Chapter 2
Dirichlet Problem in Domains of $\mathbb{C}^n$

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Abstract This lecture treats the Dirichlet problem for the homogeneous complex Monge–Ampère equation in domains $\Omega \subset \mathbb{C}^n$. The most important result, due to Bedford and Taylor [BT76], yields the optimal interior regularity of the solution when $\Omega = \mathbb{B}$ is the unit ball. We provide a complete proof, following the simplifications of Demailly [Dem93].

2.1 Introduction

The goal of this lecture is to study the Dirichlet problem in bounded domains of $\mathbb{C}^n$ for the complex Monge–Ampère operator. If $\Omega \Subset \mathbb{C}^n$ is such a domain and $\varphi : \Omega \to \mathbb{R}$ are continuous boundary values, the goal is to find a plurisubharmonic function $u : \Omega \to \mathbb{R}$ solution of the following nonlinear PDE with prescribed boundary values,

$$\mathrm{DirMA}(\Omega, \varphi) := \begin{cases} (dd^c u)^n = 0 \text{ in } \Omega \\ u_{|\partial\Omega} = \varphi \end{cases}$$

and to study regularity properties of u in terms of those of φ. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i\pi}(\partial - \overline{\partial})$ are real operators so that

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V. Guedj (ed.), *Complex Monge–Ampère Equations and Geodesics in the Space of Kähler Metrics*, Lecture Notes in Mathematics 2038,
DOI 10.1007/978-3-642-23669-3_2, © Springer-Verlag Berlin Heidelberg 2012

$$(dd^c u)^n = c \det \left(\frac{\partial^2 u}{\partial z_j \partial \overline{z_k}} \right) dV,$$

when $u \in \mathcal{C}^2(\Omega)$, for some normalizing constant $c > 0$ and dV denotes the Lebesgue volume form in $\mathbb{C}^n$.

The complex Monge–Ampère operator $(dd^c u)^n$ still makes sense when u is poorly regular, as we shall explain in Sect. 2.5.1. The property "$u_{|\partial\Omega} = \varphi$" has to be understood as

$$\lim_{\Omega \ni z \to \zeta} u(z) = \varphi(\zeta), \quad \text{for all } \zeta \in \partial\Omega.$$

Whether it holds depends both on the continuity properties of φ and on the geometry of $\partial\Omega$. We shall usually assume Ω is smooth and *strictly pseudoconvex*, a notion recalled in Sect. 2.3.1.

Nota Bene. These notes are written by Vincent Guedj and Ahmed Zeriahi after the lecture delivered by Ahmed Zeriahi in Marseille, March 2009. There is no claim for any originality, all the material presented here being quite classical. As the audience consisted of non specialists, we have tried to make these lecture notes accessible with only few prerequisites.

2.2 The Classical Dirichlet Problem in $\mathbb{C}$

In dimension one, the Monge–Ampère operator coincides with the Laplacian. It is thus much easier to study. We briefly recall here how to solve the Dirichlet problem in this case, first in the unit disk by using the Poisson representation formula – a tool not available in higher dimension-, then in general bounded domains of $\mathbb{C}$ using the method of barriers which can be adapted in higher dimension.

2.2.1 Unit Disk

We study here DirMA$(\mathbb{D}, \varphi)$ where $\mathbb{D} = \{\zeta \in \mathbb{C} \,/\, |\zeta| < 1\}$ is the unit disk. It admits a unique solution u_φ which can be expressed by averaging against the Poisson kernel.

Proposition 2.1 *Assume* $\varphi \in \mathcal{C}^0(\partial\mathbb{D})$. *Then*

$$u_\varphi(z) := \int_0^1 \frac{1 - |z|^2}{|z - e^{2i\pi\theta}|^2} \varphi(e^{2i\pi\theta}) d\theta$$

is the unique solution to DirMA$(\mathbb{D}, \varphi)$. *It is harmonic (hence real-analytic) in* $\mathbb{D}$ *and continuous up to the boundary.*

Proof. Observe that the Poisson kernel is the real part of a holomorphic function in $\mathbb{D}$,

$$\frac{1-|z|^2}{|z-e^{2i\pi\theta}|^2} = \Re\left(\frac{e^{2i\pi\theta}+z}{z-e^{2i\pi\theta}}\right).$$

This shows that u_φ is harmonic in $\mathbb{D}$, as an average of harmonic functions.

We now establish the continuity up to the boundary. Fix $\zeta = e^{2i\pi\theta_0} \in \partial\mathbb{D}$ and $\varepsilon > 0$. Since φ is assumed to be continuous at ζ, we can find $\delta > 0$ such that $|\varphi(e^{2i\pi\theta}) - \varphi(\zeta)| < \varepsilon/2$ whenever $|e^{2i\pi\theta} - \zeta| < \delta$. Observing that

$$\int_0^1 \frac{1-|z|^2}{|z-e^{2i\pi\theta}|^2} d\theta \equiv 1,$$

we infer

$$|u_\varphi(z) - \varphi(\zeta)| \leq \varepsilon/2 + 2M \int_{|e^{2i\pi\theta}-\zeta|\geq\delta} \frac{1-|z|^2}{|z-e^{2i\pi\theta}|^2} d\theta,$$

where $M = \sup_{S^1} |\varphi|$. Note that $|z - e^{2i\pi\theta}| \geq \delta/2$ if z is close enough to ζ and $|e^{2i\pi\theta} - \zeta| \geq \delta$. The latter integral is therefore bounded from above by $4(1-|z|^2)/\delta^2$ hence converges to zero as z approaches the unit circle. □

It is clear from the proof above that one can control the modulus of continuity of u_φ on $\overline{\mathbb{D}}$ in terms of that of φ. For instance if φ is Hölder continuous, then so is u_φ. Let us denote by

$$Lip_\alpha(K) := \{u : K \to \mathbb{R} \,/\, \exists C > 0, \forall x, y \in K, |u(x) - u(y)| \leq C|x-y|^\alpha\}$$

the set of α-Hölder continuous functions on a Borel set K, $0 < \alpha \leq 1$.

Exercise 2.2

(1) Show that $\varphi \in Lip_\alpha(\partial\mathbb{D}) \Rightarrow u_\varphi \in Lip_\alpha(\overline{\mathbb{D}})$ *when* $0 < \alpha < 1$.
(2) By considering $\varphi(e^{2i\pi\theta}) = |\sin\theta|$, *show that the result does not hold with* $\alpha = 1$.

Beware that the exercise is trickier than it perhaps seems at first glance: following the proof of the previous proposition, you should be able to obtain $u_\varphi \in Lip_\beta(\overline{\mathbb{D}})$ with $\beta = \alpha/(\alpha+2)$. Proving that u_φ is actually α-Hölder is slightly more subtle, give it a try!

The fact that the class Lip_1 does not behave well for the Dirichlet problem is a classical fact in the study of elliptic PDE's. Note that one can similarly show (see [GT83]) that

$$\varphi \in \mathcal{C}^{k,\alpha}(\partial\mathbb{D}) \Rightarrow u_\varphi \in \mathcal{C}^{k,\alpha}(\overline{\mathbb{D}})$$

for all $k \in \mathbb{N}$ and $0 < \alpha < 1$. In particular

$$\varphi \in \mathcal{C}^{\infty}(\partial\mathbb{D}) \Rightarrow u_\varphi \in \mathcal{C}^{\infty}(\overline{\mathbb{D}}).$$

We will soon see that this is far from being true in higher dimension.

2.2.2 Perron Envelopes

We now consider DirMA(Ω, φ), the Dirichlet problem corresponding to a bounded domain $\Omega \Subset \mathbb{C}$. Here $\varphi : \partial\Omega \to \mathbb{R}$ is a fixed continuous function on the boundary of Ω.

It follows from the maximum principle for harmonic functions that if a solution exists, it is unique. More generally if u, v are subharmonic functions on Ω such that $\Delta u \leq \Delta v$ in the weak sense of measures on Ω and $(u-v)_* \geq 0$ on $\partial\Omega$ (i.e. $u \geq v$ on $\partial\Omega$), then $u \geq v$ in Ω. Indeed, $v - u$ is subharmonic on Ω and $(v-u)^* \leq 0$ on $\partial\Omega$, so that $v - u \leq 0$ in Ω by the maximum principle for subharmonic functions.

This shows that if u is the solution of the Dirichlet problem DirMA(Ω, φ), then any "subsolution" $v \in SH(\Omega)$ such that $v^* \leq \varphi$ on $\partial\Omega$ satisfies $v \leq u$ on Ω. Therefore

$$u_\varphi := \sup\{v \,/\, v \in SH(\Omega), v^* \leq \varphi \text{ on } \partial\Omega\} \leq u.$$

Observe that u itself is a subsolution so that actually $u = u_\varphi$. In other words, if the Dirichlet problem DirMA(Ω, φ) admits a solution, then it is the "Perron envelope" u_φ defined above [Per23].

One can easily show, by "balayage" (using a max construction together with solutions of the Dirichlet problem in small disks) that u_φ is harmonic in Ω. The problem is therefore reduced to checking whether u_φ has the right boundary values. This depends on the geometry of $\partial\Omega$.

Definition 2.3 *A barrier at the point $\zeta_0 \in \partial\Omega$ is a non positive subharmonic function $b \in SH(\Omega)$ such that $\lim_{\zeta\to\zeta_0} b(\zeta) = 0$ and $b^* < 0$ in $\overline{\Omega}\setminus\{\zeta_0\}$.*

The interest in this notion lies in the following

Lemma 2.4 *If there exists a barrier at a boundary point $\zeta_0 \in \partial\Omega$, then*

$$\lim_{z\to\zeta_0} u_\varphi(z) = \varphi(\zeta_0).$$

Proof. Fix $\varepsilon > 0$. Since φ is continuous we can find $\delta > 0$ such that

$$\varphi(\zeta_0) - \varepsilon \leq \varphi(\zeta) \leq \varphi(\zeta_0) + \varepsilon \text{ for } \zeta \in \partial\Omega \text{ with } |\zeta - \zeta_0| \leq \delta.$$

Since $b^* < 0$ on the compact subset $\partial\Omega\setminus\mathbb{D}(\zeta_0, \delta)$, it follows from upper semi-continuity of b^* that for $A > 1$ large enough, $Ab^* + \varphi(\zeta_0) - \varepsilon \leq \varphi$ on $\partial\Omega$. Thus $Ab + \varphi(\zeta_0) - \varepsilon$ is a subsolution, hence

$$Ab(z) + \varphi(\zeta_0) - \varepsilon \leq u_\varphi(z), \quad \forall z \in \Omega.$$

Letting $z \to \zeta_0$ and then $\varepsilon \to 0$ shows that $\varphi(\zeta_0) \leq \liminf_{z\to\zeta_0} u_\varphi(z)$.

Consider now the Dirichlet problem $\mathrm{DirMA}(\Omega, -\varphi)$. It follows from the maximum principle that $u_\varphi + u_{-\varphi} \leq 0$ in Ω, hence $u_\varphi \leq -u_{-\varphi}$. Previous reasoning thus yields

$$\varphi(\zeta_0) \geq -(-\varphi(\zeta_0)) \geq -\liminf_{z\to\zeta_0} u_{-\varphi}(z) \geq \limsup_{z\to\zeta_0} u_\varphi(z),$$

hence finally $\lim_{z\to\zeta_0} u(z) = \varphi(\zeta_0)$. □

Constructing barriers is thus the final step towards a solution of the Dirichlet problem. It turns out that they always exist when the boundary $\partial\Omega$ is Lipschitz. Note that some hypothesis on $\partial\Omega$ has to be made: the problem $\mathrm{DirMA}(\mathbb{D}^*, \varphi)$ has no solution when $\Omega = \mathbb{D}^*$ is the unit disk minus the origin and φ is zero on the unit circle and 1 at the origin: in this case u_φ is the constant function zero, hence it does not have the right boundary value at the origin.

2.3 Maximal Plurisubharmonic Functions

We now start to consider similar questions in higher dimension. Observe that some further constraints have to be put either on the geometry of $\partial\Omega$ or on the behavior of the boundary values φ: if $f(\mathbb{D}) \subset \partial\Omega$ is a holomorphic disk (image of the unit disk by a non constant holomorphic map) lying within the boundary, then φ has to be subharmonic along $f(\mathbb{D})$ if the Dirichlet problem $\mathrm{DirMA}(\Omega, \varphi)$ ever has a solution. In order to avoid difficulties related to such questions, we restrict ourselves to considering smooth *strictly pseudoconvex* domains Ω.

2.3.1 Strictly Pseudoconvex Domains

Although it makes sense to study the Dirichlet problem for the complex Monge–Ampère operator on a general domain $\Omega \Subset \mathbb{C}^n$, we will restrict ourselves and consider only domains that are *bounded*, with smooth boundary and such that the latter has a certain convexity property:

Definition 2.5 *A bounded domain* $\Omega \Subset \mathbb{C}^n$ *is* **strictly pseudoconvex** *if there exists a smooth strictly plurisubharmonic function* ρ *on some open neighborhood* Ω' *of* $\overline{\Omega}$ *such that* $\Omega := \{z \in \Omega' \,/\, \rho(z) < 0\}$.

A classical result asserts that a bounded domain is strictly pseudoconvex if and only if it is locally biholomorphic to a strictly convex domain. Slightly more general are *weakly pseudoconvex domains* and *hyperconvex domains*. The former coincide with *domains of holomorphy* (this is the famous Levi problem), while the latter are still defined as $\{\rho < 0\}$ but for a function that is only weakly (i.e. not necessarily strictly) plurisubharmonic and exhaustive.

There do exist some interesting results concerning the Dirichlet problem on these more general domains, as well as on non pseudoconvex ones (see e.g. [Sad82, Bl00, Guan02]). These are technically more involved and beyond the scope of this lecture.

2.3.2 Perron–Bremermann Envelope

Following the one variable solution to the Dirichlet problem, it is natural to consider

$$u_\varphi := \sup\{v \,/\, v \in \mathcal{B}(\Omega, \varphi)\}$$

where

$$\mathcal{B}(\Omega, \varphi) := \left\{ v \in PSH(\Omega);\; v^*(\zeta) := \limsup_{z \to \zeta} v(z) \leq \varphi(\zeta),\; \forall \zeta \in \partial\Omega \right\},$$

is the family of subsolutions for the boundary data φ.

The function u_φ is called the Perron–Bremermann envelope associated to the boundary data φ. Bremermann [Bre59] has shown that the function u_φ is a plurisubharmonic function in Ω with boundary values φ, Walsh [Wa68] further showed that u_φ is continuous in Ω:

Theorem 2.6 *Let* $\Omega \Subset \mathbb{C}^n$ *be a smoothly bounded strictly pseudoconvex subset of* $\mathbb{C}^n$. *The upper envelope* u_φ *is a continuous plurisubharmonic function on* $\overline{\Omega}$ *with boundary values* φ *i.e.*

$$\lim_{z \to \zeta} u_\varphi(z) = \varphi(\zeta) \quad \textit{for all } \zeta \in \partial\Omega.$$

Proof. Let ρ be a strictly plurisubharmonic defining function of $\Omega = \{\rho < 0\}$. Observe that the family $\mathcal{B}(\Omega, \varphi)$ is not empty: for $A \gg 1$ large enough, the function $A(\rho - 1)$ is one of its members (recall that φ is continuous hence bounded from below on $\partial\Omega$). Note also that $\mathcal{B}(\Omega,\varphi)$ is locally uniformly bounded from above in Ω: the constant function $\sup\varphi$ dominates all members of $\mathcal{B}(\Omega,\varphi)$.

It follows that the upper semi-continuous regularization $U_\varphi := u_\varphi^*$ is plurisubharmonic in Ω. We are going to prove that U_φ has boundary values φ. This will imply that $U_\varphi \in \mathcal{B}(\Omega,\varphi)$, so that $u_\varphi = U_\varphi$ in Ω.

As in one variable, we plan to construct a plurisubharmonic barrier function at each point $\zeta_0 \in \partial\Omega$. Since ρ is strictly plurisubharmonic in a neighborhood of $\overline{\Omega}$, we can choose $A > 1$ large enough so that the function $b_0 := A\rho(z) - |z-\zeta_0|^2$ is a plurisubharmonic barrier at the point ζ_0 i.e. b_0 is plurisubharmonic in Ω, continuous up to the boundary and such that $b_0(\zeta_0) = 0$ with $b_0 < 0$ in the complement $\overline{\Omega} \setminus \{\zeta_0\}$.

Fix $\varepsilon > 0$ and take $\eta > 0$ such that $\varphi(\zeta_0) - \varepsilon \le \varphi(\zeta)$ for $\zeta \in \partial\Omega$ and $|\zeta - \zeta_0| \le \eta$. Choose $C > 1$ big enough so that $Cb_0 + \varphi(\zeta_0) - \varepsilon \le \varphi$ on $\partial\Omega$. This implies that the function $v(z) := Cb_0 + \varphi(\zeta_0) - \varepsilon$ is plurisubharmonic in a neighborhood of $\overline{\Omega}$ and such that $v \le \varphi$ on $\partial\Omega$. Thus we have $v \le u_\varphi$ on Ω, which implies that $\varphi(\zeta_0) - \varepsilon \le \liminf_{z\to\zeta_0} u_\varphi(z)$. We infer

$$\liminf_{z\to\zeta} U_\varphi(z) \ge \liminf_{z\to\zeta} u_\varphi(z) \ge \varphi(\zeta) \tag{2.1}$$

for all $\zeta \in \partial\Omega$.

In the same way, we can construct a plurisubharmonic subsolution w for the boundary data $-\varphi$ such that $\lim_{z\to\zeta_0} w(z) = -\varphi(\zeta_0) - \varepsilon$. By the maximum principle, for any $v \in \mathcal{B}(\Omega,\varphi)$, we have $v + w \le 0$ in Ω, hence $u_\varphi + w \le 0$ on Ω. By upper regularization we infer $U_\varphi + w \le 0$ in Ω, which implies

$$\limsup_{z\to\zeta_0} U_\varphi(z) \le -\liminf_{z\to\zeta_0} w(z) = \varphi(\zeta_0) + \varepsilon.$$

Therefore we have proved that

$$\limsup_{z\to\zeta} U_\varphi(z) \le \varphi(\zeta), \quad \forall\zeta \in \partial\Omega. \tag{2.2}$$

This shows that $U_\varphi \in \mathcal{B}(\Omega,\varphi)$ hence $U_\varphi \le u_\varphi$ in Ω so that $U_\varphi \equiv u_\varphi$. Inequalities (2.1), (2.2) show that the envelope u_φ has boundary values φ.

It remains to prove that $u = u_\varphi$ is lower semi-continuous in Ω. Fix $\varepsilon > 0$. Since $\partial\Omega$ is compact, we can choose $\eta > 0$ so small that

$$z \in \Omega, \quad \zeta \in \partial\Omega, \quad |z - \zeta| \le \eta \Longrightarrow |u(z) - \varphi(\zeta)| \le \varepsilon. \tag{2.3}$$

Fix $a \in \mathbb{C}^n$ with $\|a\| < \eta$ and set $\widetilde{\Omega} := \Omega - a$. Then $u(\zeta + a) \le \varphi(\zeta) + \varepsilon$ if $\zeta \in \widetilde{\Omega} \cap \partial\Omega$ and $u^*(z+a) \le \varphi(z+a) + \varepsilon \le u(z) + 2\varepsilon$ if $z \in \Omega \cap \partial\widetilde{\Omega}$. It follows that the function

$$v(z) := \begin{cases} \sup\{u(z), u(z+a) - 2\varepsilon\} & \text{for } z \in \Omega \cap \widetilde{\Omega} \\ u(z) \quad \text{for } z \in \Omega \backslash \widetilde{\Omega} \end{cases}$$

is plurisubharmonic in Ω and satisfies the condition $v^* \leq \varphi$ on $\partial\Omega$. Therefore $v \leq u_\varphi = u$ in Ω, in particular

$$u(z+a) - 2\varepsilon \leq u(z) \quad \text{for } z \in \Omega \quad \text{and} \quad a \in \mathbb{C}^n, \ \ \|a\| < \eta.$$

This shows that $u = u_\varphi$ is lower semi-continuous in Ω. □

2.3.3 Maximal Plurisubharmonic Functions

Recall that harmonic functions are "above" sub-harmonic ones. This property actually characterizes harmonicity and was illustrated in Sect. 2.2 by the fact that we could recover the harmonic solution to the Dirichlet problem as an upper envelope.

It is therefore natural to consider, among all plurisubharmonic functions, those which are maximal, a notion introduced by Sadullaev [Sad81].

Definition 2.7 *A plurisubharmonic function* $u : \Omega \to [-\infty, +\infty[$ *is said to be maximal in* Ω *if for any plurisubharmonic function* v *defined on a subdomain* $D \Subset \Omega$, $v \leq u$ *on* ∂D *implies* $v \leq u$ *in* D.

Of course a pluriharmonic function is maximal (and smooth, as it is locally the real part of holomorphic function). However, in contrast to the one variable case, maximal plurisubharmonic functions need not be continuous: any (discontinuous) subharmonic function in the unit disk $\mathbb{D}$ gives rise to a maximal plurisubharmonic in $\mathbb{D}^2$ when considered as a function of two complex variables. This is a particular case of the following criterion of maximality.

Lemma 2.8 *Let* $u : \Omega \to [-\infty, +\infty[$ *be a plurisubharmonic function in* Ω*. If for any* $z_0 \in \Omega$ *there is a complex curve* $Z \Subset \Omega$ *containing* z_0 *such that* $u|Z$ *is harmonic on* $Z \cap \Omega$*, then* u *is maximal in* Ω*.*

We leave the easy proof as an exercise. One may wonder whether maximality can always be explained by the existence of "harmonic disks". This is indeed true if the function is regular enough (by Theorem 2.20 below and Frobenius theorem), however there are less regular maximal psh functions with no harmonic disk: this is the topic of the lecture by Dujardin [DG09].

As one can guess, the Perron–Bremermann envelope is maximal:

Proposition 2.9 *Let $\Omega \Subset \mathbb{C}^n$ be a bounded strictly pseudoconvex domain in $\mathbb{C}^n$ and $\varphi \in \mathcal{C}^0(\partial\Omega)$ a continuous function on $\partial\Omega$. Then u_φ is the unique maximal plurisubharmonic function on Ω with boundary values φ.*

Proof. We first show that u_φ is maximal on Ω. Let v be a plurisubharmonic function in some subdomain $D \Subset \Omega$ such that $v \leq u_\varphi$ on ∂D. Then the function

$$w := \begin{cases} \sup\{u_\varphi, v\} & \text{in } D \\ u_\varphi & \text{in } \Omega \setminus D \end{cases}$$

is plurisubharmonic in Ω and satisfies $w^* \leq \varphi$ on $\partial\Omega$. Therefore $w \leq u_\varphi$ in Ω hence $v \leq w \leq u_\varphi$ in D, which proves our claim.

We now prove uniqueness. Let v a maximal plurisubharmonic function in Ω such that $\lim_{z\to\zeta} v(z) = \varphi(\zeta)$ for any $\zeta \in \partial\Omega$. It follows that $v \leq u_\varphi$ in Ω while for any fixed $\varepsilon > 0$, the set $\{v + \varepsilon < u_\varphi\} \Subset \Omega$ is relatively compact in Ω. Let $D \Subset \Omega$ be any domain such that $\{v+\varepsilon < u_\varphi\} \Subset D$. Then $v+\varepsilon$ is a maximal plurisubharmonic function satisfying $v + \varepsilon \geq u_\varphi$ on ∂D. Therefore $v + \varepsilon \geq u_\varphi$ in D. Letting ε decrease to zero and D increase to Ω we infer $u_\varphi \leq v$ in Ω. □

In dimension one the upper envelope u_φ is harmonic on Ω, hence it is smooth and satisfies the partial differential equation $\Delta u_\varphi = 0$ on Ω. It is natural to wonder whether a similar result holds in higher dimension as well. We study the regularity question in the next section. The PDE characterization is postponed to the last section.

2.4 Regularity of Perron–Bremermann Envelopes

In this section we study the propagation of regularity from φ to u_φ. We start by explaining the fundamental result of Bedford and Taylor [BT76], following a simplified proof due to Demailly [Dem93]. We then list various results, open questions and examples that illustrate some of the difficulties encountered with $\mathrm{DirMA}(\Omega, \varphi)$ when $n \geq 2$.

2.4.1 Unit Ball

Our goal here is to prove the following result due to Bedford and Taylor [BT76].

Theorem 2.10 *Let $\mathbb{B}$ denote the unit ball in $\mathbb{C}^n$.*
If $\varphi \in \mathcal{C}^{1,1}(\partial\mathbb{B}, \mathbb{R})$ then u_φ is a $\mathcal{C}^{1,1}$-function in $\mathbb{B}$.

Recall that a function $f : M \to \mathbb{R}$ defined on a smooth real submanifold is $\mathcal{C}^{1,1}$ if f is differentiable and df is a locally Lipschitz 1-form on M. Observe that a $\mathcal{C}^{1,1}$-function has locally bounded second order derivatives almost everywhere.

Proof. We will show in Proposition 2.12 below that $u = u_\varphi$ is Lipschitz continuous up to the boundary. We focus here on the second order estimates. By Lemma 2.11 below, it suffices to prove that for any $z \in \Omega$ and $h \in \mathbb{C}^n$ with $|h| \ll 1$ we have

$$u(z+h) + u(z-h) - 2u(z) \leq C_2 \|h\|^2.$$

The idea is to study the boundary behavior of the plurisubharmonic function $z \longmapsto \frac{1}{2}(u(z+h)+u(z-h))$ in order to compare it with the function u in Ω. This does not make sense since the translations do not preserve the boundary. We are instead going to move point z by automorphisms of the unit ball: the group of holomorphic automorphisms of the latter acts transitively on it and this is the main reason why we prove this result for the unit ball rather than for a general strictly pseudoconvex domain (which has generically few such automorphisms).

Fix a point $a \in \mathbb{B} \setminus \{0\}$ and consider the mapping

$$F_a(z) := \frac{P_a(z) - a + (1 - \|a\|^2)^{1/2}(z - P_a(z))}{1 - \langle z, a \rangle}$$

where $P_a(z) := \|a\|^{-2}\langle z, a\rangle a$ is the orthogonal projection on the complex line $\mathbb{C}\cdot a$. Here $\langle z, a \rangle = \sum_{i=1}^n z_i \overline{a_i}$ denotes the hermitian scalar product of z and a. We let the reader check that F_a is an holomorphic automorphism of the unit ball $\mathbb{B}$ which sends a to the origin. The interested reader will find further information on these automorphisms in [Ru80].

An elementary computation yields

$$F_a(z) = \frac{z - a + O(|a|^2)}{1 - \langle z, a \rangle} = z - a + \langle z, a \rangle z + O(\|a\|^2) = z - h + O(\|a\|^2),$$

where $h := a - \langle z, a \rangle z$ and $O(\|a\|^2)$ is uniform with respect to $z \in \overline{\mathbb{B}}$.

Consider the function $v(z) := u \circ F_a(z) + u \circ F_{-a}(z)$. It is plurisubharmonic in $\mathbb{B}$ and has boundary values equal to

$$g(z) := \varphi(F_a(z)) + \varphi(F_{-a}(z))$$

since F_a preserves $\partial\mathbb{B}$. We can extend φ as a $\mathcal{C}^{1,1}$-smooth function so that $\varphi(F_{\pm a}(z)) \le \varphi(z \mp h) + C_1\|a\|^2$ and

$$\varphi(z+h) + \varphi(z-h) - 2\varphi(z) \le A\|h\|^2$$

whenever $z \in \mathbb{B}$ and $\|h\| \le \delta$. Altogether this yields

$$g(z) \le \varphi(z+h) + \varphi(z-h) + 2C_1\|a\|^2 \le 2\varphi(z) + A\|h\|^2 + 2C_1\|a\|^2$$

for $z \in \partial\mathbb{B}$. We infer $v(z) \le 2u(z) + A\|h\|^2 + 2C_1\|a\|^2$ when $z \in \mathbb{B}$.

Observe that the mapping $a \mapsto h = h(a,z)$ is a local diffeomorphism in a neighborhood of the origin as long as $\|z\| < 1$. An easy computation shows that the inverse map $h \mapsto a$ has norm $\le (1-\|z\|^2)^{-1}$.

Fix a compact set $K \subset \mathbb{B}$. Then there is exists $\delta > 0$ small enough and a constant $C_2 = C_2(K) > 0$ such that for $z \in K$ and $|h| \le \delta$ we have $|a| \le C_2|h|$.

It follows that for any $z \in K$ and $|h| \le \delta$,

$$u(z+h) + u(z-h) - 2u(z) \le C_3\|h\|^2,$$

where $C_3 > 0$ is a uniform constant depending on K, which proves the required estimates. □

Let us stress that we haven't proved that u_φ is $\mathcal{C}^{1,1}$ up to the boundary of the unit ball. This would require further regularity of the boundary values (see Sect. 2.4.3). In other words the constant C_2 in the proof above depends on $\mathrm{dist}(z, \partial\mathbb{B})$.

It remains to prove the following criterion.

Lemma 2.11 *Let u be a plurisubharmonic function in a domain $\Omega \Subset \mathbb{C}^n$. Assume that there exists constants $A, \delta > 0$ such that*

$$u(z+h) + u(z-h) - 2u(z) \le A\|h\|^2, \quad \forall 0 < \|h\| < \delta$$

and for all $z \in \Omega, \mathrm{dist}(z, \partial\Omega) > \delta$. Then u is $\mathcal{C}^{1,1}$-smooth and its second derivatives, which exist almost everywhere, satisfy $\|D^2 u\|_{L^\infty(\Omega)} \le A$.

Moreover the Monge–Ampère measure $(dd^c u)^n$ is absolutely continuous w.r.t. the Lebesgue measure dV in Ω, with

$$(dd^c u)^n = c_n \det\left(\frac{\partial^2 \mathrm{u}}{\partial \mathrm{z_j} \partial \bar{\mathrm{z}}_\mathrm{k}}\right) \beta_\mathrm{n},$$

where β_n is the standard volume form on $\mathbb{C}^n$.

Proof. Let $u_\varepsilon := u \star \rho_\varepsilon$ denote the standard regularization of u defined in $\Omega_\varepsilon := \{z \in \Omega \,/\, \mathrm{dist}(\mathrm{z}, \partial\Omega) > \varepsilon\}$ for $0 < \varepsilon \ll 1$. Fix $\delta > 0$ small enough and $0 < \varepsilon < \delta/2$. Then for $\|h\| < \delta/2$, we have

$$u_\varepsilon(z+h) + u_\varepsilon(z-h) - 2u_\varepsilon(z) \leq A\|h\|^2.$$

It follows from Taylor's formula that if $z \in \Omega_\delta$,

$$\frac{d^2}{dt^2} u_\varepsilon(z+th)|_{t=0} = \lim_{t \to 0^+} \frac{u_\varepsilon(z+th) + u_\varepsilon(z-th) - 2u_\varepsilon(z)}{t^2},$$

therefore $D^2 u_\varepsilon(z) \cdot h^2 \leq A\|h\|^2$ for all $z \in \Omega_\varepsilon$ and $h \in \mathbb{C}^n$. Now for $z \in \Omega_\varepsilon$,

$$D^2 u_\varepsilon(z) \cdot h^2 = \sum_{j,k=1}^{n} \left(\frac{\partial^2 u_\varepsilon}{\partial z_j \partial z_k} h_j h_k + 2 \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k + \frac{\partial^2 u_\varepsilon}{\partial \bar{z}_j \partial \bar{z}_k} \bar{h}_j \bar{h}_k \right).$$

Recall that u_ε is plurisubharmonic in Ω_ε, hence

$$D^2 u_\varepsilon(z) \cdot h^2 + D^2 u_\varepsilon(z) \cdot [ih]^2 = \sum_{j,k=1}^{n} 4 \frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} h_j \bar{h}_k \geq 0.$$

The above upper-bound therefore also yields a lower-bound,

$$D^2 u_\varepsilon(z) \cdot h^2 \geq -D^2 u_\varepsilon(z) \cdot [ih]^2 \geq -A\|h\|^2,$$

for any $z \in \Omega_\varepsilon$ and $h \in \mathbb{C}^n$. This implies that $\|D^2 u_\varepsilon(z)\|_{L^\infty(\Omega_\varepsilon)} \leq A$.

We have thus shown that Du_ε is uniformly Lipschitz in Ω_ε. We infer that Du is Lipschitz in Ω and $Du_\varepsilon \to Du$ uniformly on compact subsets of Ω. Since the dual of L^1 is L^∞, it follows from the Alaoglu–Banach theorem that, up to extracting a subsequence, there exists a bounded function V such that $D^2 u_\varepsilon \to V$ weakly in L^∞. Now $D^2 u_\varepsilon \to D^2 u$ in the sense of distributions hence $V = D^2 u$. Therefore u is $\mathcal{C}^{1,1}$-smooth in Ω, its second order derivatives exist almost everywhere with $\|D^2 u(z)\|_{L^\infty} \leq A$.

Recall that if $f \in L^n_{loc}(\Omega)$ then $f_\varepsilon := f \star \rho_\varepsilon \to f$ in $L^n_{loc}(\Omega)$. In particular

$$\frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k} \longrightarrow \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k} \text{ in } L^n_{loc}(\Omega) \supset L^\infty(\Omega).$$

Using generalized Hölder's inequality, we infer $\det\left(\frac{\partial^2 u_\varepsilon}{\partial z_j \partial \bar{z}_k}\right) \to \det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$ in $L^1_{loc}(\Omega)$.

Recall that $(dd^c u)^n$ is well defined in the weak sense of Bedford–Taylor [BT82] as the weak limit of the smooth forms $(dd^c u_\varepsilon)^n$, since (u_ε) decreases to u as ε decreases to 0. Since convergence in L^1_{loc} implies weak convergence, we infer that

$$(dd^c u)^n = c_n det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)\beta_n. \qquad \square$$

2.4.2 Strictly Pseudoconvex Domains

We now consider the more general case of a smoothly bounded strictly pseudoconvex domain $\Omega \Subset \mathbb{C}^n$.

We show here that the upper envelope u_φ is Lipschitz up to the boundary as soon as the boundary value φ is $\mathcal{C}^{1,1}$.

Proposition 2.12 *Let $\Omega \Subset \mathbb{C}^n$ be a smoothly bounded strictly pseudoconvex set. If $\varphi \in \mathcal{C}^{1,1}(\partial\Omega, \mathbb{R})$ then the envelope u_φ is Lipschitz continuous on $\overline{\Omega}$.*

Proof. Let ρ be a smooth defining function of Ω which is strictly psh in a neighborhood Ω' of $\overline{\Omega}$.

We can find a $\mathcal{C}^{1,1}$-extension F of φ with compact support in $\mathbb{C}^n$ such that $\|F\|_{\mathcal{C}^{1,1}(\mathbb{C}^n)} \leq C\|\varphi\|_{\mathcal{C}^{1,1}(\partial\Omega)}$. Replacing F by $F + A\rho$, with $A \gg 1$, we can further assume that F is plurisubharmonic in a neighborhood Ω' of $\overline{\Omega}$.

Applying the same process to the boundary data $-\varphi$ we choose a $\mathcal{C}^{1,1}$ psh function G in Ω' such that $G = -\varphi$ on $\partial\Omega$. Observe that F is a subsolution while the function $-G$ is a supersolution, hence $F \leq u \leq -G$ in Ω.

Since $F \leq u$ in Ω, the envelope u can be extended as a psh function in Ω' by setting $u = F$ in $\Omega' \setminus \Omega$. Fix $\delta > 0$ so small that $z + h \in \Omega'$ whenever $z \in \Omega$ and $\|h\| < \delta$. Fix $h \in \mathbb{C}^n$ such that $\|h\| < \delta$. Recall that F and G are Lipschitz, thus

$$|F(z+h) - F(z)| \leq C_1\|h\| \quad \text{and} \quad |G(z+h) - G(z)| \leq C_1\|h\|$$

for any $z \in \overline{\Omega}$.

Observe that the function $v(z) := u(z+h) - C_1\|h\|$ is well defined and psh in the open set Ω. If $z \in \partial\Omega$ and $z \in \Omega_h$ (i.e. $z + h \in \Omega$), then

$$v(z) = u(z+h) - C_1\|h\| \leq -G(z+h) - C_1\|h\| \leq -G(z) = \varphi(z) = u(z).$$

This shows that the function w defined by

$$w(z) := \begin{cases} \max\{v(z), u(z)\} & \text{if } z \in \Omega \cap \Omega_h \\ u(z) & \text{if } z \in \Omega \setminus \Omega_h \end{cases}$$

is plurisubharmonic in Ω. Since $w \leq \varphi$ on $\partial\Omega$ we get $w \leq u$ in Ω, hence $v \leq u$ in Ω. We have thus shown that

$$u(z+h) - u(z) \leq C_1 \|h\|$$

whenever $z \in \Omega \cap \Omega_h$, $\|h\| \leq \delta$ and $z \in \Omega_h$. Replacing h by $-h$ shows that $|u(z+h) - u(z)| \leq C_1\|h\|$, which proves that u is Lipschitz on $\overline{\Omega}$. □

The $\mathcal{C}^{1,1}$-regularity in Ω of u_φ has been established by Krylov [Kry89] by a probabilistic approach based on controlled diffusion process, as advocated by Gaveau [Gav77]. We refer the reader to the notes by Delarue [Del09] (see Chap. 4) for an introduction to this point of view.

2.4.3 Further Results and Counterexamples

2.4.3.1 No More than $\mathcal{C}^{1,1}$

It is tempting to think that the envelope u_φ is $\mathcal{C}^\infty$-smooth when so is φ, as it is the case in dimension one. This fails when $n \geq 2$. The following example of Gamelin and Sibony shows that the envelope u_φ is not better than $\mathcal{C}^{1,1}$ even if φ is real analytic.

Example 2.13 *Let $\mathbb{B} \Subset \mathbb{C}^2$ be the open unit ball. For $(z,w) \in \partial\mathbb{B}$, set*

$$\varphi(z,w) := (|z|^2 - 1/2)^2 = (|w|^2 - 1/2)^2.$$

Observe that φ is real-analytic on $\partial\mathbb{B}$. We claim that

$$u_\varphi(z,w) = \max\{\psi(z), \psi(w)\}, \quad (z,w) \in \mathbb{B},$$

where

$$\psi(z) := \left(\max\{0, |z|^2 - 1/2\}\right)^2, \quad z \in \mathbb{C}.$$

Indeed denote by u the right hand side of the above formula. It has the right boundary values so we simply have to check that it is maximal. Now observe that if $(z,w) \in \mathbb{B}$ then either $|z|^2 < 1/2$ or $|w|^2 < 1/2$. In each case u depends only on one variable hence it is maximal. Therefore $u = u_\varphi$ and the reader will easily check that it is not $\mathcal{C}^2$-smooth.

It is perhaps worth mentioning that in the non degenerate case, the unique solution of the Dirichlet problem $MA(u) = dV$ (=volume form) with smooth boundary values φ is smooth, as was established by Caffarelli et al. [CKNS85]. The reader will find a detailed proof of this result in Boucksom's lecture [Bou09] (Chap. 7).

On the other hand when the domain is merely weakly pseudoconvex, the regularity theory breaks down dramatically as shown in [Co97, Li04].

2.4.3.2 Regularity Up to the Boundary

Looking carefully at the proof of Theorem 2.10, the reader will convince himself that the $\mathcal{C}^{1,1}$-norm of u_φ does not blow up faster than $1/\text{dist}(\cdot, \partial\Omega)^2$ as one approaches the boundary.

It is expected that u_φ is $\mathcal{C}^{1,1}$-smooth up to the boundary when $\varphi \in \mathcal{C}^{3,1}(\partial\Omega)$. This has been established by Caffarelli et al. [CNS86] for the *real* homogeneous Monge–Ampère equation. The only known approach to the complex case is due to Krylov (see Delarue's lecture [Del09, Chap. 3]).

The following example (adaptation of an example in [CNS86]) shows that there is a necessary loss in the regularity up to the boundary:

Example 2.14 *Consider* $u(z,w) = (1 + \Re(w))^{2\alpha}$, *where* $0 < \alpha < 1$. *This is a plurisubharmonic function in the unit ball* $\mathbb{B} \Subset \mathbb{C}^2$ *which is smooth and maximal, continuous up to the boundary* $\overline{\mathbb{B}}$, *hence it coincides with* u_φ *for the boundary values*

$$\varphi(z,w) = (1 + \Re(w))^{2\alpha} \in Lip_{4\alpha}(\partial\mathbb{B})$$

The only problematic point is of course $(0,-1) \in \partial\mathbb{B}$.

Observe that $u = u_\varphi$ *is only in* $Lip_{2\alpha}(\overline{\mathbb{B}})$: *this can be seen by a radial approach to the boundary point* $(0,-1)$, *while the tangential (boundary) approach allows to gain a factor 2.*

2.4.3.3 Hölder Regularity

Let Ω be a smoothly bounded strictly pseudoconvex domain in $\mathbb{C}^n$. We have given above the proof due to Bedford and Taylor [BT76] that u_φ is Lipschitz on $\overline{\Omega}$ whenever φ is $\mathcal{C}^{1,1}$-smooth. In the same vein, these authors have shown that $u_\varphi \in \text{Lip}_\beta(\overline{\Omega})$ is Hölder continuous on $\overline{\Omega}$ with exponent

$$\beta = \begin{cases} \frac{1+\alpha}{2} & \text{if } \varphi \in \mathcal{C}^{1,\alpha}(\partial\Omega), \quad 0 \le \alpha \le 1 \\ \frac{\alpha}{2} & \text{if } \varphi \in \text{Lip}_\alpha(\partial\Omega), \quad 0 \le \alpha \le 1 \end{cases}$$

When Ω is merely weakly pseudoconvex, a similar result holds with a weaker exponent β when Ω is of "finite type" [Co97]. It has been moreover proved by Coman that this propagation of Hölder regularity characterizes finite type domains (see also [Li04]).

2.5 Dirichlet Problem in Domains of $\mathbb{C}^n$

In this section we apply Bedford–Taylor's result to show that the Perron–Bremermann envelope u_φ solves the Dirichlet problem $\mathrm{DirMA}(\Omega, \varphi)$. Since u_φ is not very regular, this requires to first extend the definition of the complex Monge–Ampère operator.

2.5.1 Domain of Definition of MA

Let $\varphi \in PSH(\Omega)$ be a plurisubharmonic function. When φ is smooth, the Monge–Ampère measure $MA(\varphi)$ is absolutely continuous with respect to the Euclidean Lebesgue measure dV,

$$MA(\varphi) = (dd^c\varphi)^n := c \det\left(\frac{\partial^2\varphi}{\partial z_j \partial \overline{z_k}}\right) dV,$$

for some normalizing constant $c > 0$. We would like to extend the definition of this operator and apply it to non smooth functions φ.

It is known that one can not define the Monge–Ampère measure $MA(\varphi)$ for any such function: Kiselman gives in [Kis83] an elementary example of a function $\varphi \in PSH(\mathbb{B})$ which is smooth but along some hyperplane H, hence $MA(\varphi)$ is well defined in $\mathbb{B} \setminus H$ but it has locally infinite mass near H.

Following Bedford and Taylor [BT82], we say that φ belongs to the domain of definition of the complex Monge–Ampère operator in Ω ($\varphi \in DomMA(\Omega)$) if for every $x \in \Omega$ and for every sequence φ_j of smooth and psh functions decreasing to φ in a neighborhood V_x of x, the sequence of positive measures $MA(\varphi_j)$ converges, in the weak sense of Radon measures, to a measure μ_φ independent of the sequence (φ_j). One then sets $MA(\varphi) := \mu_\varphi$.

Although this definition may seem cumbersome, this is precisely the way one usually computes derivatives in the sense of distributions. It is moreover motivated by the following result established by Bedford and Taylor in [BT82].

Theorem 2.15 *$PSH \cap L^\infty_{loc}(\Omega) \Subset DomMA(\Omega)$.*

Thus the complex Monge–Ampère operator is well defined for psh functions that are locally bounded, which is what we basically needs here since u_φ is continuous. It follows straightforwardly from the definition that the operator MA is continuous along decreasing sequences.

More involved is the continuity along increasing sequences which was also established by Bedford and Taylor in [BT82]. Note however that MA is discontinuous along non monotonic sequences. We propose one example as an exercise for the reader.

Exercise 2.16 *Set* $\varphi_j(z,w) = \frac{1}{2j}\log[1+|z^j+w^j|^2]$.

1) Verify that the functions φ_j *are smooth, psh, with* $MA(\varphi_j)=0$ *in* $\mathbb{C}^2$.
2) Show that (φ_j) *converges in* $L^1_{loc}(\mathbb{C}^2)$ *towards*

$$\varphi(z,w) = \log\max(1,|z|,|w|) \in PSH \cap L^{\infty}_{loc}(\mathbb{C}^2)$$

and verify that $MA(\varphi)$ *is the Lebesgue measure on the real torus* $S^1 \times S^1$.

When $n=2$, it was observed by Bedford and Taylor in [BT78] that one can define $MA(\varphi)$ as soon as $\nabla\varphi \in L^2_{loc}(\Omega)$. In this case $d\varphi \wedge d^c\varphi$ is well defined hence so is the current $\varphi dd^c\varphi$ (by integration by parts) and one can thus set

$$MA(\varphi) := dd^c(\varphi dd^c \varphi)$$

where the derivatives are taken in the sense of distributions (currents). It turns out in this case that if φ_j are smooth, psh, and decrease to φ, then φ_j converge to φ in the Sobolev norm $W^{1,2}_{loc}$. It was recently shown by Blocki [Bl04] that one can not make sense of $MA(\varphi)$ when $n=2$ and $\nabla\varphi \notin L^2_{loc}(\Omega)$.

2.5.2 The Comparison Principle

The comparison principle is one of the most effective tools in pluripotential theory. It is a non linear version of the classical maximum principle. The central result, again due to Bedford and Taylor [BT87] is the following:

Theorem 2.17 *Let* u, v *be locally bounded psh functions in a domain* $\Omega \Subset \mathbb{C}^n$. *Then*

$$\mathbf{1}_{\{u>v\}}(dd^c \max\{u,v\})^n = \mathbf{1}_{\{u>v\}}(dd^c u)^n,$$

in the sense of Borel measures in Ω.

Proof. Set $D := \{u > v\}$. Observe that if u is continuous then the set D is an open subset of Ω and $\max\{u,v\} = u$ in D. Therefore we have

$$(dd^c \max\{u,v\})^n = (dd^c u)^n,$$

weakly in the open set D, as desired.

The general case proceeds by approximation: one can approximate u from above by a decreasing sequence of psh continuous functions (by local convolutions) and it suffices to establish fine convergence results in order to pass to the limit. These convergence results are of course the hard technical part of the argument and will not be reproduced here. Let us simply mention that the key properties for these to hold is that u is *quasicontinuous*, i.e. it coincides with a continuous function on a set of arbitrary large size with respect to the Monge–Ampère measures involved. □

We derive from this identity two corollaries which are often called "maximum principle" in the literature.

Corollary 2.18 *Let* $\Omega \Subset \mathbb{C}^n$ *be a bounded domain and let* u, v *be locally bounded psh functions such that* $\liminf_{z\to\partial\Omega}(u(z)-v(z)) \geq 0$. *Then*

$$\int_{\{u<v\}} (dd^c v)^n \leq \int_{\{u<v\}} (dd^c u)^n.$$

Proof. Since $\{u-\varepsilon < v\} \nearrow \{u < v\}$ as $\varepsilon \searrow 0$, we can assume that $\liminf_{z\to\partial\Omega}(u(z)-v(z)) > \varepsilon > 0$. We can thus fix a compact subset $K \Subset \Omega$ such that $u(z)-v(z) > \varepsilon$ on $\Omega\backslash K$. Therefore $\max\{u,v\} = u$ on $\Omega\backslash K$.

We infer the following "mass conservation property",

$$\int_\Omega (dd^c \max\{u,v\})^n = \int_\Omega (dd^c u)^n.$$

Indeed set $w := \max\{u,v\}$ and observe that $(dd^c w)^n - (dd^c u)^n = dd^c S$ weakly in the sense of currents on Ω, where $S := w(dd^c w)^{n-1} - u(dd^c u)^{n-1}$. Since $w = u$ on $\Omega\backslash K$, it follows that $S = 0$ in the open set $\Omega\backslash K$ thus the support of the current $dd^c S$ is contained in K. Taking a smooth test function χ on Ω such that $\chi \equiv 1$ in a neighborhood of K, we conclude that $\int_\Omega dd^c S = \int_\Omega \chi dd^c S = \int_\Omega S \wedge dd^c \chi = 0$, since $dd^c\chi = 0$ on the support of current S.

The mass conservation property together with Theorem 2.17 yields

$$\begin{aligned}
\int_{\{u<v\}} (dd^c v)^n &= \int_{\{u<v\}} (dd^c \max\{u,v\})^n \\
&= \int_\Omega (dd^c \max\{u,v\})^n - \int_{\{u\geq v\}} dd^c \max\{u,v\})^n \\
&\leq \int_\Omega (dd^c u)^n - \int_{\{u>v\}} (dd^c \max\{u,v\})^n \\
&= \int_\Omega (dd^c u)^n - \int_{\{u>v\}} (dd^c u)^n = \int_{\{u\leq v\}} (dd^c u)^n.
\end{aligned}$$

We have thus shown that $\int_{\{u<v\}}(dd^c v)^n \leq \int_{\{u\leq v\}}(dd^c u)^n$. Replacing u by $u-\varepsilon$ and letting ε decrease to zero yields the desired inequality. □

Corollary 2.19 *Let u, v be locally bounded psh functions in a bounded domain $\Omega \Subset \mathbb{C}^n$ such that $\liminf_{z\to\partial\Omega}(u(z) - v(z)) \geq 0$. Then*

$$(dd^c u)^n \leq (dd^c v)^n \Longrightarrow v \leq u \text{ in } \Omega.$$

Proof. Define for $\varepsilon > 0$, $v_\varepsilon := v + \varepsilon\rho$, where $\rho(z) := \|z\|^2 - R^2$ is chosen so that $\rho < 0$ in Ω. Observe that $\{u < v_\varepsilon\} \Subset \{u < v\} \Subset \Omega$. It follows therefore from the previous corollary that

$$\int_{\{u<v_\varepsilon\}} (dd^c v_\varepsilon)^n \leq \int_{\{u<v_\varepsilon\}} (dd^c u)^n.$$

Since $(dd^c v_\varepsilon)^n \geq (dd^c v)^n + \varepsilon^n (dd^c \rho)^n > (dd^c u)^n$, we infer $\int_{\{u<v_\varepsilon\}} (dd^c \rho)^n = 0$. This means that the sets $\{u < v_\varepsilon\}$ all have Lebesgue measure zero, $\varepsilon > 0$. Since $\{u < v\} = \bigcup_{j\geq 1}\{u < v_{1/j}\}$, it follows that the set $\{u < v\}$ also has Lebesgue measure 0 so that $v \leq u$ in Ω by the submean value inequality. □

2.5.3 Characterization of Maximal Plurisubharmonic Functions

Theorem 2.20 *A function $u \in PSH \cap L^\infty_{loc}(\Omega)$ is maximal if and only if $MA(u) = 0$. In particular the Perron–Bremermann envelope u_φ satisfies $MA(u_\varphi) = 0$ hence it is the unique solution to the Dirichlet problem* DirMA(Ω, φ).

Proof. If $(dd^c u)^n = 0$ on Ω, it follows from the comparison principle that u is a maximal plurisubharmonic function on Ω.

Conversely assume that u is maximal on Ω and let $\mathbb{B} \Subset \Omega$ be an Euclidean ball. Let φ be the restriction of u to the boundary $\partial\mathbb{B}$. Since u is maximal, it coincides with the Perron-Bremmerman envelope $u = u_\varphi$ with respect to the domain $\mathbb{B}$.

Let (φ_j) be a decreasing sequence of $\mathcal{C}^2$-smooth functions on $\partial\mathbb{B}$ which converges to φ on the boundary $\partial\mathbb{B}$. We let the reader check that $u_j := u_{\varphi_j}$ decreases to $u = u_\varphi$. By Bedford–Taylor's result, u_j is $\mathcal{C}^{1,1}(\mathbb{B})$, hence it satisfies $(dd^c u_j)^n = 0$ on $\mathbb{B}$ by Lemma 2.21 below. Since the Monge–Ampère operator is continuous along decreasing sequences we infer $(dd^c u)^n = 0$ in $\mathbb{B}$. Since $\mathbb{B}$ was arbitrary this yields $(dd^c u)^n = 0$ in all of Ω. □

It remains to check that regular maximal functions have zero Monge–Ampère measure.

Lemma 2.21 *Let $u : \Omega \to \mathbb{R}$ be a maximal plurisubharmonic function. If u is $\mathcal{C}^{1,1}$-smooth then $MA(u) = 0$.*

Proof. It follows from Lemma 2.11 that u admits second derivatives at almost every point and that its Monge–Ampère measure $MA(u)$ is absolutely continuous with respect to Lebesgue measure, with density defined almost everywhere by $\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}\right)$. We are going to show that the latter is zero whenever defined.

The second order Taylor expansion of u at z_0 gives,

$$u(z_0+h) = \Re P(h) + \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0) h_j \bar{h}_k + o(\|h\|^2),$$

where

$$P(h) := u(z_0) + 2\sum_j \frac{\partial u}{\partial z_j}(z_0) h_j + \sum_{j,k} \frac{\partial^2 u}{\partial z_j \partial z_k}(z_0) h_j h_k.$$

Assume that $\det\left(\frac{\partial^2 u}{\partial z_j \partial \bar{z}_k}(z_0)\right) > 0$. Then there exists $c > 0$ and $r > 0$ small enough such that for $\|h\| = r$, we have $u(z_0+h) = \Re P(h) + c\|h\|^2 > \Re P(h)$. Therefore the function $v(z) := \Re P(z_0 + z)$ is a plurisubharmonic function such that $v(z_0) = u(z_0)$ and $v(z) < u(z)$ on the boundary of the ball $\mathbb{B}(z_0, r)$, which contradicts the fact that u is maximal on Ω. □

Remark 2.22 *One can similarly show that a psh function φ which belongs to the domain of definition of the complex Monge–Ampère operator is maximal if and only if $MA(\varphi) = 0$ [Bl04].*

http://www.springer.com/978-3-642-23668-6

Complex Monge-Ampère Equations and Geodesics in the
Space of Kähler Metrics
(Ed.)V. Guedj
2012, VIII, 310 p. 4 illus., Softcover
ISBN: 978-3-642-23668-6