho S \boldsymbol{u}} - \rho_\mathrm{m} S_\mathrm{m} \boldsymbol{u}_\mathrm{m} = \overline{(\rho_\mathrm{m} + \rho')\,(S_\mathrm{m} + S')\,(\boldsymbol{u}_\mathrm{m} + \boldsymbol{u}')} - \rho_\mathrm{m} S_\mathrm{m} \boldsymbol{u}_\mathrm{m} \\ &= \overline{\rho'} S_\mathrm{m} \boldsymbol{u}_\mathrm{m} + \boldsymbol{u}_\mathrm{m} \left(\rho_\mathrm{m} \overline{S'} + \overline{\rho' S'} \right) + S_\mathrm{m} \left(\rho_\mathrm{m} \overline{\boldsymbol{u}'} + \overline{\rho} \boldsymbol{u}' \right) + \rho_\mathrm{m} \overline{S' \boldsymbol{u}'} + \overline{\rho' S' \boldsymbol{u}'} \end{aligned} \tag{2.145}$$

From (2.141) and (2.140) we have $\overline{\rho'} = 0$. Furthermore, from (2.141) it follows that

$$S_\mathrm{m} = \frac{\overline{(\rho_\mathrm{m} + \rho')\,(S_\mathrm{m} + S')}}{\rho_\mathrm{m}} = \frac{\rho_\mathrm{m} S_\mathrm{m} + \rho_\mathrm{m} \overline{S'} + \overline{\rho' S'}}{\rho_\mathrm{m}} = S_\mathrm{m} + \overline{S'} + \frac{\overline{\rho' S'}}{\rho_\mathrm{m}}$$

and thus

$$\overline{S'} = -\frac{\overline{\rho' S'}}{\rho_\mathrm{m}} \,, \quad \overline{\boldsymbol{u}'} = -\frac{\overline{\rho' \boldsymbol{u}'}}{\rho_\mathrm{m}} \quad \text{etc.}$$

Hence only the last two terms in (2.145) remain, and we obtain

$$\boldsymbol{J}_\mathrm{S}^\mathrm{turb} = \rho_\mathrm{m} \overline{S' \boldsymbol{u}'} + \overline{\rho' \boldsymbol{u}' S'} \tag{2.146}$$

Even when the turbulent fluctuations are strong, the density fluctuations in the ocean are always small, with a typical magnitude of $\rho'/\rho \sim 10^{-3}$. To a good approximation, the triple-correlations in (2.146) can hence be neglected, and one obtains the more common form

$$\boldsymbol{J}_\mathrm{S}^\mathrm{turb} = \rho_\mathrm{m} \overline{S' \boldsymbol{u}'} \tag{2.147}$$

It is, however, largely irrelevant whether (2.146) or (2.147) is used because, in any case, the turbulent flux ultimately has to be parameterized in terms of the mean state. Analogous considerations can be made for the temperature and momentum equations.

$$\mathrm{J}_\theta^\mathrm{turb} = \rho_\mathrm{m} \overline{\theta' \boldsymbol{u}'} + \overline{\rho' \boldsymbol{u}' \theta'} \approx \rho_\mathrm{m} \overline{\theta' \boldsymbol{u}'} \tag{2.148}$$

$$\Sigma^\mathrm{turb} = -\left(\rho_\mathrm{m} \overline{\boldsymbol{u}' \boldsymbol{u}'} + \overline{\rho' \boldsymbol{u}' \boldsymbol{u}'} \right) \approx -\rho_\mathrm{m} \overline{\boldsymbol{u}' \boldsymbol{u}'} \tag{2.149}$$

Note, however, that the definitions in (2.141) are such that no additional terms appear in the continuity equation (2.15). In fact, a term of turbulent density diffusion would be awkward in view of the absence of molecular density diffusion.

The equation of state (2.121) is nonlinear, and the outcome of averaging as defined in (2.141) is not immediately evident. We perform a Taylor expansion around the mean variables and subsequently average the result to obtain

$$\overline{\rho} = \rho_\mathrm{m} = F(S_\mathrm{m}, \theta_\mathrm{m}, p_\mathrm{m}) + \frac{\partial F}{\partial \theta}\overline{\theta'} + \cdots + \frac{\partial^2 F}{\partial \theta^2}\overline{\theta'^2} + \cdots \tag{2.150}$$

where many further terms of first and second order occur. The derivatives are taken at $(S_\mathrm{m}, \theta_\mathrm{m}, p_\mathrm{m})$. Of all contributions to (2.150), the last term is by far the largest one. With $\partial^2 F/\partial\theta^2 \approx \rho_0 10^{-5}\,\mathrm{K}^{-2}$, its magnitude depends on the strength of the temperature fluctuations. For $(\overline{\theta'^2})^{1/2} = 1\,\mathrm{K}$, the relative contribution of that term is 5×10^{-6}, which is of same order as the accuracy of the approximations used for the state equation and hence negligible.

It follows that the equations for the mean state variables according to (2.141) are to a good approximation given by (omitting the index m from all variables)

$$\frac{\mathrm{D}\rho}{\mathrm{D}t} = -\rho\nabla\cdot\boldsymbol{u} \tag{2.151}$$

$$\rho\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -2\rho\boldsymbol{\Omega}\times\boldsymbol{u} - \nabla p - \rho\nabla\Phi + \boldsymbol{\mathcal{F}} \tag{2.152}$$

$$\rho\frac{\mathrm{D}S}{\mathrm{D}t} = \mathcal{G}_\mathrm{S} \tag{2.153}$$

$$\rho\frac{\mathrm{D}\theta}{\mathrm{D}t} = \mathcal{G}_\theta \tag{2.154}$$

$$\rho = F(S, \theta, p) \tag{2.155}$$

with the flux divergencies

$$\begin{aligned}
\boldsymbol{\mathcal{F}} &= \nabla\cdot\left(\Sigma^\mathrm{turb} + \overline{\Sigma}\right)\\
\mathcal{G}_\mathrm{S} &= -\nabla\cdot\left(\boldsymbol{J}_\mathrm{S}^\mathrm{turb} + \overline{\boldsymbol{J}}_\mathrm{S}\right)\\
\mathcal{G}_\theta &= -\nabla\cdot\left(\boldsymbol{J}_\theta^\mathrm{turb} + \left(\overline{\boldsymbol{J}}_\mathrm{rad} + \overline{\boldsymbol{J}}_\mathrm{T}\right)/c_p^\star\right)
\end{aligned}$$

as the divergences of the overall (turbulent plus mean molecular) fluxes; the latter are very small as discussed above. All following considerations in this book are based on the filtered system (2.151) to (2.155), if not explicitly stated otherwise. However, for a complete description, expressions, i. e. parameterizations, for $\boldsymbol{\mathcal{F}}$, $\mathcal{G}_\mathrm{S}$ and $\mathcal{G}_\theta$ have to be provided, based on the known averaged quantities. These parameterizations, typically diffusive closures, will be discussed in the Chapters 11 and 12.

2.9 The State of Rest

The first application of the governing equations derived in the previous sections is for a state of rest without any forcing except the force of gravity. Due to the presence of gravity, pressure and density variations are closely related. Furthermore, it is discussed under which condition a density stratification caused by temperature and salinity is stable against small perturbations.

2.9.1 Hydrostatic Balance

Consider a motionless steady state which is defined by the state variables $\boldsymbol{u}_{\mathrm{r}} \equiv 0$, p_{r}, ρ_{r}, θ_{r} and S_{r} and vanishing turbulent fluxes, i. e. $\mathcal{F} = \mathcal{G}_{\mathrm{S}} = \mathcal{G}_{\theta} = 0$. The equations of motion (2.152) then reduce to

$$0 = -\nabla p_{\mathrm{r}} - \rho_{\mathrm{r}} \nabla \Phi \tag{2.156}$$

which can only be satisfied if both conditions $\nabla \rho_{\mathrm{r}} \times \nabla \Phi = 0$ and $\nabla \rho_{\mathrm{r}} \times \nabla p_{\mathrm{r}} = 0$ hold. It follows that $\rho_{\mathrm{r}} = \rho_{\mathrm{r}}(\Phi)$ and also $p_{\mathrm{r}} = p_{\mathrm{r}}(\Phi)$, and, therefore, $\mathrm{d}p_{\mathrm{r}}/\mathrm{d}\Phi = -\rho_{\mathrm{r}}$ or with $\Phi = gz$

$$\frac{\mathrm{d}p_{\mathrm{r}}}{\mathrm{d}z} = -g\rho_{\mathrm{r}} \tag{2.157}$$

Equation (2.157) is the *hydrostatic relation*. In the state of rest there are no forces in the surfaces $\Phi = \text{const}$ to accelerate the fluid. The equation of state (2.121) takes

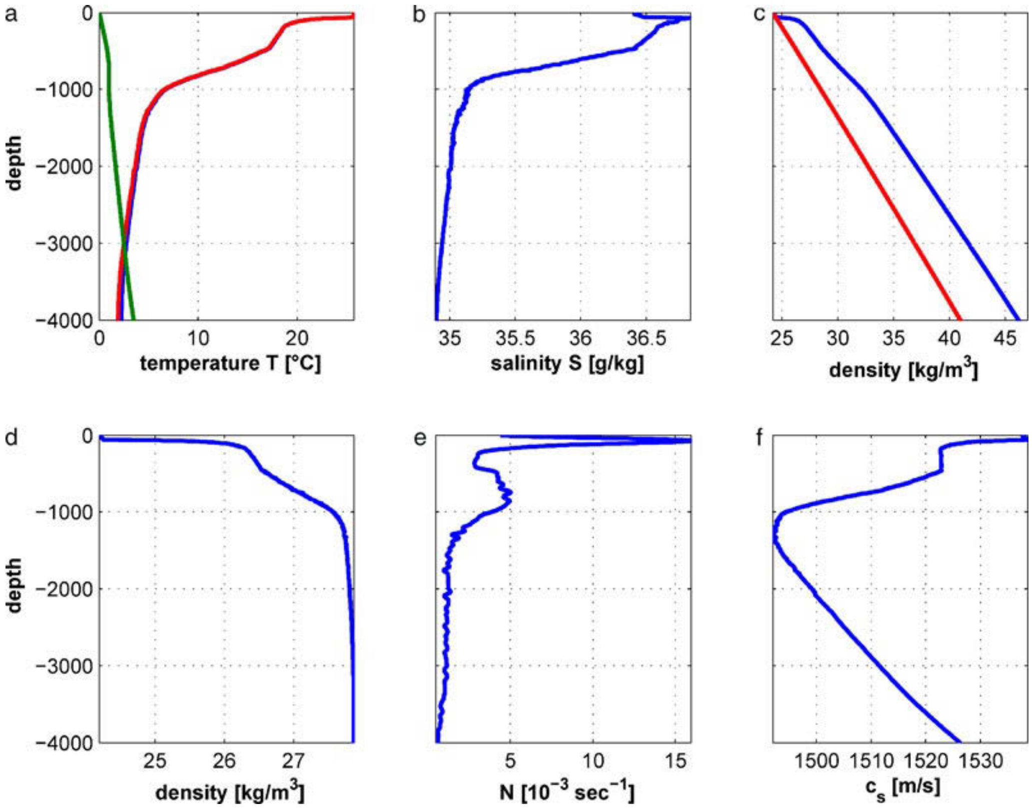

Fig. 2.17 Profiles of *in-situ* temperature (blue) and potential temperature (red) (**a**) and salinity (**b**) in the subtropical North Atlantic near Bermuda. **a** also shows the difference between *in-situ* and conservative temperature (green, amplified by a factor of 10). A profile of density (in $\mathrm{kg\,m^{-3}}$; a reference value of $1{,}000\,\mathrm{kg\,m^{-3}}$ is subtracted), taken at the *in-situ* pressure is shown in the blue curve in **c**. The straight red line in **c** is the *in-situ* density for constant temperature and salinity (the values are those at the surface). It demonstrates the pressure dependence of the density. In **d**, the density is evaluated at the surface pressure. **e** shows the Brunt–Väisälä frequency (in $10^{-3}\,\mathrm{s^{-1}}$) at the station, **f** shows the profile of the sound velocity (in $\mathrm{m\,s^{-1}}$)

the form

$$\rho_r(z) = F(S_r, \theta_r, p_r(z)) \tag{2.158}$$

As the equation of state is nonlinear, it follows that both potential temperature θ_r and salinity S_r can only depend on z but not on x, y, t. For given profiles of $S_r(z)$ and $\theta_r(z)$, the vertical distribution of density and pressure can be calculated by combining (2.157) and the equation of state. By vertical differentiation of (2.158), and invoking (2.125) to (2.128), one obtains

$$\frac{1}{\rho_r}\frac{d\rho_r}{dz} = \gamma\frac{dS_r}{dz} - \alpha\frac{d\theta_r}{dz} - \frac{g}{c_r^2} \tag{2.159}$$

where $c_r(z)$ is the sound velocity for the hydrostatic state[21]. For typical oceanic conditions, the density changes due to the compressibility of sea water (last term in (2.159)) are approximately an order of magnitude larger than those due to the salinity/temperature variations, as demonstrated in Figure 2.17 where profiles of temperature, salinity and appropriate density variables are shown for a CTD cast near Bermuda in the subtropical North Atlantic.

As shown in the box on p. 74, the density variations in the ocean do not exceed 2–3%. Inserting (B14.1) into the hydrostatic balance (2.157), we obtain the depth

14. Scale Depth

According to (2.159), the density profile can be calculated in a situation without the stratification of temperature and salinity so that $S_r, \theta_r = \text{const}$. It is convenient to introduce instead of the sound velocity the *scale depth* D by

$$\mathrm{D}(z) = c_r^2(z)/g$$

where $c_r = c_s(S_r, \theta_r, p_r(z))$ is the reference state of the sound speed. It follows from (2.159) that $d\rho_r/dz = -\rho_r/\mathrm{D}$ which has the solution

$$\rho_r(z) = \rho_r(0) \exp \int_z^0 \frac{1}{\mathrm{D}(z')} dz'$$

The scale depth increases with temperature and pressure and is almost independent of salinity. Typical oceanic values are around 210 km in the upper ocean and 240 km at $5{,}000$ m depth. Ignoring the weak dependency of the scale depth on z, we may write

$$\rho_r(z) = \rho_r(0)e^{-z/\mathrm{D}} \approx \rho_r(0)(1 - z/\mathrm{D}) \tag{B14.1}$$

The last approximation holds because the water depth is always much smaller than the scale depth. In the state of rest, the relative variations of density hence do not exceed $H/\mathrm{D} \sim 2{-}3\%$. Most oceanic densities range between $\rho \approx 1025{-}1050\,\mathrm{kg\,m^{-3}}$.
Note the analogy of the oceanic scale depth to the atmospheric *scale height* H_s which for an isothermal atmosphere of mean temperature T_m is given as $H_s = R_m T_c/g \approx 7.4$ km (R_m is the gas constant). Thus, while in the ocean the scale depth is much larger than the ocean depth, in the atmosphere the scale height is of similar magnitude as the height of the troposphere (see also Section 2.10).

---

[21] In the following, α, γ and κ are the modified coefficients (2.122)–(2.124). They are to be evaluated at the local values of θ_r, S_r and p_r.

dependence of the reference pressure

$$p_{\mathrm{r}}(z)=p_{\mathrm{r}}(0)+g\mathrm{D}\rho_{\mathrm{r}}(0)\left[\exp(-z/D)-1\right]\approx p_{\mathrm{r}}(0)-gz\rho_{\mathrm{r}}(0)\left(1-\frac{z}{2\mathrm{D}}\right) \tag{2.160}$$

which can be further approximated (to an accuracy of 2–3%) to the commonly used form

$$p_{\mathrm{r}}(z)\approx p_{\mathrm{r}}(0)-gz\rho_0 \tag{2.161}$$

where $\rho_0=\rho_{\mathrm{r}}(0)$. Hence the pressure of 1 bar corresponds approximately to a water column of 9.81 m (see also the box on p. 20).

2.9.2 Static Stability

The state of rest as described in the previous section constitutes an equilibrium state of the ocean. As we have seen, the stratification of temperature and salinity has only a minor influence on the overall density. Nevertheless, the stratification is fundamental for ocean dynamics as it controls the vertical stability.

Consider a fluid particle which is displaced *adiabatically* by a small amount ξ in the vertical direction from its level z to $z+\xi$. If the displacement is *upwards*, and the particle at its new position is denser than the surrounding fluid, the force of gravity will act to bring the particle back towards its previous level, and the stratification is said to be *stable*. If, on the other hand, the particle at its new position is less dense than the surrounding fluid, gravity will force the particle to continue moving upwards, and the stratification is said to be *unstable*. If, on the other hand, the displacement is *downwards*, the situation is reversed, and a less dense particle will return to its previous position.

Since the particle is displaced adiabatically, its salinity and potential temperature remain constant, and its density can only change due to the different pressure at the new position $z+\xi$. The pressure at the new position is $p_{\mathrm{r}}(z+\xi)\approx p_{\mathrm{r}}(z)-g\rho_{\mathrm{r}}\xi$. The density change $\Delta\rho_{\mathrm{ad}}$ of the particle hence follows from (2.158) as

$$\Delta\rho_{\mathrm{ad}}=\rho_{\mathrm{r}}\,\kappa\,\Delta p=-\frac{g\rho_{\mathrm{r}}}{c_{\mathrm{r}}^2}\xi \tag{2.162}$$

In the surrounding fluid, not only the pressure but also salinity and potential temperature change, and its density change $\Delta\rho_{\mathrm{env}}$ is hence given by

$$\Delta\rho_{\mathrm{env}}\approx\frac{\partial\rho_{\mathrm{r}}}{\partial z}\xi=\rho_{\mathrm{r}}\left(-\frac{g}{c_{\mathrm{r}}^2}-\alpha\frac{\partial\theta_{\mathrm{r}}}{\partial z}+\gamma\frac{\partial S_{\mathrm{r}}}{\partial z}\right)\xi$$

The density difference of the particle and the surrounding fluid at the new position is thus

$$\Delta\rho_{\mathrm{ad}}-\Delta\rho_{\mathrm{env}}=\left(\alpha\frac{\partial\theta_{\mathrm{r}}}{\partial z}-\gamma\frac{\partial S_{\mathrm{r}}}{\partial z}\right)\rho\xi \tag{2.163}$$

For $\xi>0$, stability requires that the particle is denser than its new environment. Therefore, the stratification is stable provided that $\Delta\rho_{\mathrm{ad}}>\Delta\rho_{\mathrm{env}}$, and unstable for

$\Delta\rho_{\mathrm{ad}} < \Delta\rho_{\mathrm{env}}$. Conversely, for negative ξ the stratification is stable for $\Delta\rho_{\mathrm{ad}} < \Delta\rho_{\mathrm{env}}$, and unstable for $\Delta\rho_{\mathrm{ad}} > \Delta\rho_{\mathrm{env}}$. In both cases, stability therefore requires that the sign of $\Delta\rho_{\mathrm{ad}} - \Delta\rho_{\mathrm{env}}$ is the same as the sign of ξ. From (2.163) it follows then that a stability criterion can be conveniently formulated in terms of

$$N^2 = g\left(\alpha\frac{\partial\theta_{\mathrm{r}}}{\partial z} - \gamma\frac{\partial S_{\mathrm{r}}}{\partial z}\right) \tag{2.164}$$

The stratification is stable if $N^2 > 0$, unstable if $N^2 < 0$, and neutrally stable if $N^2 = 0$. The variable N has the dimension of an inverse time, and is called BRUNT–VÄISÄLÄ[22]–frequency (also *stability* or *buoyancy* frequency). The momentum balance for the particle is given by $\rho\ddot{\xi} = g(\Delta\rho_{\mathrm{env}} - \Delta\rho_{\mathrm{ad}})$, or

$$\ddot{\xi} = -N^2\xi \quad \text{with the solution} \quad \xi = \xi_0 \mathrm{e}^{\mathrm{i}Nt}$$

For stable stratification, the particle undergoes *stability oscillations* with a period $2\pi/N$. In the ocean, the stability period varies approximately between 10 min (upper ocean) and 2 h (in the deep ocean).

15. Other Forms of the Brunt–Väisälä frequency

For an adiabatic displacement, the *in-situ* temperature is obviously not constant, and its changes are given by

$$(\Delta T)_{\mathrm{ad}} = \Gamma\Delta p = -g\rho\Gamma\xi \tag{B15.1}$$

with the lapse-rate Γ. The stability frequency can be expressed in terms of T and Γ as

$$N^2 = g\left[\tilde{\alpha}\left(\frac{\partial T}{\partial z} + g\rho\Gamma\right) - \tilde{\gamma}\frac{\partial S}{\partial z}\right] \tag{B15.2}$$

which is, of course, equivalent to the formulation (2.164). The tilde is used here to indicate the use of the conventional density equation $\rho = \rho(S, T, p)$ using the *in-situ* temperature instead of the conservative temperature as state variable. Another equivalent form for N^2 is given by

$$N^2 = -\frac{g}{\rho}\left[\frac{\partial\rho}{\partial z} - \left(\frac{\partial\rho}{\partial z}\right)_{\mathrm{ad}}\right] = -\frac{g}{\rho}\frac{\partial\rho}{\partial z} - \frac{g^2}{c_{\mathrm{s}}^2} \tag{B15.3}$$

2.10 * Some Differences to Atmospheric Thermodynamics

In this section, we briefly discuss some differences of the previous results of ocean thermodynamics to the atmospheric case, in particular with respect to thermodynamics. The lesson from atmospheric thermodynamics is illuminating since the phase transition between vapor and liquid water (or ice) occurs in principle in seawater in the form of the transition between freshwater and sea ice. Such phase transitions have been neglected for the oceanic case; however, they can easily be incorporated when necessary. The way to do so can be learned from the atmospheric thermodynamics.

---

[22] SIR DAVID BRUNT, *1886 at Staylittle/Wales, †1965, meteorologist. VILHO VÄISÄLÄ, *1889 in Kontiolahti, †1969 in Helsinki, meteorologist and physicist.

2.10.1 Differences in Thermodynamics

Besides phase transitions, which are of great importance for the circulation of the atmosphere, the thermodynamics of the atmosphere is interesting with respect to a further aspect. Some of the thermodynamic properties like the equation of state can be given for the atmospheric case in an (approximate) closed analytical form, since air can be treated approximately as an ideal gas. We briefly review the thermodynamics of the atmosphere in this section.

Atmospheric Fields

The air of the Earth's atmosphere is here considered simply as a mixture of dry air (basically oxygen and nitrogen) and water vapor with concentration q. The partial masses are described by

q	specific humidity	in kg water vapor/kg moist air
ρ_q	density of vapor	in kg water vapor/m^3
a	content of dry air	in kg dry air/kg moist air
ρ_a	density of dry air	in kg dry air/m^3
f	liquid water concentration	kg water/kg moist air

Particularly for the specification of the water vapor there are various other concepts such as the mixing ratio $r = q/(1-q)$. For the atmospheric partial masses we define the density variables

$$\rho_\mathrm{q} = q\rho\ , \quad \rho_\mathrm{a} = a\rho = (1-q)\rho\ , \quad \rho = \rho_\mathrm{q} + \rho_\mathrm{a}$$

The water vapor is highly variable in the atmosphere. As a total it accounts for about 0.25% of the mass of the atmosphere, but local concentrations range from a few parts per million by volume in the cold high regions of the Earth's atmosphere up to as much as 5% by volume in hot, humid air masses.

Partial Masses

Air is a mixture of dry air (78% $\mathrm{N_2}$, 21% $\mathrm{O_2}$, 1% Ar and further trace gases), water vapor, liquid water and ice. It is normally sufficient to consider only water vapor (concentration $m_\mathrm{q} = q$) and dry air (concentration $m_\mathrm{a} = 1-q$) explicitly, and regard liquid water and ice as external components. Formally, counting the number of relevant partial masses, the situation corresponds thus to that in the ocean, with specific humidity q replacing salinity S. The independent state variables of the atmospheric flow are thus q, T, p. The atmospheric equation of state is thus $\upsilon = \upsilon(q, T, p)$, and we obtain the condition

$$\frac{\partial \upsilon}{\partial q} = \upsilon_\mathrm{q} - \upsilon_\mathrm{a} \tag{2.165}$$

and corresponding relations for all additive variables. As before, $\upsilon = 1/\rho$.

First Law

The internal energy of moist air – i. e. the mixture of dry air and water vapor – contains thermal energy of dry air and water vapor. For the first law we may hence use an analogous expression to that in the ocean, given in (1.23). In addition, however, we have to account for energy transfer during phase transitions as the enthalpy H_f of liquid water differs from the enthalpy H_q of water vapor. Since we do not account for the enthalpy stored in liquid water, that process has to be considered as an external source/sink. If a net amount $\delta^{\mathrm{e-c}}q$ of water vapor is added (through evaporation of liquid water) to or removed (through condensation) from a fluid element, the energy necessary to vaporize the water is given by $-L_\mathrm{q}\delta^{\mathrm{e-c}}q$ (positive when condensation is dominant) where the enthalpy difference liquid water–water vapor $H_\mathrm{f} - H_\mathrm{q} = L_\mathrm{q}$ as defined in (2.85) is the *latent heat* of condensation. Hence the first law has the form

$$\mathrm{d}E = \delta Q - p\,\mathrm{d}\upsilon + \frac{\partial H}{\partial q}\mathrm{d}q - L_\mathrm{q}\delta^{\mathrm{e-c}}q \tag{2.166}$$

where δQ is the total energy exchange that is not connected to diffusive exchange of mass and to phase transitions.

Second Law

In the atmospheric case of the second law we have to include the difference of the entropies of vapor and liquid water if condensation/evaporation occurs. Hence we have from (1.26)

$$\mathrm{d}\eta = \frac{\delta Q}{T} + \frac{\partial \eta}{\partial q}\mathrm{d}q + \left(\eta_\mathrm{q} - \eta_\mathrm{f}\right)\delta^{\mathrm{e-c}}q \tag{2.167}$$

Equation of State

For an ideal gas, the state equation is a consequence of the empirical laws of BOYLE–MARIOTTE, GAY–LUSSAC, and AVOGADRO[23]. In retrospect of the corresponding laws and hypotheses, which bear their names, we can define a mole of a substance as the weight in grams of $N_\mathrm{A} = 6.022 \times 10^{23}$ of its molecules, N_A being the Avogadro number. For example, a mole of pure water is 18.016 g and the molar weight is $M = 18.016$ (dimensionless). For a mole of any (ideal) gas, the equation of state is $pV = R^\star T$ where V is the occupied volume and $R^\star = 8314.36\,\mathrm{J\,kmol^{-1}\,K^{-1}}$ is the universal gas constant. For n moles with mass $m = nM$ the volume increases by the factor n (keeping pressure and temperature constant) and the relation becomes

$$pV = nR^\star T = \frac{m}{M}R^\star T \quad \text{or} \quad \frac{p}{\rho} = \frac{R^\star}{M}T = RT$$

where $R = R^\star/M$ is now specific for the particular gas.

---

[23] ROBERT BOYLE, *1627 in Lismore, †1691 in London, physicist and chemist. One of the founders of the Royal Society of London. EDME MARIOTTE, *1620 in Dijon, †1684 in Paris, physicist. LORENZO ROMANO AMEDEO CARLO AVOGADRO, *1776 in Turin, †1858 in Turin, physicist.

We thus define the respective gas constants of water vapor and dry air, $R_{\mathrm{q}} = R^{\star}/M^{\star}_{\mathrm{q}}$ and $R_{\mathrm{a}} = R^{\star}/M^{\star}_{\mathrm{a}}$, with the molar weights $M^{\star}_{\mathrm{q}} = 18.016$ and $M^{\star}_{\mathrm{a}} = 28.966$. The equations of state of the individual constituents are then $p_{\mathrm{q}}/\rho_{\mathrm{q}} = R_{\mathrm{q}}T$ and $p_{\mathrm{a}}/\rho_{\mathrm{a}} = R_{\mathrm{a}}T$, where p_{q} and p_{a} are the respective partial pressures. The equation of state of the mixture of dry air and water vapor follows from the additivity of the partial pressures to obtain the total pressure, $p = p_{\mathrm{q}} + p_{\mathrm{a}}$ (DALTON's[24] law), and the additivity of partial mass or densities to the total density, $\rho = \rho_{\mathrm{q}} + \rho_{\mathrm{a}}$. It becomes

$$\rho(q,T,p) = \frac{1}{\upsilon(q,T,p)} = \frac{p}{R_{\mathrm{m}}T} \tag{2.168}$$

which is identical to the state equation for an ideal gas except for the q-dependence of the gas constant R_{m} of the mixture,

$$R_{\mathrm{m}} = R_{\mathrm{m}}(q) = R_{\mathrm{a}} + \left(R_{\mathrm{q}} - R_{\mathrm{a}}\right) q = R_{\mathrm{a}} \left[1 + \left(M^{\star}_{\mathrm{a}}/M^{\star}_{\mathrm{q}} - 1\right) q\right] \tag{2.169}$$

Gibbs Function

An advantage of atmospheric thermodynamics is that moist air can be treated as an ideal gas, at least away from conditions where phase transitions occur. Both constituents, dry air and vapor, behave approximately as ideal gases because intermolecular forces as well as the finite volume of molecules can be neglected, and both components behave as if no other component were present. For an ideal gas, the Gibbs function can be derived in closed form, based on the results of kinetic gas theory, as

$$G^{\mathrm{id}}(T,p) = RT\left(\ln p - \frac{1}{\kappa}\ln T\right) \tag{2.170}$$

where R is the gas constant and $\kappa = 1/(1+\nu/2)$, with ν being the number of degrees of freedom for the molecular kinetic energy ($\kappa_{\mathrm{a}} = 2/7$ for the two-atomic dry air and $\kappa_{\mathrm{q}} = 2/8$ for the three-atomic water vapor). The Gibbs function for moist air, by additivity, can then be constructed as

$$G(q,T,p) = (1-q)\, G^{\mathrm{id}}(T,p_{\mathrm{a}}) + q\, G^{\mathrm{id}}(T,p_{\mathrm{q}}) \tag{2.171}$$

We express the partial pressures in terms of the total pressure as $p_{\mathrm{a}} = p(1-q)R_{\mathrm{a}}/R_{\mathrm{m}}$ respectively $p_{\mathrm{q}} = pqR_{\mathrm{q}}/R_{\mathrm{m}}$ with $R_{\mathrm{m}} = (1-q)R_{\mathrm{a}} + qR_{\mathrm{q}}$, and introduce the mass-weighted ratio κ_{m} for the mixture according to

$$\kappa_{\mathrm{m}} = \kappa_{\mathrm{m}}(q) = \frac{R_{\mathrm{a}} + \left(R_{\mathrm{q}} - R_{\mathrm{a}}\right) q}{R_{\mathrm{a}}/\kappa_{\mathrm{a}} + \left(R_{\mathrm{q}}/\kappa_{\mathrm{q}} - R_{\mathrm{a}}/\kappa_{\mathrm{a}}\right) q} \tag{2.172}$$

This equation uses Dalton's law $p = p_{\mathrm{q}} + p_{\mathrm{a}}$ and the atmospheric equation of state. The general form (2.171) results in

$$G(q,T,p) = R_{\mathrm{m}}\, T\left(\ln p - \frac{1}{\kappa_{\mathrm{m}}}\ln T\right) - T\Delta_{\mathrm{m}} \tag{2.173}$$

---

[24] JOHN DALTON, *1766 in Eaglesfield/Cumberland, †1844 in Manchester, chemist, meteorologist and physicist.

The expression for the Gibbs function of humid air hence corresponds to the ideal gas form (2.170), except for the last term in (2.173) which is

$$\Delta_\mathrm{m} = \Delta_\mathrm{m}(q) = -qR_\mathrm{q} \ln\left(qR_\mathrm{q}\right) + R_\mathrm{m}(q) \ln R_\mathrm{m}(q) - (1-q)R_\mathrm{a} \ln\left[(1-q)R_\mathrm{a}\right] \tag{2.174}$$

It is always positive, $\Delta_\mathrm{m}(q) \geq 0$, and specifically $\Delta_\mathrm{m}(q) \approx -qR_\mathrm{q} \ln q$ for $q \ll 1$. The entropy of moist air follows through differentiation according to (1.49) as

$$\eta(q, T, p) = \frac{R_\mathrm{m}}{\kappa_\mathrm{m}} (1 + \ln T - \kappa_\mathrm{m} \ln p) + \Delta_\mathrm{m} \tag{2.175}$$

and corresponds to the entropy of an ideal gas with 'constants' $R_\mathrm{m}(q)$ and $\kappa_\mathrm{m}(q)$, except for $\Delta_\mathrm{m}(q)$ which is the additional entropy due to mixing of the two gases. Other thermodynamic potentials can likewise be derived through differentiation according to (1.49). In particular one obtains the enthalpy, internal energy and the 'chemical potential' as

$$H(q, T, p) = \frac{R_\mathrm{m}}{\kappa_\mathrm{m}} T \tag{2.176}$$

$$E(q, T, p) = \frac{R_\mathrm{m}}{\kappa_\mathrm{m}} (1 - \kappa_\mathrm{m}) T \tag{2.177}$$

$$\mu(q, T, p) = (R_\mathrm{q} - R_\mathrm{a}) T \ln p - \left(\frac{R_\mathrm{q}}{\kappa_\mathrm{q}} - \frac{R_\mathrm{a}}{\kappa_\mathrm{a}}\right) T \ln T - T \frac{\partial \Delta_\mathrm{m}}{\partial q} \tag{2.178}$$

For $q = 0$ one recovers the corresponding relations for dry air. For typical values of $q = O(0.01)$ in the atmosphere, $R_\mathrm{m}(q) \approx R_\mathrm{a}$ with an error of approximately 1%, and likewise $\kappa_\mathrm{m}(q) \approx \kappa_\mathrm{a} = 2/7$. Hence for many problems the effect of q on the thermodynamics can be ignored (except when phase transitions are involved where the above consideration of ideal gas behavior is not valid).

Specific Heat

The specific heat of moist air is evaluated from (1.57) and (1.58),

$$c_p = \frac{\partial H}{\partial T} = \frac{R_\mathrm{m}}{\kappa_\mathrm{m}} = \frac{7}{2} \left[R_\mathrm{a} + \left(\frac{8}{7} R_\mathrm{q} - R_\mathrm{a}\right) q\right] \tag{2.179}$$

$$c_v = \left(\frac{\partial E}{\partial T}\right)_\mathrm{v} = \frac{5}{2} \left[R_\mathrm{a} + \left(\frac{6}{5} R_\mathrm{q} - R_\mathrm{a}\right) q\right] \tag{2.180}$$

Both specific heats are strictly independent of temperature and pressure, as to be expected for a mixture of two ideal gases. Their dependence on specific humidity is very weak, and their ratio is to a very good approximation given as $c_p/c_v \approx 7/5$, i. e. its value for dry air.

2.10.2 Differences in Conservation Laws

The differences in the thermodynamics of the atmosphere with respect to the ocean have consequences for the conservation laws for water vapor (replacing salt conservation), energy and the temperature equations. These budgets are discussed in this section.

Water Vapor Balance

For the atmospheric counterpart, the water vapor q must satisfy a conservation law corresponding to (2.17), with a diffusive water vapor flux $\boldsymbol{J}_\mathrm{q}$ and a source term $C_\mathrm{q} = e - c$, accounting for the phase transition (evaporation and condensation), where e is the rate of evaporation and c the rate of condensation (both in $\mathrm{kg\,m^{-3}\,s^{-1}}$ of liquid water). Hence

$$\rho \frac{\mathrm{D}q}{\mathrm{D}t} = -\nabla \cdot \boldsymbol{J}_\mathrm{q} + C_\mathrm{q} \tag{2.181}$$

The diffusive flux $\boldsymbol{J}_\mathrm{q}$ of water vapor can be expressed in a similar form as outlined above for salt, with a diffusivity $\kappa_\mathrm{q} \approx 2.4 \times 10^{-5}\,\mathrm{m^2\,s^{-1}}$ (at 8°C).

Dry air satisfies a sourceless balance, and strictly speaking, the 'total' mass with density $\rho = \rho_\mathrm{a} + \rho_\mathrm{q}$ would then not be conserved: there is a loss of moist air to the liquid or ice phase. This loss is small and generally ignored in the total mass balance. Strictly, however, two more partial mass compartments must be included, the liquid and the frozen phase of water. Equivalently, an ice phase could be considered for seawater with corresponding source terms in the balances of freshwater and ice.

Energy Changes by Phase Transitions

Writing $C_\mathrm{q} = e - c$ for the difference between evaporations and condensation (i.e. the source of vapor) per volume and time, we obtain $-L_\mathrm{q} C_\mathrm{q}$ as an additional source term in the enthalpy budget as compared to the oceanic case (2.76) (positive if condensation prevails, i.e. $C_\mathrm{q} < 0$). Similar contributions arise from freezing of melting processes.

Latent Energy

The conversion term $-L_\mathrm{q} C_\mathrm{q}$, representing the energy exchange due to phase transitions, has no equivalent in the ocean energetics because ice freezing/melting occurs only very localized at the surface and is usually treated in the boundary conditions and not in the interior balances. It is, however, quite important in the atmosphere. With a minor assumption (taking L_q as constant, i.e. neglecting its temperature dependence), this term can be brought to a flux form as well. The *latent* energy is defined as $E^\mathrm{lat} = L_\mathrm{q} q$, and is that part of internal energy of the liquid phase which is lost in the phase transition and gained by the internal energy of the vapor. From the conservation of water vapor we then obtain

$$\rho \frac{\mathrm{D}E^\mathrm{lat}}{\mathrm{D}t} \approx \rho L_\mathrm{q} \frac{\mathrm{D}q}{\mathrm{D}t} = -L_\mathrm{q} \nabla \cdot \boldsymbol{J}_\mathrm{q} + L_\mathrm{q} C_\mathrm{q} \approx -\nabla \cdot (L_\mathrm{q} \boldsymbol{J}_\mathrm{q}) + L_\mathrm{q} C_\mathrm{q} \tag{2.182}$$

where $L_\mathrm{q} \boldsymbol{J}_\mathrm{v}$ is defined as *latent heat flux*. The combined energy $E^\mathrm{tot} + E^\mathrm{lat}$ is then governed by

$$\rho \frac{\mathrm{D}}{\mathrm{D}t} \left(E^\mathrm{tot} + E^\mathrm{lat} \right) \approx -\nabla \cdot \left(\boldsymbol{J}^\mathrm{tot} + L_\mathrm{q} \boldsymbol{J}_\mathrm{q} \right) + \rho \frac{\partial \Phi_\mathrm{tide}}{\partial t} \tag{2.183}$$

which reveals a total energy flux $\boldsymbol{J}^\mathrm{tot} + L_\mathrm{q} \boldsymbol{J}_\mathrm{q}$ for this energy variable. Only the external tidal forces and the radiative flux at the outer rim of the atmosphere remain to change this form of energy.

Temperature Equation

The temperature equation for the atmosphere is derived in complete analogy, starting with (2.76) and incorporating the energy transfer expressions for phase transitions and evaporation/condensation terms in the conservation of water vapor. Again, the dissipation of energy and the 'chemical' terms are small and one finds

$$Q_{\mathrm{T}} \approx -\nabla \cdot (\boldsymbol{J}_{\mathrm{T}} + \boldsymbol{J}_{\mathrm{rad}}) - L_{\mathrm{q}} C_{\mathrm{q}} \tag{2.184}$$

Potential Temperature

Using the form of entropy (2.175) for a moist atmosphere, we obtain an implicit definition of the potential temperature from the entropy relation (2.113), i. e.

$$\begin{aligned} \eta(q,T,p) &= R\left[\frac{1}{\kappa_{\mathrm{m}}}\left(1+\ln\frac{T}{T_0}\right) - \ln\frac{p}{p_0}\right] + \xi(q) \\ &= R\left[\frac{1}{\kappa_{\mathrm{m}}}\left(1+\ln\frac{\theta}{T_0}\right)\right] + \xi(q) \end{aligned}$$

where $\xi(q)$ is easily computed from $\Delta_{\mathrm{m}}(q)$. Evaluating for θ yields

$$\theta = T\left(\frac{p_0}{p}\right)^{\kappa_{\mathrm{m}}} \tag{2.185}$$

which is in accordance with the considerations of the ideal gas physics as discussed above. The only difference is that here $\kappa_{\mathrm{m}} = \kappa_{\mathrm{m}}(q)$ refers to the mixture of dry air and water vapor. For the definition of $\kappa_{\mathrm{m}}(q)$ we refer to (2.172). In fact, the dependence on q is weak. In contrast to the ocean case, however, θ and T differ quite substantially in the atmosphere. Following (2.179), the specific heat is almost constant, $c_p^0 = c_p$. Neglecting the 'chemical' terms here as well we hence find

$$\rho c_p \frac{\mathrm{D}\theta}{\mathrm{D}t} = -\frac{\theta}{T}\left(\nabla \cdot \boldsymbol{J}_{\mathrm{H}} + L_{\mathrm{q}} C_{\mathrm{q}}\right) \tag{2.186}$$

This form is usually applied. The equivalent form

$$\rho \frac{\mathrm{D}\theta}{\mathrm{D}t} = -\nabla \cdot \left(\frac{\theta}{c_p T}\boldsymbol{J}_{\mathrm{H}}\right) + \frac{\boldsymbol{J}_{\mathrm{H}}}{c_p} \cdot \nabla \frac{\theta}{T} - \frac{\theta}{c_p T} L_{\mathrm{q}} C_{\mathrm{q}} \tag{2.187}$$

oriented at the general form of the conservation equation, is less customary.

The potential temperature is not conserved in the moist atmosphere, not only because of the evaporation/precipitation term, but also because θ/T is significantly not constant. The use of θ instead of T, therefore, has less advantages than for the ocean. In atmospheric models one thus usually applies the *in-situ* temperature equation (2.101) with the source term (2.184).

Potential Density

For a mixture of ideal gases, as we treat the atmosphere, the use of θ instead of T as a variable in the equation of state is straightforward. By combination of (2.185) and (2.168) one obtains

$$\rho(q,T,p) = \frac{p}{R_{\mathrm{m}}(q)\theta}\left(\frac{p_0}{p}\right)^{\kappa_{\mathrm{m}}(q)}$$

Hesselberg Average

For the atmosphere, the mass-weighted average (see Section 2.8.2) has the advantage of exact validity of the ideal gas law for the averaged fields: indeed, from the equation $p = \rho RT$ we obtain $\overline{p} = \rho_\mathrm{m} RT_\mathrm{m}$ with the mass-averaged temperature $T_\mathrm{m} = \overline{\rho T}/\rho_\mathrm{m}$. This can be used for each partial mass (dry air and water vapor) and then transferred to the equation of state.

State of Rest

It is also interesting to compare the oceanic state of rest with the atmospheric one. With the state equation (2.168), the hydrostatic equation (2.157) takes the form

$$\frac{1}{p}\frac{\mathrm{d}p}{\mathrm{d}z} = -\frac{g}{R_\mathrm{m} T} \tag{2.188}$$

which can be used to determine the pressure distribution for a given temperature profile. A simple form is obtained when the atmosphere is assumed isothermal i. e. $T \approx T_\mathrm{c} = 250\,\mathrm{K}$ (the error is $< 15\,\%$). It follows that

$$p(z) = p(0)\exp(-z/H_\mathrm{s}) \tag{2.189}$$

where $H_\mathrm{s} = R_\mathrm{m} T_\mathrm{c}/g \approx 7.4\,\mathrm{km}$ is the atmospheric *scale height*. In this approximation, the density decreases exponentially with increasing height but more substantially than the oceanic counterpart because H_s is of the order of the tropospheric height.

2.11 Vorticity

We have seen in the previous sections that salinity and potential temperature are materially conserved under the restrictive conditions of an adiabatic flow. The quantities keep their value in a moving fluid parcel and the corresponding conservation laws are trivially solved. For momentum such a situation does not occur, even when all external forces vanish and adiabatic conditions apply. The reason is the ubiquitous presence of the pressure force. However, there are materially conserved quantities which relate to momentum.

Forming the curl of the momentum balance, gradient forces cancel out and the balance of vorticity is derived. The concept of vorticity leads to associated quantities which are materially conserved – or very approximately so – in many nontrivial and interesting dynamical situations. In Section 1.1.2 we have shown the similarity of vorticity and angular momentum, and indeed there is some analogy between the vorticity-related conservation theorems introduced in this section and the conservation of angular momentum of a solid body. The name ‘vorticity’ was introduced by LAMB[25] in 1916.

The vorticity theorems apply to locally defined quantities relating the vorticity vector to integral properties of closed material loops of fluid. In Cartesian coordinates

---

[25] SIR HORACE LAMB, *1849 in Stockport, †1934 in Cambridge, fluid dynamicist.

the vorticity vector is written as (cf. (1.12))

$$\boldsymbol{\omega} = (\omega_1, \omega_2, \omega_3) = \left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3} \,, \quad \frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1} \,, \quad \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2} \right) \tag{2.190}$$

In Section 1.1.2 we have related the vorticity vector $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$ to the angular velocity $(1/2)\boldsymbol{\omega} \cdot \boldsymbol{n}$ of an infinitesimally small disk with normal vector $\boldsymbol{n}$. In fact, for a rigid body rotation where $\boldsymbol{u} = \boldsymbol{\omega}_0 \times \boldsymbol{r}$ with angular velocity $\boldsymbol{\omega}_0$ we find $\boldsymbol{\omega} = 2\boldsymbol{\omega}_0$. Here, parcel trajectories are clearly curved, but this property is not mandatory for nonzero vorticity: in a simple unidirectional shear flow $\boldsymbol{u} = (0, \kappa x, 0)$ the vorticity is nonzero, $\boldsymbol{\omega} = (0, 0, \kappa)$ (see Figure 2.18 for the shape of the shear flow).

The *circulation* C around a closed curve Γ in the fluid is defined by the line integral of the tangential velocity vector,

$$C = \oint_{\Gamma} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s} \tag{2.191}$$

The closed curve Γ can be defined at any initial time anywhere in the fluid. Following the fluid motion of the individual fluid parcels on Γ for subsequent times, $C = C(t)$ becomes a material property of the parcel group $\Gamma = \Gamma(t)$. The circulation around a rectangle of dimension $L \times B$ with above specified velocity field is $C = \kappa L B$, in this case equal to vorticity times the enclosed area. In the course of time the rectangle will be deformed to a parallelogram, but the area and the circulation will remain constant. These properties are, of course, more general, as will be investigated in this chapter.

On the other hand, curved trajectories do not imply nonzero vorticity, as can be seen in the two-dimensional point vortex (see right panel of Figure 2.18) described by the stream function (in cylindrical coordinates) $\Psi(r, \varphi) = -(\kappa/2\pi) \ln r$. Here, the velocity is tangential, $u^{\varphi} = \kappa/(2\pi r)$, and the vorticity $\boldsymbol{\omega} = \boldsymbol{k}(1/r)\partial(r u^{\varphi})/\partial r$ is vertical to the two-dimensional motion. It vanishes everywhere except in the center where it is infinite. The circulation around any loop enclosing the center equals κ, and it vanishes for loops that do not contain the center. Again, $C(t)$ is conserved since material loops cannot cross the center. In these examples we find, with $\kappa > 0$, a counterclockwise local rotation ($\omega_3 > 0$) about the vertical. A vorticity and the corresponding flow is called *cyclonic* (cum sole) if the rotation is the same as the Earth rotation (i. e. positive in the northern hemisphere). An *anticyclone* rotates the other way.

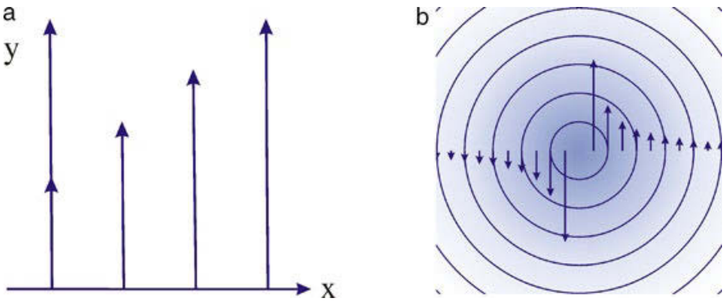

Fig. 2.18 Sketch of linear shear flow with vorticity (**a**) and a point vortex with zero vorticity (**b**)

2.11.1 Kinematical Properties

There is a number of simple theorems which relate the circulation C of a loop Γ at a certain instant of time to other physical quantities related to the same loop or to other loops. The theorems derived below require that at any instant Γ is reducible to a point by subsequent shrinking, without leaving the fluid. Thus there always exists a simply connected surface A in the fluid with the rim Γ. Then, for any such A we may use Stokes' theorem (see Appendix A) to express C in terms of the vorticity vector,

$$C = \oint_{\Gamma} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s} = \int_{A} \mathrm{d}A\boldsymbol{n} \cdot \boldsymbol{\omega} \tag{2.192}$$

The latter integral is referred to as the *vortex flux* through the area A. Notice that the vortex flux is a kinematic concept; it should not be confused with the dynamical concept of the flux of vorticity (by advection or diffusion) considered later in this chapter.

The vortex flux has the same value for any surface attached to Γ. This follows immediately from (2.192), but it also may be inferred from Gauss' theorem applied to the vorticity vector. Since it is nondivergent by construction, i. e. $\nabla \cdot \boldsymbol{\omega} = 0$, there are no sources of vorticity inside the fluid, and we find

$$\int_{V} \mathrm{d}^3 x \nabla \cdot \boldsymbol{\omega} = \oint_{A} \mathrm{d}A\boldsymbol{n} \cdot \boldsymbol{\omega} = 0 \tag{2.193}$$

where A is now the surface surrounding the entire volume V. Hence circulation and vortex flux through a loop are synonymous concepts.

The quantities $\boldsymbol{\omega}$ and C use the relative velocity $\boldsymbol{u}$ referenced to the rotating Earth whereas the absolute velocity $\boldsymbol{u}_{\mathrm{a}} = \boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r}$ is seldom of interest in dynamical considerations. It is for some purposes useful to consider the vorticity and circulation referred to an inertial system. We thus define

$$\boldsymbol{\omega}_{\mathrm{a}} = \nabla \times \boldsymbol{u}_{\mathrm{a}} = \nabla \times (\boldsymbol{u} + \boldsymbol{\Omega} \times \boldsymbol{r}) = \boldsymbol{\omega} + 2\boldsymbol{\Omega}$$

and

$$C_{\mathrm{a}} = C + \oint_{\Gamma} \boldsymbol{\Omega} \times \boldsymbol{r} \cdot \mathrm{d}\boldsymbol{s}$$

correspondingly. We refer to $\boldsymbol{\omega}_{\mathrm{a}}$ and $\boldsymbol{\omega}$ as absolute and relative vorticity vectors, respectively. As $2\boldsymbol{\Omega}$ is the vorticity of the planetary rotation, it is referred to as planetary vorticity. In the same way we speak of absolute and relative circulation.

There is a simple measure which classifies the importance of the relative and planetary vorticity in the sum, the absolute vorticity. Consider the vertical component of the vorticity vectors,

$$\omega_{\mathrm{a}}^{(z)} = \eta + f \ , \quad \eta = \boldsymbol{k} \cdot \boldsymbol{\omega} = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \ , \quad f = 2\boldsymbol{k} \cdot \boldsymbol{\Omega} = 2\Omega \sin\varphi$$

where φ is the geographic latitude and the vertical $\boldsymbol{k}$ is defined as the radial direction. Notice that with $\eta = O(U/L)$ the ratio of the relative and planetary contribution to the absolute vorticity $\omega_{\mathrm{a}}^{(z)}$,

$$\frac{\eta}{f} = O\left(\frac{U}{fL}\right) = O(\mathrm{Ro})$$

is found to be of order of the ROSSBY[26]-number $\mathrm{Ro} = U/(2\Omega L)$. In a large-scale oceanic flow we have $U = 0.1\,\mathrm{m\,s^{-1}}$, $L = 10^3\,\mathrm{km}$ so that with $f = 10^{-4}\,\mathrm{s^{-1}}$ (corresponding to 30° latitude) we find $\mathrm{Ro} = 10^{-3}$. In the atmosphere larger values occur ($\eta \sim 10^{-5}\,\mathrm{s^{-1}}$, $\mathrm{Ro} \sim 10^{-2}\ldots-10^{-1}$). In any case, in a large-scale flow the relative vorticity is small compared to the planetary vorticity, and thus the sign of the absolute vorticity corresponds to the sign of the Coriolis parameter f.

The above kinematic properties of relative vorticity and circulation are carried over to the absolute quantities since $\nabla\cdot\boldsymbol{\omega}_{\mathrm{a}} = 0$ as well. These properties are cast into a similar phrasing by use of the concepts of the *vortex line or filament* and the *vortex tube*. A vortex line is a continuous line of fluid parcels which is everywhere tangent to the instantaneous local vorticity vector. A vortex tube, sketched in Figure 2.19, is the ensemble of vortex lines passing through a given loop Γ at a certain instant. These groups of fluid parcels may be defined for the absolute or relative vorticity vectors but, of course, the ensembles differ. Notice, moreover, that neither vortex lines nor vortex tubes are material in general. It is only under rather restricted conditions that a vortex line remains intact in the course of the fluid motion. This will be investigated in the next section.

It is clear, however, that the vortex flux is the same through any intersection of a vortex tube (first Helmholtz theorem), i. e.

$$C_{\mathrm{a}} = \int\limits_{A_1} \mathrm{d}A\boldsymbol{n}_1\cdot\boldsymbol{\omega}_{\mathrm{a}} = \int\limits_{A_2} \mathrm{d}A\boldsymbol{n}_2\cdot\boldsymbol{\omega}_{\mathrm{a}} \tag{2.194}$$

where A_i and $\boldsymbol{n}_i$ are two intersecting surfaces with the corresponding normal vectors oriented towards the same side of the tube. The spatial constancy of C_{a} in a given tube follows directly from (2.193) since the contribution from the tube mantle is

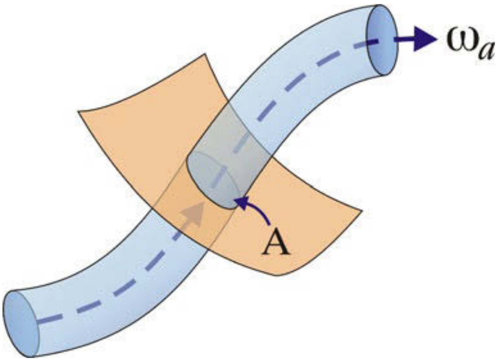

Fig. 2.19 Illustration of a vortex tube. The configuration is restricted to have the form of a simple 'tube', i. e. cross sections shall be simple closed curves. Redrawn after Vallis (2006)

---

[26] CARL-GUSTAV ARVID ROSSBY, *1898 in Stockholm, †1957 in Stockholm, meteorologist. The dimensionless number named after Rossby was used earlier by the Russian fluid dynamicist I.A. Kibel.

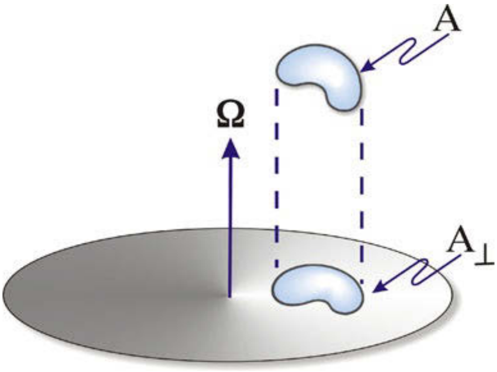

Fig. 2.20 Schematic showing the projection onto the equatorial plane. Redrawn after Vallis (2006)

identical to zero. This theorem may as well be formulated for the relative vortex flux (with a different tube, however). An immediate consequence of the constant strength of the vortex flux is that vortex lines and tubes cannot end somewhere in the fluid; they must close on themselves, extend to infinity or end at boundaries.

The flux of planetary vorticity

$$\int_A \mathrm{d}A 2\boldsymbol{\Omega} \cdot \boldsymbol{n} = 2\boldsymbol{\Omega} \cdot \int_A \mathrm{d}A\boldsymbol{n}$$

through an intersection A of a vortex tube may expressed in terms of the area $A_\perp$ which is the projection of A onto the plane perpendicular to $\boldsymbol{\Omega}$ which is the equatorial plane (see Figure 2.20). We thus may express the absolute vortex flux or circulation in the form

$$C_\mathrm{a} = C + 2\Omega A_\perp$$

Notice that while C_a is spatially constant along the absolute vortex tube, the individual contributions C and $2\Omega A_\perp$ vary in general. Remember also that all terms may vary in the course of time.

2.11.2 Dynamical Properties

The kinematic theorems discussed above are quite general but not very powerful since they do not say anything about the temporal evolution. The vorticity and the circulation (following a material loop) change in the course of time. There is a sequence of vorticity and circulation theorems due to HELMHOLTZ (1858), KELVIN[27] (Thomson, W. (Lord Kelvin), 1869) and BJERKNES[28] (Bjerknes, 1898). For a historical perspective see Thorpe et al. (2003). The physics revealed in these theorems is reflected in the complete evolution equation of the vorticity vector but is not immediately evident from the underlying balance of momentum.

---

[27] WILLIAM THOMSON, LORD KELVIN, *1824 in Belfast, †1907 in Netherhall near Largs, mathematician and physicist.

[28] VILHELM BJERKNES, *1862 in Kristiania, †1951 in Oslo, physicist and hydrodynamicist.

Barotropic Fluid

Helmholtz' dynamical vorticity theorem applies to a homogeneous fluid (i. e. constant density ρ) and a strictly nondivergent motion or, more general, to a flow in which pressure surfaces and density surfaces coincide. This may be an inherent property of the fluid, as e. g. for an equation of state $\rho = \rho(p)$, or a property of the flow in a certain area of the fluid. This latter case is called a *barotropic* state of flow. If, moreover, the external forces are derivable from a potential (implying that the fluid is frictionless), the acceleration is the gradient of a potential,

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} + 2\boldsymbol{\Omega} \times \boldsymbol{u} = -\nabla\left(\frac{p}{\rho} + \Phi\right) \tag{2.195}$$

and – after some mathematical manipulations outlined in more detail later in a more general setting (page 91) – the curl of the momentum balance becomes

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_{\mathrm{a}}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}_{\mathrm{a}}}{\rho} \cdot \nabla\right)\boldsymbol{u} \tag{2.196}$$

This is the Helmholtz equation, here written for the absolute vorticity $\boldsymbol{\omega}_{\mathrm{a}}$. In this specific set-up the equation is identical to the evolution equation of a material line element $\boldsymbol{\ell}$ derived in (1.7) and repeated here,

$$\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{\ell} = (\boldsymbol{\ell} \cdot \nabla)\boldsymbol{u} \tag{2.197}$$

If we choose $\boldsymbol{\ell}$ parallel to a vortex filament at a certain position at some initial time, we conclude from (2.196) and (2.197) that this vortex line coincides with the material line element defined by $\boldsymbol{\ell}$ at all later times (second Helmholtz theorem). Evidently, vortex tubes are material, and the strength of a vortex tube is an integral of the motion (third Helmholtz theorem), i. e. the circulation and vortex flux are not only constant along a specified vortex tube but also constant in time. Notice that (2.196) also implies that a line element which is free of vorticity will never acquire vorticity (Lagrange–Cauchy theorem). The generalization of these Helmholtz' theorems is considered further below.

It is worth mentioning that (2.196) (or (2.197)) may be integrated exactly as shown first by Cauchy in 1815. Using the Lagrangian framework where $\boldsymbol{X}(\boldsymbol{a}, t)$ denotes the particle position with initial value $\boldsymbol{a}$ and $\nabla_{\boldsymbol{a}}\boldsymbol{X}$ is the JACOBIAN[29], it is easily verified that

$$\boldsymbol{\ell} = \boldsymbol{c} \cdot \nabla_{\boldsymbol{a}}\boldsymbol{X}$$

with a materially conserved vector $\boldsymbol{c} = \boldsymbol{c}(\boldsymbol{a})$ is the solution of (2.197) for which $\boldsymbol{\ell}(t = 0) = \boldsymbol{c}$. Hence $\boldsymbol{\omega}_{\mathrm{a}} = (\rho/\rho_0)\boldsymbol{\omega}_0 \cdot \nabla_{\boldsymbol{a}}\boldsymbol{X}$ solves the Helmholtz equation (2.196). Here $\boldsymbol{\omega}_0$ and ρ_0 are the initial values.

Circulation Theorems

The temporal change

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \frac{\mathrm{D}}{\mathrm{D}t}\oint_{\Gamma} \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s} = \oint_{\Gamma}\left(\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \cdot \mathrm{d}\boldsymbol{s} + \boldsymbol{u} \cdot \frac{\mathrm{D}\mathrm{d}\boldsymbol{s}}{\mathrm{D}t}\right) \tag{2.198}$$

---

[29] CARL JACOBI, *1804 in Potsdam, †1851 in Berlin, mathematician.

of the circulation around a material loop is readily evaluated. Since d$\boldsymbol{s}$ is a material line element, we may use (2.197) to convert the last term on the right hand side of (2.198) as follows,

$$\oint_{\Gamma} \boldsymbol{u} \cdot \frac{\mathrm{D}\mathrm{d}\boldsymbol{s}}{\mathrm{D}t} = \oint_{\Gamma} \boldsymbol{u} \cdot (\mathrm{d}\boldsymbol{s} \cdot \nabla)\, \boldsymbol{u} = \oint_{\Gamma} \mathrm{d}\boldsymbol{s} \cdot \nabla \left(\frac{1}{2} \boldsymbol{u}^2 \right) = 0$$

The change of the circulation is thus determined by the line integral of the acceleration,

$$\frac{\mathrm{D}C}{\mathrm{D}t} = \oint_{\Gamma} \mathrm{d}\boldsymbol{s} \cdot \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} \tag{2.199}$$

and if the acceleration is derivable from a potential as under conditions leading to (2.195), we find that C remains constant; the circulation is materially conserved for frictionless, barotropic flow in case of no rotation (Kelvin's theorem).

In general, however, we have a conservation of momentum of the form (2.47), written here as

$$\rho \frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -2\rho \boldsymbol{\Omega} \times \boldsymbol{u} - \nabla p - \rho \nabla \Phi + \boldsymbol{\mathcal{F}} \tag{2.200}$$

where the effect of friction is contained in the force term $\boldsymbol{\mathcal{F}}$. Then

$$\frac{\mathrm{D}C}{\mathrm{D}t} = -\oint_{\Gamma} \mathrm{d}\boldsymbol{s} \cdot \left(2\boldsymbol{\Omega} \times \boldsymbol{u} + \frac{1}{\rho} \nabla p + \nabla \Phi - \frac{1}{\rho} \boldsymbol{\mathcal{F}} \right) \tag{2.201}$$

There are three forces that may lead to a change in the circulation (notice that gravity does not contribute). The Coriolis term may be expressed in terms of the rate of change of the area $A_{\perp}$ introduced above,

$$-\oint_{\Gamma} \mathrm{d}\boldsymbol{s} \cdot (2\boldsymbol{\Omega} \times \boldsymbol{u}) = -2\Omega \frac{\mathrm{D}A_{\perp}}{\mathrm{D}t} \tag{2.202}$$

The pressure term is conveniently rewritten in either of the following forms: since $\mathrm{d}\boldsymbol{s} \cdot \nabla p = \mathrm{d}p$, we have

$$-\oint_{\Gamma} \mathrm{d}\boldsymbol{s} \cdot \frac{1}{\rho} \nabla p = -\oint_{\Gamma} \frac{\mathrm{d}p}{\rho} = -\oint_{\Gamma} \upsilon \mathrm{d}p = \oint_{\Gamma} p \mathrm{d}\upsilon \tag{2.203}$$

or, using Stokes' theorem, we find

$$-\oint_{\Gamma} \mathrm{d}\boldsymbol{s} \cdot \frac{1}{\rho} \nabla p = -\int_{A} \mathrm{d}A \boldsymbol{n} \cdot \nabla \times \left(\frac{\nabla p}{\rho} \right) = \int_{A} \mathrm{d}A \boldsymbol{n} \cdot \boldsymbol{B} \tag{2.204}$$

where

$$\boldsymbol{B} = \frac{1}{\rho^2} \nabla \rho \times \nabla p$$

is called the *baroclinic vector*, also solenoidal vector, of the flow field. Both formulations are, of course, equivalent. In (2.203) the pressure term is found equal to the expansion work which the ensemble of fluid parcels along the material loop have to perform against the pressure field. In (2.204) it appears that the ‘torque’ induced by the pressure forces, acting on mass distribution in the surface which is normal to the pressure force, spins up the circulation. It is easy to see that either term vanishes for a barotropic state where the surfaces of constant density and constant pressure coincide and the baroclinic vector $\boldsymbol{B}$ vanishes everywhere. The complementary, more general state with pressure changing on density surfaces is called a *baroclinic* state, sketched in Figure 2.21.

The rate of change of the circulation is then written as

$$\frac{\mathrm{D}C}{\mathrm{D}t} + 2\Omega\frac{\mathrm{D}A_\perp}{\mathrm{D}t} = \frac{\mathrm{D}C_\mathrm{a}}{\mathrm{D}t} = \oint_\Gamma p\,\mathrm{d}\upsilon + \oint_\Gamma \mathrm{d}\boldsymbol{s}\cdot\frac{\mathcal{F}}{\rho} \qquad (2.205)$$

This is the famous circulation theorem of Bjerknes (1898) which shows that the absolute circulation is only changed by the baroclinic torque and friction.

There are various other equivalent forms of the baroclinic torque. As an example, with the first law of thermodynamics, (1.16), in the form $-p\mathrm{d}\upsilon = \mathrm{d}E - T\mathrm{d}\eta$, we have

$$-\oint_\Gamma \upsilon\,\mathrm{d}p = \oint_\Gamma T\,\mathrm{d}\eta = \int_A \mathrm{d}A\boldsymbol{\nabla}T\times\boldsymbol{\nabla}\eta \qquad (2.206)$$

A perfect (ideal and frictionless) gas where $\eta = \eta(T)$ would thus follow Helmholtz theorems. Plotting the loop Γ in the (υ, p)-plane, the torque is found equal to the surrounded area (which evidently contracts to a curve for a barotropic state). Plotting in this plane grid lines of $\upsilon = n\Delta\upsilon$, $p = m\Delta p$ with $n, m = 0, \pm1, \pm2\ldots$, we find that the torque is evaluated as the number of elementary (υ, p)-cells times the area $\Delta\upsilon\Delta p$. The grid and cells appear and may be counted as well on any surface A spanned by Γ. Corresponding cells may be defined in the (T, η)-plane.

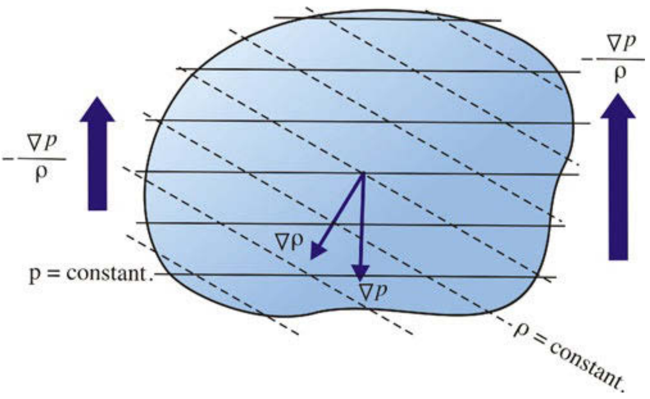

Fig. 2.21 Baroclinic conditions and direction of gradients. Whereas $\boldsymbol{\nabla}\rho$ and $\boldsymbol{\nabla}p$ are constant in this example, the vector $-(1/\rho)\boldsymbol{\nabla}p$ is increasing from left to right

Vorticity Equation

The local form of the vorticity balance is derived by applying the curl to the momentum balance. This operation is simplified by using the vector identities

$$\boldsymbol{\omega} \times \boldsymbol{u} = (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} - \nabla \boldsymbol{u}^2/2$$
$$\nabla \times (\boldsymbol{\omega} \times \boldsymbol{u}) = \boldsymbol{\omega}(\nabla \cdot \boldsymbol{u}) + (\boldsymbol{u} \cdot \nabla)\boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\boldsymbol{u}$$

The first uses $\boldsymbol{\omega} = \nabla \times \boldsymbol{u}$, the second $\nabla \cdot \boldsymbol{\omega} = 0$. The momentum balance thus takes the equivalent form

$$\frac{\partial \boldsymbol{u}}{\partial t} + \boldsymbol{\omega}_{\mathrm{a}} \times \boldsymbol{u} = -\nabla\left(\frac{1}{2}\boldsymbol{u}^2 + \Phi\right) - \frac{1}{\rho}\nabla p + \frac{\mathcal{F}}{\rho} \tag{2.207}$$

and the vorticity balance becomes

$$\frac{\mathrm{D}\boldsymbol{\omega}_{\mathrm{a}}}{\mathrm{D}t} = (\boldsymbol{\omega}_{\mathrm{a}} \cdot \nabla)\boldsymbol{u} - \boldsymbol{\omega}_{\mathrm{a}}(\nabla \cdot \boldsymbol{u}) + \boldsymbol{B} + \nabla \times \frac{\mathcal{F}}{\rho} \tag{2.208}$$

For barotropic steady flows, (2.208) leads to the TAYLOR–PROUDMAN[30] theorem discussed in the box on p. 91.

The divergence term (second on the right-hand side of (2.208)) describes changes of vorticity which arise when the volume and thus density of a parcel changes. Thus,

16. Taylor–Proudman Theorem

If the flow is frictionless, barotropic, and steady, the vorticity equation (2.208) takes the simple form

$$\boldsymbol{\nabla} \times (\boldsymbol{\omega}_{\mathrm{a}} \times \boldsymbol{u}) = \boldsymbol{\nabla} \times [(\boldsymbol{\omega} + 2\boldsymbol{\Omega}) \times \boldsymbol{u}] = 0$$

and if, in addition, the flow is slow, in the sense that the relative vorticity is small compared to the planetary part, we find

$$\boldsymbol{\Omega}(\boldsymbol{\nabla} \cdot \boldsymbol{u}) - (\boldsymbol{\Omega} \cdot \boldsymbol{\nabla})\boldsymbol{u} = 0$$

Evaluating this in a Cartesian coordinate system where $\boldsymbol{\Omega} = (0, 0, \Omega)$ it is found that

$$\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0\,, \quad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{B16.1}$$

In a barotropic steady, slow, and frictionless motion, the normal velocity thus cannot vary in the direction of the planetary rotation vector, which is the celebrated Taylor–Proudman theorem. Fluid parcels lying on a line which is parallel to $\boldsymbol{\Omega}$ will always remain in such a state. In addition, the normal velocity vector (u, v) is nondivergent. The fluid moves in a column-like way and we speak of Taylor columns. It is customary to consider the first condition in (B16.1) as a definition or at least a necessary prerequisite of a barotropic flow, though it is a mere consequence in a rather restrictive situation.

Notice that (B16.1) does not imply a two-dimensional flow since the vertical velocity is not constrained at all. If, however, the three-dimensional velocity is nondivergent, $\boldsymbol{\nabla} \cdot \boldsymbol{u} = 0$, this requires a vertically constant w, and fluid parcels oriented along $\boldsymbol{\Omega}$ cannot change their distance. Still, $w = w(x, y)$ may vary laterally. An example is the barotropic motion over a fast rotating terrain with topography.

---

[30] GEOFFREY INGRAM TAYLOR, *1886 in St. John's Wood, †1975 in Cambridge, physicist. JOSEPH PROUDMAN, *1888 in Unsworth near Bury (Lancashire), †1975 in Fordingbridge, Hampshire, mathematician and oceanographer.

if the flow is for instance divergent ($\nabla \cdot \boldsymbol{u} > 0$), the volume expands, the density decreases, and mass is moved away from the center of mass. Consequently the local rate of rotation must diminish. In general, this effect is small in oceanic and even atmospheric flows. It is anyhow an easy matter to eliminate the divergence term by use of the mass conservation, which casts the balance of vorticity into the form of BELTRAMI'S[31] equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_{\mathrm{a}}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}_{\mathrm{a}}}{\rho} \cdot \nabla\right)\boldsymbol{u} + \frac{1}{\rho}\left(\boldsymbol{B} + \nabla \times \frac{1}{\rho}\mathcal{F}\right) \tag{2.209}$$

We are already familiar with the terms which act here as sources of vorticity, and for $\boldsymbol{B} = 0$ and $\mathcal{F} = 0$, (2.209) is identical to (2.196). The baroclinic vector $\boldsymbol{B}$ and the curl of the frictional force appear in the local balance as they appear in the circulation theorem (2.205) in the integrated form. The first term on the right-hand side is identified with the tilting and stretching of the infinitesimal material line element $\boldsymbol{\ell}$ which is oriented as the local absolute vorticity vector, as explained below. In contrast to the other sources it cannot generate vorticity in a nonrotating state, i. e. only the presence of a nonzero vector $\boldsymbol{\omega}_{\mathrm{a}}$ may lead to a change in the components of $\boldsymbol{\omega}_{\mathrm{a}}$.

Consider a particular component of the vorticity balance (2.209) in a Cartesian frame of reference, say the balance of the vertical component $\omega_{\mathrm{a}}^{(z)} = \partial v/\partial x - \partial u/\partial y + \Omega^{(z)}$,

$$\begin{aligned}\frac{\mathrm{D}}{\mathrm{D}t}\frac{\omega_{\mathrm{a}}^{(z)}}{\rho} &= \frac{\omega_{\mathrm{a}}^{(x)}}{\rho}\frac{\partial w}{\partial x} + \frac{\omega_{\mathrm{a}}^{(y)}}{\rho}\frac{\partial w}{\partial y} + \frac{\omega_{\mathrm{a}}^{(z)}}{\rho}\frac{\partial w}{\partial z} \\ &\quad + \frac{1}{\rho}\left[B^{(z)} + \frac{\partial}{\partial x}(\mathcal{F}^{(y)}/\rho) - \frac{\partial}{\partial y}(\mathcal{F}^{(x)}/\rho)\right]\end{aligned} \tag{2.210}$$

Apparently, if the fluid expands (or shrinks) in the vertical direction, i. e. $\partial w/\partial z \neq 0$, a nonzero $\omega_{\mathrm{a}}^{(z)}$ of a fluid parcel will be changed. This mechanism is accordingly called *vortex stretching*, and the corresponding term in (2.210) is the *stretching term*. In contrast, the first two terms on the right-hand side act even if $\omega_{\mathrm{a}}^{(z)}$ vanishes; they describe the generation of vertical vorticity of the parcel by tilting of vorticity components of other coordinate directions into the vertical direction. This is induced by lateral changes of the vertical velocity. These terms are called the *tilting terms*. This concept is applied accordingly to the other components of the vorticity balance. We will see, however, in later chapters, that for a large-scale geophysical flow on the rotating Earth, the vertical component (i. e. radial with respect to the Earth geometry) of vorticity plays a particularly important role, and the stretching term in turn is the most important term. The three sources of vorticity – stretching, tilting and solenoidal production – are illustrated in Figure 2.22.

2.11.3 Ertel's Potential Vorticity

The most complete form of potential vorticity conservation has been found by ERTEL[32]. It is based on the same principle, the vorticity balance (2.209), now written

---

[31] EUGENIO BELTRAMI, *1835 in Cremona, †1900 in Rome, mathematician.

[32] HANS ERTEL, *1904 in Berlin, †1971 in Berlin, meteorologist and geophysicist.

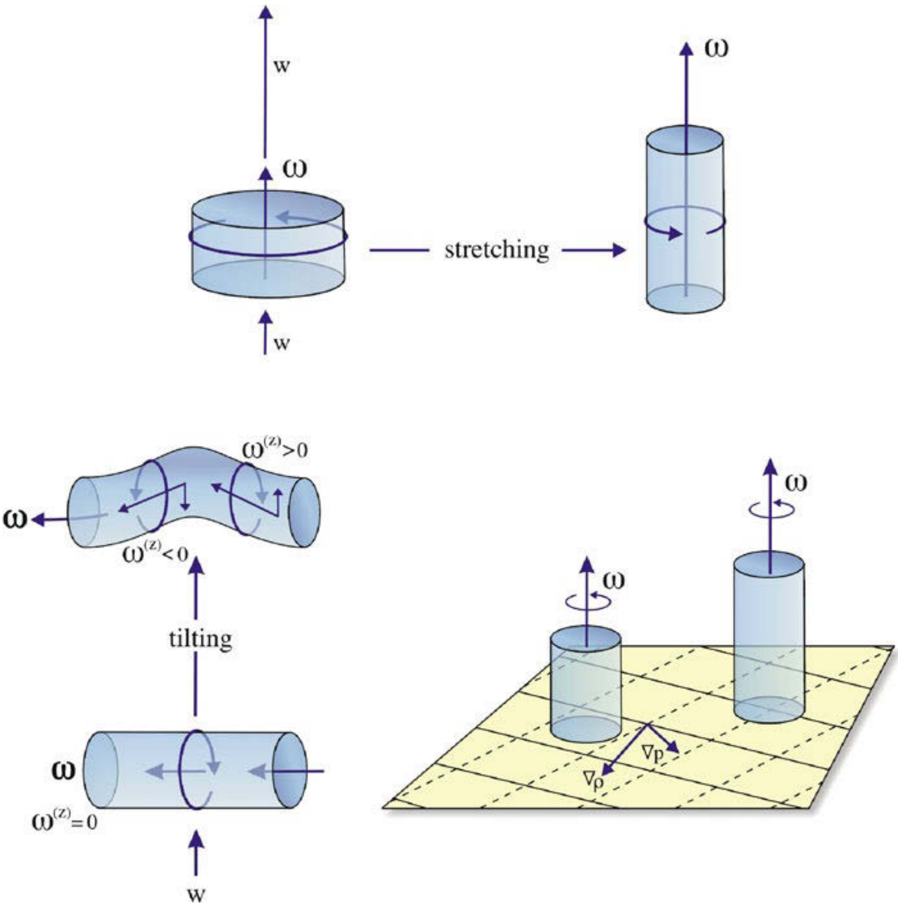

Fig. 2.22 Sketch showing the change of vorticity by stretching, tilting, and solenoidal production by the baroclinic vector. Redrawn after Vallis (2006) and Dutton (1976)

with an expanded baroclinic vector,

$$\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\right) = \left(\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\cdot\nabla\right)\boldsymbol{u} + \frac{1}{\rho^3}\nabla\rho\times\nabla p + \frac{1}{\rho}\nabla\times\mathcal{F} \tag{2.211}$$

with the mass conservation being incorporated already. In addition, consider a tracer χ which satisfies the conservation $\rho\mathrm{D}\chi/\mathrm{D}t = \mathcal{G}_\chi$. In practical applications, χ is usually taken as potential density, leading to the name ‘potential vorticity’. We will, however, retain the diabatic terms $\mathcal{F}$ and $\mathcal{G}_\chi$. Projection of (2.211) onto $\nabla\chi$ is

$$\nabla\chi\cdot\frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\right) = \left[\left(\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\right)\cdot\nabla\boldsymbol{u}\right]\cdot\nabla\chi + \nabla\chi\cdot\frac{\nabla\rho\times\nabla p}{\rho^3} + \frac{1}{\rho}\nabla\chi\cdot(\nabla\times\mathcal{F})$$

Using the fairly complicated and nonobvious vector identity

$$\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\cdot\frac{\mathrm{D}}{\mathrm{D}t}\nabla\chi = \left(\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\cdot\nabla\right)\frac{\mathrm{D}}{\mathrm{D}t}\chi - \left[\left(\frac{\boldsymbol{\omega}_\mathrm{a}}{\rho}\cdot\nabla\right)\boldsymbol{u}\right]\cdot\nabla\chi$$

it is then easy to derive Ertel's theorem (Ertel, 1942)

$$\rho \frac{\mathrm{D}}{\mathrm{D}t}\left(\frac{\boldsymbol{\omega}_{\mathrm{a}}}{\rho}\cdot\nabla\chi\right)=\boldsymbol{\omega}_a\cdot\nabla\left(\mathcal{G}_\chi/\rho\right)+\frac{1}{\rho^2}\nabla\chi\cdot\nabla\rho\times\nabla p+\nabla\chi\cdot(\nabla\times\boldsymbol{\mathcal{F}}) \quad (2.212)$$

The theorem states that a frictionless (i. e. $\boldsymbol{\mathcal{F}}=0$) and barotropic (i. e. $\nabla\rho\times\nabla p=0$) flow which conserves the tracer χ (i. e. $\mathcal{G}_\chi=0$) also conserves Ertel's potential vorticity Q given by

$$Q=\frac{1}{\rho}\boldsymbol{\omega}_{\mathrm{a}}\cdot\nabla\chi=\frac{1}{\rho}(\boldsymbol{\omega}+2\boldsymbol{\Omega})\cdot\nabla\chi$$

The condition of barotropicity may be relaxed for a tracer which is a function of ρ and p only, i. e. $\chi=\chi(\rho,p)$ because for any such tracer the triple product of gradients in (2.212) vanishes as well.

An illustrative proof of Ertel's theorem can be obtained by direct use of Helmholtz's vorticity equation (2.196) which applies to a frictionless barotropic fluid. Consider two particles with trajectories $\boldsymbol{X}_i(t), i=1,2$ such that the $\boldsymbol{\ell}=\boldsymbol{X}_2-\boldsymbol{X}_1$ is oriented along the local vorticity vector $\boldsymbol{\omega}_{\mathrm{a}}/\rho$ at $\boldsymbol{X}_1$ at the initial time $t=0$. According to Helmholtz' theorem this continues to be true when the particles proceed on their paths. Thus $\boldsymbol{\ell}=\epsilon\boldsymbol{\omega}_{\mathrm{a}}/\rho$ with a constant ϵ. Consider the time evolution of the tracer difference over the separation $\boldsymbol{\ell}$,

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\left[\chi(\boldsymbol{X}_2(t),t)-\chi(\boldsymbol{X}_1(t),t)\right]=\mathcal{G}_\chi(\boldsymbol{X}_2(t),t)-\mathcal{G}_\chi(\boldsymbol{X}_1(t),t)$$

which takes the form

$$\rho\frac{\mathrm{D}}{\mathrm{D}t}\boldsymbol{\ell}\cdot\nabla\chi=\boldsymbol{\ell}\cdot\nabla\mathcal{G}_\chi$$

for infinitesimal separation. By this we have indeed derived (2.212) for the case that the baroclinic vector and the friction vanish.

It becomes obvious that the vorticity and circulation theorems are closely connected. In fact, Ertel's theorem is a local formulation of Kelvin's theorem, including the effect of rotation. To see this, consider an infinitesimal closed curve C on a surface where $\chi=$ const. From Kelvin's theorem we find that $\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{n}\delta A$ is materially conserved. Because the normal vector is $\boldsymbol{n}=\nabla\chi/|\nabla\chi|$ and the mass in the volume between neighboring χ-surfaces and $\chi+\Delta\chi$ is conserved, $\delta m=\rho\delta A\cdot h=$ const, with $h=\Delta\chi/|\nabla\chi|$, we immediately find

$$\boldsymbol{\omega}\cdot\boldsymbol{n}\delta A=\frac{\boldsymbol{\omega}_{\mathrm{a}}\cdot\nabla\chi}{|\nabla\chi|}\frac{\delta m}{\rho h}=Q\frac{\delta m}{\Delta\chi}$$

Hence Q must be conserved as well.

2.12 * Lagrangian Concepts in Fluid Mechanics

So far we have considered the governing equations of motion in Eulerian coordinates. It is instructive to consider the principal form of these equations also in Lagrangian coordinates, which we will do in this section. As well known from classical mechanics, the use of a variational principle greatly simplifies the derivation of conservation

17. Impermeability Theorem for Potential Vorticity

As noted by Obukhov (1962), the terms on the right-hand side of (2.212) may be cast into the form of a divergence and Ertel's theorem takes the form

$$\rho\frac{\mathrm{D}Q}{\mathrm{D}t} = -\boldsymbol{\nabla}\cdot\boldsymbol{J}_{\mathrm{Q}} \tag{B17.1}$$

with flux vector of potential vorticity

$$\boldsymbol{J}_{\mathrm{Q}} = \frac{1}{\rho}\boldsymbol{\nabla} p \times \boldsymbol{\nabla}\chi - \boldsymbol{\omega}_{\mathrm{a}}\mathcal{G}_\chi - \boldsymbol{\mathcal{F}}\times\boldsymbol{\nabla}\chi \tag{B17.2}$$

The proof is easy: $\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla}\mathcal{G}_\chi = \boldsymbol{\nabla}\cdot\boldsymbol{\omega}_{\mathrm{a}}\mathcal{G}_\chi$ because $\boldsymbol{\omega}_{\mathrm{a}}$ is solenoidal, $\boldsymbol{\nabla}\chi\cdot\boldsymbol{\nabla}\times\boldsymbol{\mathcal{F}} = \boldsymbol{\nabla}\cdot(\boldsymbol{\mathcal{F}}\times\boldsymbol{\nabla}\chi)$ because $\boldsymbol{\nabla}\times\boldsymbol{\nabla}\chi = \boldsymbol{0}$, and $\boldsymbol{\nabla}\chi\cdot\boldsymbol{\nabla}1/\rho\times\boldsymbol{\nabla}p = \boldsymbol{\nabla}\cdot 1/\rho(\boldsymbol{\nabla}p\times\boldsymbol{\nabla}\chi)$ because $\boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}a\times\boldsymbol{\nabla}b) = \boldsymbol{0}$ for any a, b. Then (B17.1) has the form of the general conservation equation discussed in Chapter 2. In flux form we have

$$\frac{\partial}{\partial t}\rho Q + \boldsymbol{\nabla}\cdot\left(\boldsymbol{J}_{\mathrm{Q}} + \rho\boldsymbol{u}Q\right) = 0 \tag{B17.3}$$

The result is a bit surprising because there is no restriction as usual for a sourceless conservation equation. In fact, (B17.3) is valid for a diabatic flow in the presence of friction and baroclinicity. The rate of change of the Q-content in an arbitrary volume is obtained by integration and use of Gauss' theorem. We find

$$\frac{\partial}{\partial t}\int_V \rho Q\,\mathrm{d}V = -\oint\left(\boldsymbol{J}_{\mathrm{Q}} + \rho\boldsymbol{u}Q - \rho Q\frac{\mathrm{d}\boldsymbol{n}}{\mathrm{d}t}\right)\cdot\mathrm{d}\boldsymbol{A} \tag{B17.4}$$

where $\mathrm{d}\boldsymbol{n}/\mathrm{d}t$ denotes the velocity of the boundary (positive for the outward direction), and $\mathrm{d}\boldsymbol{A} = \boldsymbol{n}\mathrm{d}A$ is the surface element. For a volume which is bounded by surfaces $\chi = \chi_1$ (the 'bottom') and $\chi = \chi_2$ (the 'top') this integral budget attains a simpler form because all fluxes through the top and bottom boundaries vanish. The proof uses the fact that the normal vector at the top and the bottom is aligned along the gradient of the tracer, i. e. $\boldsymbol{n} = \boldsymbol{\nabla}\chi/|\boldsymbol{\nabla}\chi|$, and

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\frac{\partial\chi/\partial t}{|\boldsymbol{\nabla}\chi|}$$

which follows because $\mathrm{d}\chi = (\partial\chi/\partial n)\mathrm{d}n + (\partial\chi/\partial t)\mathrm{d}t$ vanishes on the bounding surfaces $\chi = \text{const}$. Furthermore,

$$\boldsymbol{n}\cdot\boldsymbol{u} = \frac{\boldsymbol{u}\cdot\boldsymbol{\nabla}\chi}{|\boldsymbol{\nabla}\chi|} = \frac{\mathcal{G}_\chi - \partial\chi/\partial t}{|\boldsymbol{\nabla}\chi|} \quad\text{and}\quad \boldsymbol{n}\cdot\boldsymbol{J}_{\mathrm{Q}} = -\frac{\mathcal{G}_\chi\boldsymbol{\omega}_{\mathrm{a}}\cdot\boldsymbol{\nabla}\chi}{|\boldsymbol{\nabla}\chi|} = -\frac{\mathcal{G}_\chi\rho Q}{|\boldsymbol{\nabla}\chi|}$$

by use of the tracer balance equation, the definition of $\boldsymbol{J}_{\mathrm{Q}}$ and the potential vorticity. Hence

$$\boldsymbol{n}\cdot(\boldsymbol{J}_{\mathrm{Q}} + \rho\boldsymbol{u}Q) = -\rho Q\frac{\partial\chi/\partial t}{|\boldsymbol{\nabla}\chi|} = \rho Q\frac{\mathrm{d}n}{\mathrm{d}t}$$

for the top and bottom surfaces in the integral (B17.4). It follows that the content of potential vorticity in a volume bounded by surfaces $\chi = \text{const}$ is conserved, except for fluxes through lateral boundaries (in the oceanic case, with χ taken as potential density, the potential vorticity is then controlled by mixing at continental boundaries, which is unfortunately an area of major ignorance). This important theorem (*impermeability theorem*) was found by Haynes and McIntyre (1990). Notice that this includes all effects of friction, mixing and baroclinicity. In particular, the surfaces $\chi = \text{const}$ do not need to be material surfaces.

equations and is always possible with Lagrangian coordinates. Furthermore, the use of a variational principle can also be useful in the context of Eulerian coordinates. While not generally possible, it is shown in the following that the complete set of equations of motion – in their adiabatic form – can indeed be derived from a variational principle. Note that we return to the Lagrangian concepts in Chapter 9, where the procedure is used to obtain wave solutions.

We begin with a discussion of the variational principle for discrete systems, which we then extend to continuous systems. Consider a discrete system with n degrees of freedom and which is described by coordinates $q_i(t), i = 1, \ldots, n$ where t is the time. In classical mechanics the action is defined by

$$A[q_i] = \int_{t_1}^{t_2} L \mathrm{d}t \quad \text{with} \quad L = E_\mathrm{k} - E_\mathrm{p}$$

where L is called *Lagrangian* and E_k and E_p are kinetic and potential energy of the system, respectively. The coordinates do not need to be the physical coordinates of the system and, therefore, are usually called generalized coordinates. In general, $L = L(\dot{q}_i, q_i, t)$ is a function of the $\dot{q}_i, q_i$ and the time t. The system is Lagrangian if its equations of motion can be derived from HAMILTON'S[33] variational principle

$$\delta A[q_i] = \lim_{\epsilon \to 0} \frac{A[q_i + \epsilon \delta q_i] - A[q_i]}{\epsilon} = 0 \tag{2.213}$$

involving the action $A[q_i]$ which is a functional of the coordinates q_i. The variation in (2.213) is performed with $\delta q_i(t_1) = \delta q_i(t_2) = 0$ and yields the *Lagrange equations*

$$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0 \tag{2.214}$$

in terms of the Lagrangian. These are the n equations of motion of the system. As a simple example, take the one-dimensional harmonic oscillator where $K = m\dot{q}^2/2$ and $V = kq^2/2$. The Lagrangian formalism then yields via (2.214) the familiar oscillator equation $m\ddot{q} + kq = 0$. When dealing with systems having many degrees of freedom, the handling by the one scalar functional L is clearly of advantage.

An extension of the discrete Lagrange–Hamilton formalism (outlined above and in the box on p. 98), which is important for our purposes of describing fluid systems, concerns the step from discrete to continuous systems, i. e. instead of variables q_i characterized by a discrete index i the system is now described by functions $\phi(x, t)$ of a continuous variable x (usually the position coordinate). The functional L in (2.215) is now a *Lagrangian density* in x-space. The appropriate action is

$$A[\phi] = \int_{t_1}^{t_2} \mathrm{d}t \int_D \mathrm{d}x L\left(\frac{\partial \phi}{\partial t}, \frac{\partial \phi}{\partial x}, \phi, x, t\right) \tag{2.215}$$

and variations $\delta\phi(x, t)$ are considered which vanish at limits of the time interval and on the boundary D of the spatial domain. Then, the so-called *Euler–Lagrange equation* results,

$$\frac{\partial}{\partial t} \frac{\partial L}{\partial \phi_t} + \frac{\partial}{\partial x} \frac{\partial L}{\partial \phi_x} - \frac{\partial L}{\partial \phi} = 0 \tag{2.216}$$

---

[33] SIR WILLIAM ROWAN HAMILTON, *1805 in Dublin, †1865 in Dunsink near Dublin, mathematician and physicist.

with shorthand notation $\phi_\mathrm{t} = \partial\phi/\partial t$ etc. Worth mentioning is that higher order derivatives $\phi_\mathrm{tt}, \phi_{\mathrm{t}x}, \phi_{xx}, \ldots$ may be considered in the Lagrange function, resulting then in corresponding higher derivatives in the Euler–Lagrange equations. We may actually deal with a vector field $\phi_\alpha(\boldsymbol{x}, t), \alpha = 1, \ldots, m$ of dimension m defined in two- or three-dimensional space which yields

$$\frac{\partial}{\partial t}\frac{\partial L}{\partial \phi_{\alpha,t}} + \frac{\partial}{\partial x_j}\frac{\partial L}{\partial \phi_{\alpha,j}} - \frac{\partial L}{\partial \phi_\alpha} = 0 \,, \quad \alpha = 1, \ldots, m \tag{2.217}$$

where the comma notation abbreviates the spatial derivative, and the sum convention is used for the index j.

A Hamiltonian theory may be developed for continuous systems as well. In fact, a complete set of balance equations for energy and momentum conservation may be formulated,

$$\begin{aligned} \frac{\partial T_{00}}{\partial t} + \frac{\partial T_{0i}}{\partial x_i} &= -\frac{\partial L}{\partial t} \\ \frac{\partial T_{i0}}{\partial t} + \frac{\partial T_{ij}}{\partial x_j} &= -\frac{\partial L}{\partial x_i} \end{aligned} \tag{2.218}$$

where

$$T_{00} = \sum_\alpha \phi_{\alpha,\mathrm{t}} \frac{\partial L}{\partial \phi_{\alpha,\mathrm{t}}} - L \quad \text{energy density} \tag{2.219}$$

$$T_{0i} = \sum_\alpha \phi_{\alpha,\mathrm{t}} \frac{\partial L}{\partial \phi_{\alpha,i}} \quad \text{energy flux density} \tag{2.220}$$

$$T_{i0} = \sum_\alpha \phi_{\alpha,i} \frac{\partial L}{\partial \phi_{\alpha,\mathrm{t}}} \quad \text{momentum density} \tag{2.221}$$

$$T_{ij} = \sum_\alpha \phi_{\alpha,i} \frac{\partial L}{\partial \phi_{\alpha,j}} - \delta_{ij} L \quad \text{stress tensor} \tag{2.222}$$

Note that the meaning of 'energy' and 'momentum' coincides with the physical energy and momentum only if the Lagrangian is built from physical energies.

Following (2.218), energy is conserved if the Lagrangian does not explicitly depend on time, and momentum is conserved if no explicit spatial dependence occurs. There is a deeper reason for these conservation laws: according to NOETHER's[34] theorem a (sourceless) conservation law exists for each symmetry in the Lagrange function L. A symmetry exists if L is invariant to a continuous transformation of its dependent or independent variables. In the above cases (2.218), these are translations in time or space.

In the following we present some examples of fluid mechanical systems which can be treated by a Lagrangian.

2.12.1 Incompressible Fluid

In the Lagrangian representation, outlined in Section 1.1, the position of a fluid parcel at time t was denoted by $\boldsymbol{X}(\boldsymbol{a}, t)$ where the initial position $\boldsymbol{a} = \boldsymbol{X}(\boldsymbol{a}, t = 0)$

---

[34] AMALIE EMMY NOETHER, *1882 in Erlangen, †1935 in Bryn Mawr in Pennsylvania, mathematician.

18. Discrete Systems – Hamilton's Equations

From the Lagrangian L we may proceed to the *Hamiltonian* H defined by

$$H = \sum_i p_i \dot{q}_i - L \quad \text{with} \quad p_i = \frac{\partial L}{\partial \dot{q}_i} \tag{B18.1}$$

The p_i is the momentum conjugate to the coordinate q_i. To obtain the correct form of the Hamiltonian, the last relation must be inverted to get $\dot{q}_i$ in terms of the p_i. Then, H is a function of the p_i, q_i and t, and inserting L from (B18.1) into (2.213) and varying now q_i and p_i yields $2n$ first-order equations

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \quad \text{and} \quad \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Two implications are immediately evident: (1) the value of H is conserved if the Hamiltonian is time-independent because the sum in

$$\frac{\mathrm{d}H}{\mathrm{d}t} = \frac{\partial H}{\partial t} + \sum_i \left(\dot{p}_i \frac{\partial H}{\partial p_i} + \dot{q}_i \frac{\partial H}{\partial q_i} \right) = \frac{\partial H}{\partial t} \tag{B18.2}$$

is identical to zero; and (2) for the classical kinetic energy $K = 1/2 \sum m_i \dot{q}_i^2$ we find that H is the total energy because $\dot{q}_i = p_i / m_i$ and

$$H = \sum_i \frac{p_i^2}{2m_i} + V = K + V$$

Considering again the harmonic oscillator, we find from the Hamiltonian formalism $p = \partial L / \partial \dot{q} = m\dot{q}$ and thus $H = p^2/2m + kq^2/2$ and then the two coupled equations $\dot{q} = \partial H / \partial p = p/m, \dot{p} = -\partial H / \partial q = -kq$.

is used as particle label. As $\boldsymbol{u}(\boldsymbol{x}, t) = \partial \boldsymbol{X} / \partial t$ is the Eulerian velocity at the position $\boldsymbol{x} = \boldsymbol{X}(\boldsymbol{a}, t)$, the equations of motion are – in a mixed Lagrangian-Eulerian expression – given by

$$\rho \ddot{X}_i = -\frac{\partial p}{\partial x_i}\,, \quad i = 1, 2, 3$$

where $\rho(\boldsymbol{x}, t)$ and $p(\boldsymbol{x}, t)$ are Eulerian fields. To simplify the notation, we use the dot for the partial derivative with respect to time. When the initial value $\rho_0(\boldsymbol{a}) = \rho(\boldsymbol{a}, t = 0)$ is taken as Lagrangian variable for the density (which is constant in time), conservation of mass requires $\rho_0 \mathrm{d}^3 a = \rho \mathrm{d}^3 x$ where the volume elements $\mathrm{d}^3 a$ and $\mathrm{d}^3 x$ are mapped onto each other by corresponding parcel trajectories. Write $\mathrm{d}^3 a = \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3$ for the volume of a small cube at the initial time. With the transformation $X_i(\boldsymbol{a}, t)$ we find for the small increment $\mathrm{d}a_1$ the displacement $\mathrm{d}X_i(\boldsymbol{a}, t) = (\partial X_i / \partial a_1)\mathrm{d}a_1$ and write this as a vector $\mathrm{d}\boldsymbol{X}^{(1)}$, similarly for $\mathrm{d}\boldsymbol{X}^{(2)}$ and $\mathrm{d}\boldsymbol{X}^{(3)}$. These span the volume at the time t

$$\mathrm{d}\boldsymbol{X}^{(1)} \cdot (\mathrm{d}\boldsymbol{X}^{(2)} \times \mathrm{d}\boldsymbol{X}^{(3)}) = \epsilon_{ijk} \frac{\partial X_i}{\partial a_1} \mathrm{d}a_1 \frac{\partial X_j}{\partial a_2} \mathrm{d}a_2 \frac{\partial X_k}{\partial a_3} \mathrm{d}a_3 = J(\boldsymbol{a}, t) \mathrm{d}a_1 \mathrm{d}a_2 \mathrm{d}a_3$$

evaluated by the familiar triple product, with

$$\epsilon_{ijk} = \begin{cases} +1 & \text{if } ijk \text{ is an even permutation of 1,2,3} \\ -1 & \text{if } ijk \text{ is an uneven permutation of 1,2,3} \\ 0 & \text{otherwise} \end{cases}$$

This defines the Jacobian determinant J of the transformation from the initial state $\boldsymbol{a} = \boldsymbol{X}(\boldsymbol{a}, t = 0)$ to the state $\boldsymbol{X}(\boldsymbol{a}, t)$ at time t. Hence

$$\rho_0(\boldsymbol{a}) = \rho(\boldsymbol{X}, t) J(\boldsymbol{a}, t) \quad \text{with} \quad J(\boldsymbol{a}, t) = \frac{\partial(X_1, X_2, X_3)}{\partial(a_1, a_2, a_3)}$$

Introducing a Lagrangian pressure variable $\pi(\boldsymbol{a}, t) = p(\boldsymbol{x}, t)$ by use of the inverse trajectory mapping $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{X}, t) = \boldsymbol{A}(\boldsymbol{x}, t)$ (see (1.2)), we find for the pressure gradient

$$\frac{\partial p}{\partial x_i} = \frac{\partial A_j}{\partial x_i} \frac{\partial \pi}{\partial a_j}$$

and the equations of motion are completely expressed in the Lagrangian variables by

$$\rho_0 \ddot{X}_i = -J \frac{\partial A_j}{\partial x_i} \frac{\partial \pi}{\partial a_j} = -C_{ij} \frac{\partial \pi}{\partial a_j} \tag{2.223}$$

C_{ij} is the cofactor of $\partial X_i / \partial a_j$ in the expansion of the determinant $J = C_{ij} \partial X_i / \partial a_j$. To arrive at the last relation, we have used the identity $\partial A_i / \partial x_j = C_{ij} / J$.

As shown in the box on p. 119 below, the volume is conserved for an *incompressible* flow, $J = 1$, hence $\rho_0 = \rho$. The above equations of motion (2.223) follow for this incompressible state from the Lagrangian density (in $\boldsymbol{a}$-space)

$$L = \frac{1}{2} \rho_0 \dot{X}_i^2 + \pi (J - 1) \tag{2.224}$$

Variation with respect to the Lagrangian multiplier π for the incompressibility constraint $J = 1$ trivially recovers this constraint, and variation with respect to X_i yields

$$\rho_0 \ddot{X}_i + \frac{\partial}{\partial a_j} \left(\pi \frac{\partial J}{\partial X_{i,j}} \right) = 0 \tag{2.225}$$

Since each element occurs only once in J, we have $\partial J / \partial X_{i,j} = C_{ij}$, and $\partial C_{ij} / \partial a_j = 0$ is used to equate (2.225) with (2.223).

2.12.2 Compressible Isentropic Fluid

Note that (2.223) is valid for compressible flow as well, but replacing the constraint $J - 1 = 0$ in the Lagrangian by the compressible version $\rho J - \rho_0 = 0$ does not lead to these equations. An adequate Lagrangian for isentropic (adiabatic) conditions may be constructed by utilization of the internal energy $E = E(\upsilon, \eta)$ where $\upsilon = 1/\rho$ is the specific volume and η the specific entropy. In Section 1.2 it was shown that the derivatives of E yield the pressure and the temperature, respectively,

$$\frac{\partial E}{\partial \upsilon} = -p \,, \quad \frac{\partial E}{\partial \eta} = T \tag{2.226}$$

As η is conserved, it enters as $\eta(\boldsymbol{x}, t) = \eta_0(\boldsymbol{a})$. This constraint as well as the constraint of mass conservation is implemented by Lagrangian multipliers,

$$L = \frac{1}{2} \rho_0 \dot{X}_i^2 - \rho_0 E(\upsilon, \eta) + \lambda_1 (\eta - \eta_0) + \lambda_2 (\upsilon - J \upsilon_0) \tag{2.227}$$

and variations of $X_i, \lambda_1, \lambda_2, \upsilon$ and η yield

$$\begin{aligned}\rho_0 \ddot{X}_i + \frac{\partial}{\partial a_j}\left(\rho_0 \lambda_2 \frac{\partial J}{\partial X_{i,j}}\right) &= 0\\ \eta &= \eta_0\\ \upsilon &= J\upsilon_0\\ \rho_0 \frac{\partial E}{\partial \upsilon} &= \lambda_2\\ \rho_0 \frac{\partial E}{\partial \eta} &= \lambda_1\end{aligned}$$

Use of (2.226) and elimination of $\lambda_2 = -\rho_0 p$ lead to the momentum balance in its proper form. The other multiplier is found to relate to the temperature, $\lambda_1 = \rho_0 T$. With a given thermodynamic potential $E = E(\upsilon, \eta)$, the equation of state follows from (2.226). Note that it is straightforward to include other conserved quantities such as salinity S: add $\lambda_3(S - S_0)$ to the Lagrangian and use $E = E(\upsilon, \eta, S)$ with $\mu = -\partial E/\partial S$ as chemical potential (see (1.47)).

2.12.3 Rotating Fluid with Gravity

The Lagrangians (2.224) for an incompressible fluid or (2.227) for an isentropic fluid are readily extended to include gravity and rotation. We proceed with (2.224) and take coordinates relative to the rotating frame with a rotation vector $\Omega_i, i = 1, 2, 3$. The coordinate transformation to the rotating frame follows the same route outlined in Section 2.3.3. The appropriate equations of motion are

$$\rho\left(\ddot{X}_i + 2\epsilon_{ijk}\Omega_j \dot{X}_k\right) = -\frac{\partial p}{\partial x_i} - \rho\frac{\partial \Phi}{\partial x_i} \tag{2.228}$$

where $\Phi(x_i)$ is the apparent gravitational potential (including the centrifugal part as described in Section 2.3.4). The Lagrangian form of (2.228) is

$$\rho_0\left(\ddot{X}_i + 2\epsilon_{ijk}\Omega_j \dot{X}_k + \frac{\partial \Phi}{\partial X_i}\right) = -C_{ij}\frac{\partial \pi}{\partial a_j}$$

It follows from varying the Lagrangian

$$L = \frac{1}{2}\rho_0\left(\dot{X}_i^2 + 2\epsilon_{ijk}\Omega_i X_j \dot{X}_k\right) - \rho_0 \Phi(X_i) + \pi(J - 1) \tag{2.229}$$

with respect to X_i and π. The inertial part can be obtained be replacing $\dot{X}_i$ in the Lagrangian (2.224) for the nonrotating frame by $\dot{X}_i + \epsilon_{ijk}\Omega_j X_k$; the term which is quadratic in Ω_i is contained in the apparent gravity potential.

2.12.4 Rotating Stratified Fluid

Next we consider waves residing on a given background stratification on an f-plane with $\Omega_i = \delta_{i3} f/2$ with a constant f and gravity acting along the vertical direction,

i. e. $\Phi = gX_3$. We transform from the field variables x_i, π to perturbations ξ of displacement and ϖ of pressure about a state of rest,

$$\begin{aligned}\xi_i(\boldsymbol{a},t) &= X_i(\boldsymbol{a},t) - a_i \\ \varpi(\boldsymbol{a},t) &= \pi(\boldsymbol{a},t) - p_0(a_3)\end{aligned}$$

where p_0 is hydrostatically balanced with a stratification $\rho_0(a_3)$,

$$\frac{\mathrm{d}p_0}{\mathrm{d}a_3} = -\rho_0(a_3)$$

The effect of $p_0(a_3)(J-1)$ in the transformed Lagrangian is identical to a term $p_0(X_3)$ in the Lagrangian, and hence, omitting terms with no effect on the equations of motion, we arrive at

$$L = \frac{1}{2}\rho_0\left[\dot{\xi}_i^2 + f(\xi_1\dot{\xi}_2 - \dot{\xi}_1\xi_2)\right] - U(\xi_3) + \varpi(J-1) \tag{2.230}$$

with a potential U and Jacobian determinant J, given by

$$\begin{aligned}U(\xi_3) &= p_0(\xi_3 + a_3) + \rho_0 g \xi_3 \\ J &= 1 + \frac{\partial \xi_i}{\partial a_i} + \Delta_{ii} + \frac{\partial(\xi_1,\xi_2,\xi_3)}{\partial(a_1,a_2,a_3)}\end{aligned}$$

Here, Δ_{ij} is the cofactor of $\partial\xi_i/\partial a_j$ in the Jacobian determinant $|\partial\xi/\partial a|$. The equations of motion in the new coordinates are obtained by varying with respect to ξ_i and ϖ. The potential may be expanded as

$$U(\xi_3) = p_0(a_3) + \frac{1}{2}\rho_0 N^2(a_3)\xi_3^2 + \frac{1}{3!}\xi_3^3\frac{\mathrm{d}^3 p_0}{\mathrm{d}a_3^3} + \cdots$$

with the squared Brunt-Väisälä frequency $N^2 = -(g/\rho_0)\mathrm{d}\rho_0/\mathrm{d}a_3$. The p_0-term can be omitted. This form reveals the ordinary quadratic potential energy $N^2\xi_3^2/2$ resulting from a displacement in stratified environment. Note that U leads to nonlinearities in the equations of motion in addition to those arising from the incompressibility constraint.

2.12.5 A Variational Principle for Eulerian Coordinates

Lagrangian variables have the disadvantage that local interactions, as e. g. acceleration by the local pressure gradient force, are not immediately reflected in the governing equations. The Eulerian representation appears, therefore, more useful as it corresponds to a local field theory[35]. Various attempts have been made to construct a Lagrangian theory in Eulerian coordinates (pioneering papers are Lin (1963) and Seliger and Whitham (1968)). A simple-minded translation of the ingredients of the above Lagrangian is given by

$$L_1 = \frac{1}{2}\rho\boldsymbol{u}^2 - \rho E(\upsilon,\eta) + \rho\lambda_1\left(\frac{\partial\eta}{\partial t} + \boldsymbol{u}\cdot\nabla\eta\right) + \lambda_2\left(\frac{\partial\rho}{\partial t} + \nabla\cdot\rho\boldsymbol{u}\right)$$

---

[35] The transformation from Eulerian to Lagrangian variables is not canonical. Therefore, it is not immediately evident that the Eulerian equations are derivable from a variational principle.

which takes mass conservation and material conservation of entropy as side constraints into account. Because there is no dependence on $\partial \boldsymbol{u}/\partial t$, the evolution equation of $\boldsymbol{u}$ does not follow directly by variation. Instead, variation with respect to $\boldsymbol{u}$ yields a Clebsch representation of the velocity field,

$$\boldsymbol{u} = \nabla\lambda_2 - \lambda_1 \nabla\eta \tag{2.231}$$

which seems at first general enough but, in fact, is rather restrictive: the circulation $\oint \boldsymbol{u} \cdot \mathrm{d}\boldsymbol{s}$ for any closed loop in an isentropic surface $\eta = \text{const}$ vanishes according to (2.231). This dilemma is mediated by introduction of additional constraints, so-called Lin constraints. Lin used the initial position $\boldsymbol{A}(\boldsymbol{x}, t)$ of a parcel as additional variable: it is materially conserved, and

$$L_2 = L_1 + \rho\boldsymbol{\alpha} \cdot \left(\frac{\partial \boldsymbol{A}}{\partial t} + \boldsymbol{u} \cdot \nabla \boldsymbol{A} \right)$$

yields a representation for the velocity field

$$\boldsymbol{u} = \nabla\lambda_2 - \lambda_1 \nabla\eta - \sum_j \alpha_j \nabla A_j \tag{2.232}$$

Actually, using one additional coordinate as constraint is already sufficiently general (Bretherton, 1970) because three scalar fields are only needed to guarantee arbitrary initial conditions for the velocity field. Instead of singling out a particular coordinate it seems thus reasonable to look for an additional scalar field which is conserved.

For a binary fluid like seawater salinity S is such a variable. Consider material salinity conservation as constraint; hence we use

$$L = L_1 + \rho\lambda_3 \left(\frac{\partial S}{\partial t} + \boldsymbol{u} \cdot \nabla S \right)$$

Variation of the Lagrangian with respect to all fields yields the equations

$$\delta\boldsymbol{u} : \boldsymbol{u} = \nabla\lambda_2 - \lambda_1 \nabla\eta - \lambda_3 \nabla S \tag{2.233}$$

$$\delta\lambda_1 : \frac{\partial \eta}{\partial t} + \boldsymbol{u} \cdot \nabla\eta = \frac{\mathrm{D}\eta}{\mathrm{D}t} = 0 \tag{2.234}$$

$$\delta\lambda_3 : \frac{\partial S}{\partial t} + \boldsymbol{u} \cdot \nabla S = \frac{\mathrm{D}S}{\mathrm{D}t} = 0 \tag{2.235}$$

$$\delta\lambda_2 : \frac{\partial \rho}{\partial t} + \nabla \cdot \rho\boldsymbol{u} = 0 \tag{2.236}$$

$$\delta S : \frac{\partial \lambda_3}{\partial t} + \boldsymbol{u} \cdot \nabla\lambda_3 = \frac{\mathrm{D}\lambda_3}{\mathrm{D}t} = \mu \tag{2.237}$$

$$\delta\eta : \frac{\partial \lambda_1}{\partial t} + \boldsymbol{u} \cdot \nabla\lambda_1 = \frac{\mathrm{D}\lambda_1}{\mathrm{D}t} = -T \tag{2.238}$$

$$\delta\rho : \frac{1}{2}\boldsymbol{u}^2 - E - pv + \lambda_1 \frac{\mathrm{D}\eta}{\mathrm{D}t} + \lambda_3 \frac{\mathrm{D}S}{\mathrm{D}t} - \frac{\mathrm{D}\lambda_2}{\mathrm{D}t} = 0 \tag{2.239}$$

where $\mu = -\partial E/\partial S$ is the chemical potential (see (1.47)). The last equation may be written as

$$\frac{\mathrm{D}\lambda_2}{\mathrm{D}t} = \frac{1}{2}\boldsymbol{u}^2 - H \tag{2.240}$$

where $H = E + pv$ is the enthalpy. It remains to show that in the realm of the above set of equations the conventional momentum balance is valid. After some nontrivial eliminations (see the box on p. 103) we indeed arrive at

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -v\nabla p \tag{2.241}$$

The set of Eulerian balances (2.234)–(2.236) and (2.241) must be supplemented by the equation of state in the form $\rho = F(S, \eta, p)$. It may be derived from (2.226) with salinity as additional variable.

19. Derivation of (2.241)

Applying $\mathrm{D}/\mathrm{D}t$ to (2.233) yields

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = \frac{\mathrm{D}\boldsymbol{\nabla}\lambda_2}{\mathrm{D}t} - \lambda_1\frac{\mathrm{D}\boldsymbol{\nabla}\eta}{\mathrm{D}t} - \lambda_3\frac{\mathrm{D}\boldsymbol{\nabla}S}{\mathrm{D}t} + T\boldsymbol{\nabla}\eta - \mu\boldsymbol{\nabla}S \tag{B19.1}$$

where (2.237) and (2.238) has been used. The last two terms will be replaced by the relation for gradient of the enthalpy

$$\boldsymbol{\nabla}H = T\boldsymbol{\nabla}\eta - \mu\boldsymbol{\nabla}S + v\boldsymbol{\nabla}p$$

The first three terms in (B19.1) are evaluated according to the identity

$$\frac{\mathrm{D}\boldsymbol{\nabla}\lambda_2}{\mathrm{D}t} = \boldsymbol{\nabla}\frac{\mathrm{D}\lambda_2}{\mathrm{D}t} - \frac{\partial\lambda_2}{\partial x}\boldsymbol{\nabla}u - \frac{\partial\lambda_2}{\partial y}\boldsymbol{\nabla}v - \frac{\partial\lambda_2}{\partial z}\boldsymbol{\nabla}w$$

and correspondingly for η and S. Implementing the result into (B19.1), collecting then the term proportional to $\boldsymbol{\nabla}u$, and using the first component of (2.233), and proceeding similarly with $\boldsymbol{\nabla}v$ and $\boldsymbol{\nabla}w$, we find

$$\frac{\mathrm{D}\boldsymbol{u}}{\mathrm{D}t} = -v\boldsymbol{\nabla}p + \boldsymbol{\nabla}\frac{\mathrm{D}\lambda_2}{\mathrm{D}t} + \boldsymbol{\nabla}H - u\boldsymbol{\nabla}u - v\boldsymbol{\nabla}v - w\boldsymbol{\nabla}w \tag{B19.2}$$

where (2.234) and (2.235) was used. Finally, (2.241) results by implementing (2.240).

Further Reading

The manual *International Thermodynamic Equation of Seawater–2010* (IOC, SCOR and IAPSO, 2010) contains the currently most accurate and comprehensive information on all relevant aspects of the thermodynamics of seawater, including the algorithms for the practical computation of thermodynamic parameters.

The principles of ocean thermodynamics is treated in an accurate though rather condensed way in the first chapters of *Fundamentals of Ocean Dynamics* by Kamenkovich (1977).

Thermodynamics of Atmospheres and Oceans by Curry and Webster (1999) covers the subject for both ocean and atmosphere.

Classical textbooks on fluid mechanics are *An Introduction to Fluid Dynamics* by Batchelor (1977) and *Fluid Mechanics* by Landau and Lifshitz (1987). More easy for the beginner is *Fluid Mechanics* by Kundu et al. (2004) which gives an overview on basic conservation laws of fluids and discusses many applications for irrotational, laminar, turbulent and compressible flows.

An overview on the Hamiltonian formulation of fluid dynamics can be found in the *Lectures on Geophysical Fluid Dynamics* by Salmon (1998).

The mathematical tools for the description of hydrodynamic fields are described e. g. in *Mathematical Principles of Classical Fluid Mechanics* by Serrin (1959), *Vectors, Tensors, and the Basic Equations of Fluid Mechanics* by Aris (1989), and at a more elementary level in *Fundamentals of Atmospheric Dynamics and Thermodynamics* by Riegel (1992).

The article *The First Law of Thermodynamics in a Salty Ocean* by Warren (2006) is an excellent review of the first law from an oceanic perspective.

For a historical review of fluid dynamics see *Worlds of Flow: a History of Hydrodynamics from the Bernoullis to Prandtl* by Darrigol (2009).