Simplicial Methods for Operads and Algebraic Geometry

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Lecture 2

Trees as operads

In this lecture, we introduce convenient categories of trees that will be used for the definition of dendroidal sets. These categories are generalizations of the simplicial category $\Delta$ used to define simplicial sets. First we consider the case of planar trees and then the more general case of non-planar trees.

2.1 A formalism of trees

A tree is a non-empty connected finite graph with no loops. A vertex in a graph is called outer if it has only one edge attached to it. All the trees we will consider are rooted trees, i.e., equipped with a distinguished outer vertex called the output and a (possibly empty) set of outer vertices (not containing the output vertex) called the set of inputs.

When drawing trees, we will delete the output and input vertices from the picture. From now on, the term ‘vertex’ in a tree will always refer to a remaining vertex. Given a tree $T$, we denote by $V(T)$ the set of vertices of $T$ and by $E(T)$ the set of edges of $T$.

The edges attached to the deleted input vertices are called input edges or leaves; the edge attached to the deleted output vertex is called output edge or root. The rest of the edges are called inner edges. The root induces an obvious direction in the tree, ‘from the leaves towards the root’. If $v$ is a vertex of a finite rooted tree, we denote by $\text{out}(v)$ the unique outgoing edge and by $\text{in}(v)$ the set of incoming edges (note that $\text{in}(v)$ can be empty). The cardinality of $\text{in}(v)$ is called the valence of $v$, the element of $\text{out}(v)$ is the output of $v$, and the elements of $\text{in}(v)$ are the inputs of $v$.

As an example, consider the following picture of a tree:
The output vertex at the edge $a$ and the input vertices at $e$, $f$ and $c$ have been deleted. This tree has three vertices $r$, $v$ and $w$ of respective valences 3, 2, and 0. It also has three input edges or leaves, namely $e$, $f$ and $c$. The edges $b$ and $d$ are inner edges and the edge $a$ is the root. A tree with no vertices

whose input edge (which we denote by $e$) coincides with its output edge will be denoted by $\eta_e$, or simply by $\eta$.

**Definition 2.1.1.** A **planar rooted tree** is a rooted tree $T$ together with a linear ordering of $\text{in}(v)$ for each vertex $v$ of $T$.

The ordering of $\text{in}(v)$ for each vertex is equivalent to drawing the tree on the plane. When we draw a tree we will always put the root at the bottom. One drawback of drawing a tree on the plane is that it immediately becomes a planar tree; we thus may have many different ‘pictures’ for the same tree. For example, the two trees

are two different planar representations of the same tree.

### 2.2 Planar trees

Let $T$ be a planar rooted tree. Any such tree generates a non-$\Sigma$ operad, which we denote by $\Omega_p(T)$. The set of colours of $\Omega_p(T)$ is the set $E(T)$ of edges of $T$, and the operations are generated by the vertices of the tree. More explicitly, each vertex $v$ with input edges $e_1, \ldots, e_n$ and output edge $e$ defines an operation $v \in \Omega_p(T)(e_1, \ldots, e_n; e)$. The other operations are the unit operations and the operations obtained by compositions. This operad has the property that, for
all \(e_1, \ldots, e_n, e\), the set of operations \(\Omega_p(T)(e_1, \ldots, e_n; e)\) contains at most one element. For example, consider the same tree \(T\) pictured before:

![Diagram of a tree with vertices labeled as follows: from top to bottom, right to left are labels 6, 5, 4, 3, 2, 1, 0. The edges connect these vertices in a linear fashion.]

The operad \(\Omega_p(T)\) has six colours \(a, b, c, d, e,\) and \(f\). Then \(v \in \Omega_p(T)(e, f; b)\), \(w \in \Omega_p(T)(; d)\), and \(r \in \Omega_p(b, c, d; a)\) are the generators, while the other operations are the units \(1_a, 1_b, \ldots, 1_f\) and the operations obtained by compositions, namely

\[
 r(v, 1_c, w) = (r \circ_1 v) \circ_4 w = (r \circ_3 w) \circ_1 v \in \Omega_p(T)(e, f, c; a).
\]

This is a complete description of the operad \(\Omega_p(T)\).

**Definition 2.2.1.** The category of planar rooted trees \(\Omega_p\) is the full subcategory of the category of non-\(\Sigma\) coloured operads whose objects are \(\Omega_p(T)\) for any tree \(T\).

We can view \(\Omega_p\) as the category whose objects are planar rooted trees. The set of morphisms from a tree \(S\) to a tree \(T\) is given by the set of non-\(\Sigma\) coloured operad maps from \(\Omega_p(S)\) to \(\Omega_p(T)\). Observe that any morphism \(S \to T\) in \(\Omega_p\) is completely determined by its effect on the colours (i.e., edges).

The category \(\Omega_p\) extends the simplicial category \(\Delta\). Indeed, any \(n \geq 0\) defines a linear tree

![Diagram of a linear tree with vertices labeled as follows: from top to bottom are labels 6, 5, 4, 3, 2, 1, 0. The edges connect these vertices in a linear fashion.]  

with \(n + 1\) edges and \(n\) vertices \(v_1, \ldots, v_n\). We denote this tree by \([n]\) or \(L_n\). Any order-preserving map \(\{0, \ldots, n\} \to \{0, \ldots, m\}\) defines an arrow \([n] \to [m]\) in the category \(\Omega_p\). In this way, we obtain an embedding

\[
\Delta \xrightarrow{u} \Omega_p.
\]

This embedding is fully faithful. Moreover, it describes \(\Delta\) as a sieve (or ideal) in \(\Omega_p\), in the sense that for any arrow \(S \to T\) in \(\Omega_p\), if \(T\) is linear then so is \(S\). In the next sections we give a more explicit description of the morphisms in \(\Omega_p\).
2.2.1 Face maps

Let $T$ be a planar rooted tree and $b$ an inner edge in $T$. Let us denote by $T/b$ the tree obtained from $T$ by contracting $b$. Then there is a natural map $\partial_b : T/b \to T$ in $\Omega_p$, called the *inner face* map associated with $b$. This map is the inclusion on both the colours and the generating operations of $\Omega_p(T/b)$, except for the operation $u$, which is sent to $r \circ_b v$. Here $r$ and $v$ are the two vertices in $T$ at the two ends of $b$, and $u$ is the corresponding vertex in $T/b$, as in the picture:

$$
\begin{align*}
T/b & \quad \xrightarrow{\partial_b} \quad T \\
\begin{tikzpicture}
\node (a) at (0,0) [circle, draw] {$a$};
\node (b) at (1,1) [circle, draw] {$b$};
\node (c) at (1.5,1) [circle, draw] {$c$};
\node (d) at (2,0) [circle, draw] {$d$};
\node (e) at (2.5,1) [circle, draw] {$e$};
\node (f) at (3,0) [circle, draw] {$f$};
\node (w) at (3.5,1) [circle, draw] {$w$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (b) -- (d);
\draw (b) -- (e);
\draw (b) -- (f);
\end{tikzpicture}
\end{align*}
$$

Now let $T$ be a planar rooted tree and $v$ a vertex of $T$ with exactly one inner edge attached to it. Let $T/v$ be the tree obtained from $T$ by removing the vertex $v$ and all the outer edges. There is a face map associated to this operation, denoted $\partial_v : T/v \to T$, which is the inclusion both on the colours and on the generating operations of $\Omega_p(T/v)$. These types of face maps are called the *outer faces* of $T$. The following are two outer face maps:

$$
\begin{align*}
T/v & \quad \xrightarrow{\partial_v} \quad T \\
\begin{tikzpicture}
\node (a) at (0,0) [circle, draw] {$a$};
\node (b) at (1,1) [circle, draw] {$b$};
\node (c) at (1.5,1) [circle, draw] {$c$};
\node (d) at (2,0) [circle, draw] {$d$};
\node (e) at (2.5,1) [circle, draw] {$e$};
\node (w) at (3,0) [circle, draw] {$w$};
\draw (a) -- (b);
\draw (b) -- (c);
\draw (b) -- (d);
\draw (b) -- (e);
\draw (b) -- (w);
\end{tikzpicture}
\end{align*}
$$

Note that the possibility of removing the root vertex of $T$ is included in this definition. This situation can happen only if the root vertex is attached to exactly one inner edge, thus not every tree $T$ has an outer face induced by its root. There is another particular situation which requires special attention, namely the inclusion of the tree with no vertices $\eta$ into a tree with one vertex, called a *corolla*. In this case we get $n + 1$ face maps if the corolla has $n$ leaves. The operad $\Omega_p(\eta)$ consists of only one colour and the identity operation on it. Then a map of operads $\Omega_p(\eta) \to \Omega_p(T)$ is just a choice of an edge of $T$.

We will use the term *face map* to refer to an inner or outer face map.

2.2.2 Degeneracy maps

There is one more type of map that can be associated with a vertex $v$ of valence one in $T$ as follows. Let $T\setminus v$ be the tree obtained from $T$ by removing the vertex $v$ and merging the two edges incident to it into one edge $e$. Then there is a map
σ_v: T \rightarrow T \setminus v in Ω_p called the degeneracy map associated with v, which sends the colours e_1 and e_2 of Ω_p(T) to e, sends the generating operation v to id_e, and is the identity for the other colours and operations. It can be pictured like this:

![Diagram](image)

Face maps and degeneracy maps generate the whole category Ω_p. The following lemma is the generalization to Ω_p of the well-known fact that in Δ each arrow can be written as a composition of degeneracy maps followed by face maps. For the proof of this fact we refer the reader to Lemma 2.3.2, where we prove a similar statement in the category of non-planar trees.

**Lemma 2.2.2.** Any arrow f: A \rightarrow B in Ω_p decomposes (up to isomorphism) as

![Diagram](image)

where σ: A \rightarrow C is a composition of degeneracy maps and δ: C \rightarrow B is a composition of face maps. □

### 2.2.3 Dendroidal identities

In this section we are going to make explicit the relations between the generating maps (faces and degeneracies) of Ω_p. The identities that we obtain generalize the simplicial ones in the category Δ.

**Elementary face relations**

Let \partial_a: T/a \rightarrow T and \partial_b: T/b \rightarrow T be distinct inner faces of T. It follows that the inner faces \partial_b: (T/b)/a \rightarrow T/b and \partial_b: (T/a)/b \rightarrow T/a exist, we have (T/a)/b = (T/b)/a, and the following diagram commutes:

![Diagram](image)
Lecture 2. Trees as operads

Let $\partial_v: T/v \to T$ and $\partial_w: T/w \to T$ be distinct outer faces of $T$, and assume that $T$ has at least three vertices. Then the outer faces $\partial_w: (T/v)/w \to T/v$ and $\partial_v: (T/w)/v \to T/w$ also exist, $(T/v)/w = (T/w)/v$, and the following diagram commutes:

\[
\begin{array}{ccc}
(T/v)/w & \xrightarrow{\partial_w} & T/v \\
\downarrow{\partial_v} & & \downarrow{\partial_v} \\
T/w & \xrightarrow{\partial_w} & T.
\end{array}
\]

In case that $T$ has only two vertices, there is a similar commutative diagram involving the inclusion of $\eta$ into the $n$-th corolla.

The last remaining case is when we compose an inner face with an outer one in any order. There are several possibilities and in all of them we suppose that $\partial_v: T/v \to T$ is an outer face and $\partial_e: T/e \to T$ is an inner face.

- If in $T$ the edge $e$ is not adjacent to the vertex $v$, then the outer face $\partial_v: (T/e)/v \to T/e$ and the inner face $\partial_e: (T/v)/e \to T/v$ exist, $(T/e)/v = (T/v)/e$, and the following diagram commutes:

\[
\begin{array}{ccc}
(T/v)/e & \xrightarrow{\partial_e} & T/v \\
\downarrow{\partial_v} & & \downarrow{\partial_v} \\
T/e & \xrightarrow{\partial_e} & T.
\end{array}
\]

- Suppose that in $T$ the inner edge $e$ is adjacent to the vertex $v$ and denote the other adjacent vertex to $e$ by $w$. Observe that $v$ and $w$ contribute a vertex $v \circ_e w$ or $w \circ_e v$ to $T/e$. Let us denote this vertex by $z$. Then the outer face $\partial_z: (T/e)/z \to T/e$ exists if and only if the outer face $\partial_w: (T/v)/w \to T/v$ exists, and in this case $(T/e)/z = (T/v)/w$. Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
(T/v)/w & \xrightarrow{\partial_w} & (T/e)/z \xrightarrow{\partial_e} T/e \\
\downarrow{\partial_v} & & \downarrow{\partial_v} \\
T/v & \xrightarrow{\partial_v} & T.
\end{array}
\]

It follows that we can write $\partial_v \partial_w = \partial_e \partial_z$, where $z = v \circ_e w$ if $v$ is ‘closer’ to the root of $T$ or $z = w \circ_e v$ if $w$ is ‘closer’ to the root of $T$.

Elementary degeneracy relations

Let $\sigma_v: T \to T\vline v$ and $\sigma_w: T \to T\vline w$ be two degeneracies of $T$. Then the degeneracies $\sigma_v: T\vline w \to (T\vline w)\vline v$ and $\sigma_w: T\vline v \to (T\vline v)\vline w$ exist, we have
2.3. Non-planar trees

\[(T \setminus v) \setminus w = (T \setminus w) \setminus v,\] and the following diagram commutes:

\[
\begin{array}{ccc}
T & \xrightarrow{\sigma_v} & T \setminus v \\
\downarrow{\sigma_w} & & \downarrow{\sigma_w} \\
T \setminus w & \xrightarrow{\sigma_v} & (T \setminus v) \setminus w.
\end{array}
\]

**Combined relations**

Let \(\sigma_v: T \rightarrow T \setminus v\) be a degeneracy and \(\partial: T' \rightarrow T\) be a face map such that \(\sigma_v: T' \rightarrow T' \setminus v\) makes sense (i.e., \(T'\) still contains \(v\) and its two adjacent edges as a subtree). Then there exists an induced face map \(\partial: T' \setminus v \rightarrow T \setminus v\) determined by the same vertex or edge as \(\partial: T' \rightarrow T\). Moreover, the following diagram commutes:

\[
\begin{array}{ccc}
T & \xrightarrow{\sigma_v} & T \setminus v \\
\uparrow{\partial} & & \uparrow{\partial} \\
T' & \xrightarrow{\sigma_v} & T' \setminus v.
\end{array}
\]

Let \(\sigma_v: T \rightarrow T \setminus v\) be a degeneracy and \(\partial: T' \rightarrow T\) be a face map induced by one of the adjacent edges to \(v\) or the removal of \(v\), if that is possible. It follows that \(T' = T \setminus v\) and the composition

\[
T \setminus v \xrightarrow{\partial} T \xrightarrow{\sigma_v} T \setminus v
\]

is the identity map \(\text{id}_{T \setminus v}\).

### 2.3 Non-planar trees

Any non-planar tree \(T\) generates a (symmetric) coloured operad \(\Omega(T)\). Similarly as in the case of planar trees, the set of colours of \(\Omega(T)\) is the set of edges \(E(T)\) of \(T\). The operations are generated by the vertices of the tree, and the symmetric group on \(n\) letters \(\Sigma_n\) acts on each operation with \(n\) inputs by permuting the order of its inputs. Each vertex \(v\) of the tree with output edge \(e\) and a numbering of its input edges \(e_1, \ldots, e_n\) defines an operation \(v \in \Omega(e_1, \ldots, e_n; e)\). The other operations are the unit operations and the operations obtained by compositions and the action of the symmetric group. For example, consider the tree

\[
\begin{align*}
\text{\includegraphics[width=0.2\textwidth]{tree.png}}
\end{align*}
\]
The operad $\Omega(T)$ has six colours $a, b, c, d, e,$ and $f.$ The generating operations are the same as the generating operations of $\Omega_p(T).$ All the operations of $\Omega_p(T)$ are operations of $\Omega(T),$ but there are more operations in $\Omega(T)$ obtained by the action of the symmetric group. For example if $\sigma$ is the transposition of two elements of $\Sigma_2,$ we have an operation $v \circ \sigma \in \Omega(f, c; b).$ Similarly if $\sigma$ is the transposition of $\Sigma_3$ that interchanges the first and third elements, then there is an operation $r \circ \sigma \in \Omega(d, c, b; a).

More formally, if $T$ is any tree, then $\Omega(T) = \Sigma(\Omega_p(T)),$ where $\overline{T}$ is a planar representative of $T.$ In fact, a choice of a planar structure on $T$ is precisely a choice of generators for $\Omega(T).

**Definition 2.3.1.** The category of rooted trees $\Omega$ is the full subcategory of the category of coloured operads whose objects are $\Omega(T)$ for any tree $T.$

We can view $\Omega$ as the category whose objects are rooted trees. The set of morphisms from a tree $S$ to a tree $T$ is given by the set of coloured operad maps from $\Omega(S)$ to $\Omega(T).$ Note that any morphism $S \rightarrow T$ in $\Omega$ is completely determined by its effect on the colours (i.e., edges).

The morphisms in $\Omega$ are generated by faces and degeneracies (as in the planar case) and also by (non-planar) isomorphisms.

**Lemma 2.3.2.** Any arrow $f : S \rightarrow T$ in $\Omega$ decomposes as

$$
\begin{array}{ccc}
S & \xrightarrow{f} & T \\
\downarrow{\sigma} & & \downarrow{\delta} \\
S' & \xrightarrow{\varphi} & T'
\end{array}
$$

where $\sigma : S \rightarrow S'$ is a composition of degeneracy maps, $\varphi : S' \rightarrow T'$ is an isomorphism, and $\delta : T' \rightarrow T$ is a composition of face maps.

**Proof.** We proceed by induction on the sum of the number of vertices of $S$ and $T.$ If $T$ and $S$ have no vertices, then $T = S = \eta$ and $f$ is the identity. Note that, without loss of generality, we can assume that $f$ sends the root of $S$ to the root of $T;$ otherwise we can factor it as a map $S \rightarrow T'$ that preserves the root followed by a map $T' \rightarrow T$ that is a composition of outer faces. Also, we can assume that $f$ is an epimorphism on the leaves since, if this is not the case, $f$ factors as $S \rightarrow T/v \xrightarrow{\partial_v} T,$ where $v$ is the vertex below the leaf in $T$ that is not in the image of $f.$

If $a$ and $b$ are edges of $S$ such that $f(a) = f(b),$ then $a$ and $b$ must be on the same (linear) branch of $S$ and $f$ sends intermediate vertices to identities.

Since $f$ is a map of coloured operads, we can factor it in a unique way as a surjection followed by an injection on the colours. This corresponds to a factorization in $\Omega,$

$$
\begin{array}{ccc}
S & \xrightarrow{\psi} & S' \\
& \xrightarrow{\xi} & T
\end{array}
$$
where $\psi$ is a composition of degeneracies and $\xi$ is bijective on leaves, sends the root of $S'$ to the root of $T$, and is injective on the colours (by the previous observations).

If $\xi$ is surjective on colours, then $\xi$ is an isomorphism. If $\xi$ is not surjective, then there is an edge $e$ in $T$ not in the image of $\xi$. Since $e$ is an internal edge (not a leaf), $\xi$ factors as

$$S' \xrightarrow{\xi'} T/e \xrightarrow{\partial_e} T.$$ 

Now we continue by induction on the map $\xi'$.

In general, limits and colimits do not exist in the category $\Omega$; for example, $\Omega$ lacks sums and products. However, certain pushouts do exist in $\Omega$, as expressed in the following lemma:

**Lemma 2.3.3.** Let $f: R \rightarrow S$ and $g: R \rightarrow T$ be two surjective maps in $\Omega$. Then the pushout

\[
\begin{array}{ccc}
R & \xrightarrow{f} & S \\
\downarrow{g} & & \downarrow{\text{Id}_S} \\
T & \xrightarrow{\text{Id}_T} & P
\end{array}
\]

exists in $\Omega$.

**Proof.** The maps $f$ and $g$ can each be written as a composition of an isomorphism and a sequence of degeneracy maps by Lemma 2.3.2. Since pushout squares can be pasted together to get larger pushout squares, it thus suffices to prove the lemma in the case where $f$ and $g$ are degeneracy maps given by unary vertices $v$ and $w$ in $R$, i.e., $f: R \rightarrow S$ is $\sigma_v: R \rightarrow R \setminus v$ and $g: R \rightarrow T$ is $\sigma_w: R \rightarrow R \setminus w$. If $v = w$, then the following diagram is a pushout:

\[
\begin{array}{ccc}
R & \xrightarrow{\sigma_v} & R \setminus v \\
\downarrow{\sigma_v} & & \downarrow{\text{Id}_{R \setminus v}} \\
R \setminus v & \xrightarrow{\text{Id}_{R \setminus v}} & R \setminus v.
\end{array}
\]

If $v \neq w$, then the commutative square

\[
\begin{array}{ccc}
R & \xrightarrow{\sigma_v} & R \setminus v \\
\downarrow{\sigma_v} & & \downarrow{\text{Id}_{R \setminus v}} \\
R \setminus w & \xrightarrow{\sigma_v} & (R \setminus v) \setminus w = (R \setminus w) \setminus v
\end{array}
\]

is also a pushout, as one easily checks. \qed
2.3.1 Dendroidal identities with isomorphisms

The dendroidal identities for the category $\Omega$ are the same as for the category $\Omega_p$ plus some more relations involving the isomorphisms in $\Omega$. As an example, we give the following relation, that involves inner faces and isomorphisms. Let $T$ be a tree with an inner edge $a$ and let $f: T \to T'$ be a (non-planar) isomorphism. Then the trees $T/a$ and $T'/b$ exist, where $b = f(a)$, the map $f$ restricts to an isomorphism $f: T/a \to T'/b$, and the following diagram commutes:

\[
\begin{array}{ccc}
T/a & \xrightarrow{\partial_a} & T \\
\downarrow f & & \downarrow f \\
T'/b & \xrightarrow{\partial_b} & T'.
\end{array}
\]

Similar relations hold for outer faces and degeneracies.

2.3.2 Isomorphisms along faces and degeneracies

For any tree $T$ in $\Omega$, let $P(T)$ be the set of planar structures of $T$. Note that $P(T) \neq \emptyset$ for every tree $T$. Thus, the category $\Omega$ is equivalent to the category $\Omega'$ whose objects are planar trees, i.e., pairs $(T, p)$ where $T$ is an object of $\Omega$ and $p \in P(T)$, and whose morphisms are given by

\[\Omega'((T, p), (T', p')) = \Omega(T, T').\]

A morphism $\varphi: (T, p) \to (T', p')$ in $\Omega'$ is called planar if, when we pull back the planar structure $p'$ on $T'$ to one on $T$ along $\varphi$, then it coincides with $p$. Using this equivalent formulation of $\Omega$, the category $\Omega_p$ is then the subcategory of $\Omega$ consisting of the same objects and planar maps only, i.e., compositions of faces and degeneracies. In $\Omega_p$, the only automorphisms are identities.

If $\delta: T \to S$ is a composition of faces and $\alpha: S \to S'$ is an isomorphism, there is a factorization

\[T \xrightarrow{\delta} S \xrightarrow{\alpha} S',\]

where $\delta'$ is again a composition of faces and $\alpha'$ is an isomorphism. This factorization is unique if one fixes some conventions, e.g., one takes the objects of $\Omega$ to be planar trees, and takes faces and degeneracies to be planar maps. Similarly, isomorphisms can be pushed forward and pulled back along a composition of degeneracies. Let $\sigma: T \to S$ be a composition of degeneracies and $\alpha: S \to S'$ and
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\( \beta : T \rightarrow T' \) be two isomorphisms. Then there are factorizations

\[
\begin{array}{ccc}
T & \xrightarrow{\sigma} & S \\
\downarrow{\alpha'} & & \downarrow{\alpha} \\
T' & \xrightarrow{\sigma'} & S'
\end{array}
\quad \quad
\begin{array}{ccc}
T & \xrightarrow{\sigma} & S \\
\downarrow{\beta} & & \downarrow{\beta'} \\
T' & \xrightarrow{\sigma''} & S'
\end{array}
\]

where \( \alpha' \) and \( \beta' \) are isomorphisms and \( \sigma' \) and \( \sigma'' \) are compositions of degeneracies.

Thus, any arrow in \( \Omega \) can be written in the form \( \delta \sigma \alpha \) or \( \delta \alpha \sigma \) with \( \delta \) a composition of faces, \( \sigma \) a composition of degeneracies, and \( \alpha \) an isomorphism.

2.3.3 The presheaf of planar structures

Let \( P : \Omega^{\text{op}} \rightarrow \text{Sets} \) be the presheaf on \( \Omega \) that sends each tree to its set of planar structures. Observe that \( P(T) \) is a torsor under \( \text{Aut}(T) \) for every tree \( T \), where \( \text{Aut}(T) \) denotes the set of automorphisms of \( T \). Recall that the category of elements \( \Omega/P \) is the category whose objects are pairs \( (T, x) \) with \( x \in P(T) \). A morphism between two objects \( (T, x) \) and \( (S, y) \) is given by a morphism \( f : T \rightarrow S \) in \( \Omega \) such that \( P(f)(y) = x \). Hence, \( \Omega/P = \Omega_p \) and we have a projection \( v : \Omega_p \rightarrow \Omega \).

There is a commutative triangle

\[
\begin{array}{ccc}
\Delta & \xrightarrow{u} & \Omega_p \\
\downarrow{i} & & \downarrow{v} \\
\Omega & & \Omega_p
\end{array}
\]

where \( i \) is the fully faithful embedding of \( \Delta \) into \( \Omega \) which sends the object \( [n] \) in \( \Delta \) to the linear tree \( L_n \) with \( n \) vertices and \( n + 1 \) edges for every \( n \geq 0 \).

2.3.4 Relation with the simplicial category

We have seen that both the categories \( \Omega \) and \( \Omega_p \) extend the category \( \Delta \), by viewing the objects of \( \Delta \) as linear trees. In fact, it is possible to obtain \( \Delta \) as a comma category of \( \Omega \) or of \( \Omega_p \) as follows.

Let \( \eta \) be the tree in \( \Omega \) consisting of no vertices and one edge, and let \( \eta_p \) be the planar representative of \( \eta \) in \( \Omega_p \). If \( T \) is any tree in \( \Omega \), then \( \Omega(T, \eta) \) consists of only one morphism if \( T \) is a linear tree, or it is the empty set otherwise. The same happens for \( \Omega_p \) and \( \eta_p \). Thus, \( \Omega/\eta = \Omega_p/\eta_p = \Delta \).
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