

# Contents

|          |                                                                                                |           |
|----------|------------------------------------------------------------------------------------------------|-----------|
| <b>1</b> | <b>Introduction to Oscillation Theory</b>                                                      | <b>1</b>  |
| 1.1      | Introduction                                                                                   | 1         |
| 1.2      | Nonoscillation of Autonomous Delay Equations with Positive Coefficients                        | 1         |
| 1.3      | Nonlinear Equations of Mathematical Biology                                                    | 9         |
| 1.3.1    | Linearization of Nonlinear Delay Equations                                                     | 9         |
| 1.3.2    | Hutchinson's Equation                                                                          | 10        |
| 1.3.3    | Lasota-Ważewska Equation                                                                       | 12        |
| 1.3.4    | Nicholson's Blowflies Equation                                                                 | 13        |
| 1.3.5    | Mackey-Glass Equations                                                                         | 15        |
| 1.4      | Impulsive Equations                                                                            | 16        |
| 1.5      | Some Other Classes of Equations                                                                | 18        |
| 1.6      | Discussion and Open Problems                                                                   | 20        |
| <b>2</b> | <b>Scalar Delay Differential Equations on Semiaxes</b>                                         | <b>23</b> |
| 2.1      | Introduction                                                                                   | 23        |
| 2.2      | Preliminaries                                                                                  | 24        |
| 2.3      | Nonoscillation Criteria                                                                        | 25        |
| 2.4      | Comparison Theorems                                                                            | 28        |
| 2.5      | Nonoscillation Conditions, Part 1                                                              | 32        |
| 2.6      | Nonoscillation Conditions, Part 2                                                              | 37        |
| 2.7      | Oscillation Conditions                                                                         | 43        |
| 2.8      | Estimations of Solutions                                                                       | 46        |
| 2.9      | Positivity of Solutions                                                                        | 49        |
| 2.10     | Slowly Oscillating Solutions for Delay Differential Equations                                  | 51        |
| 2.11     | Stability and Nonoscillation                                                                   | 52        |
| 2.12     | Discussion and Open Problems                                                                   | 52        |
| <b>3</b> | <b>Scalar Delay Differential Equations on Semiaxis with Positive and Negative Coefficients</b> | <b>59</b> |
| 3.1      | Introduction                                                                                   | 59        |
| 3.2      | Nonoscillation Criteria                                                                        | 59        |

|          |                                                                                        |            |
|----------|----------------------------------------------------------------------------------------|------------|
| 3.3      | Nonoscillation Conditions, Part 1 . . . . .                                            | 66         |
| 3.4      | Nonoscillation Conditions, Part 2 . . . . .                                            | 71         |
| 3.5      | Equations with an Oscillatory Coefficient . . . . .                                    | 77         |
| 3.6      | Discussion and Open Problems . . . . .                                                 | 79         |
| <b>4</b> | <b>Oscillation of Equations with Distributed Delays . . . . .</b>                      | <b>83</b>  |
| 4.1      | Introduction . . . . .                                                                 | 83         |
| 4.2      | Preliminaries . . . . .                                                                | 84         |
| 4.3      | Existence of a Positive Solution—General Results . . . . .                             | 86         |
| 4.4      | Comparison Theorems . . . . .                                                          | 92         |
| 4.5      | Nonoscillation Criteria for Some Autonomous Integrodifferential<br>Equations . . . . . | 97         |
| 4.6      | Explicit Nonoscillation and Oscillation Conditions . . . . .                           | 101        |
| 4.7      | Slowly Oscillating Solutions . . . . .                                                 | 107        |
| 4.8      | Equations with Positive and Negative Coefficients . . . . .                            | 108        |
| 4.9      | Discussion and Open Problems . . . . .                                                 | 118        |
| <b>5</b> | <b>Scalar Advanced and Mixed Differential Equations on Semiaxes . . . . .</b>          | <b>123</b> |
| 5.1      | Introduction . . . . .                                                                 | 123        |
| 5.2      | Advanced Equations . . . . .                                                           | 123        |
| 5.3      | Mixed Equations with Positive Coefficients . . . . .                                   | 132        |
| 5.4      | Mixed Equation with Negative Coefficients . . . . .                                    | 134        |
| 5.5      | Positive Delay Term, Negative Advanced Term . . . . .                                  | 135        |
| 5.6      | Negative Delay Term, Positive Advanced Term . . . . .                                  | 141        |
| 5.7      | Discussion and Open Problems . . . . .                                                 | 144        |
| <b>6</b> | <b>Neutral Differential Equations . . . . .</b>                                        | <b>149</b> |
| 6.1      | Introduction and Preliminaries . . . . .                                               | 149        |
| 6.2      | Nonoscillation Criteria . . . . .                                                      | 151        |
| 6.3      | Efficient Nonoscillation Conditions . . . . .                                          | 156        |
| 6.4      | Explicit Oscillation Conditions . . . . .                                              | 160        |
| 6.5      | Positivity of Solutions . . . . .                                                      | 164        |
| 6.6      | Slowly Oscillating Solutions . . . . .                                                 | 165        |
| 6.7      | Neutral Equations with Positive and Negative Coefficients . . . . .                    | 166        |
| 6.8      | Discussion and Open Problems . . . . .                                                 | 168        |
| <b>7</b> | <b>Second-Order Delay Differential Equations . . . . .</b>                             | <b>171</b> |
| 7.1      | Introduction . . . . .                                                                 | 171        |
| 7.2      | Preliminaries . . . . .                                                                | 171        |
| 7.3      | Nonoscillation Criteria . . . . .                                                      | 172        |
| 7.4      | Comparison Theorems . . . . .                                                          | 176        |
| 7.5      | Explicit Nonoscillation and Oscillation Conditions . . . . .                           | 182        |
| 7.6      | Slowly Oscillating Solutions . . . . .                                                 | 187        |
| 7.7      | Existence of a Positive Solution . . . . .                                             | 188        |
| 7.8      | Discussion and Open Problems . . . . .                                                 | 190        |

|           |                                                                                             |            |
|-----------|---------------------------------------------------------------------------------------------|------------|
| <b>8</b>  | <b>Second-Order Delay Differential Equations with Damping Terms . .</b>                     | <b>193</b> |
| 8.1       | Introduction . . . . .                                                                      | 193        |
| 8.2       | Preliminaries . . . . .                                                                     | 193        |
| 8.3       | Nonoscillation Criteria . . . . .                                                           | 195        |
| 8.4       | Comparison Theorems . . . . .                                                               | 200        |
| 8.5       | Explicit Nonoscillation Conditions . . . . .                                                | 203        |
| 8.6       | Discussion and Open Problems . . . . .                                                      | 204        |
| <b>9</b>  | <b>Vector Delay Differential Equations . . . . .</b>                                        | <b>207</b> |
| 9.1       | Introduction . . . . .                                                                      | 207        |
| 9.2       | Preliminaries . . . . .                                                                     | 208        |
| 9.3       | Main Results . . . . .                                                                      | 210        |
| 9.4       | Comparison Results . . . . .                                                                | 214        |
| 9.5       | Higher-Order Scalar Delay Differential Equations . . . . .                                  | 217        |
| 9.6       | Positivity and Solution Estimates . . . . .                                                 | 220        |
| 9.7       | Positive Solutions and Stability . . . . .                                                  | 223        |
| 9.8       | Systems of Differential Equations with a Distributed Delay . . . .                          | 229        |
| 9.8.1     | Nonnegativity of Fundamental Matrices . . . . .                                             | 229        |
| 9.8.2     | Comparison Results and Positivity of Solutions . . . . .                                    | 232        |
| 9.8.3     | Solution Estimates . . . . .                                                                | 233        |
| 9.8.4     | Nonoscillation and Stability . . . . .                                                      | 236        |
| 9.9       | Discussion and Open Problems . . . . .                                                      | 238        |
| <b>10</b> | <b>Linearization Methods for Nonlinear Equations with a Distributed<br/>Delay . . . . .</b> | <b>241</b> |
| 10.1      | Introduction . . . . .                                                                      | 241        |
| 10.2      | Preliminaries . . . . .                                                                     | 241        |
| 10.3      | Linearized Oscillation . . . . .                                                            | 244        |
| 10.4      | Applications . . . . .                                                                      | 248        |
| 10.4.1    | Logistic Equation . . . . .                                                                 | 249        |
| 10.4.2    | Lasota-Ważewska Equation . . . . .                                                          | 250        |
| 10.4.3    | Nicholson's Blowflies Equation . . . . .                                                    | 253        |
| 10.5      | "Mean Value Theorem" for Equations with a Distributed Delay .                               | 258        |
| 10.6      | Discussion and Open Problems . . . . .                                                      | 261        |
| <b>11</b> | <b>Nonlinear Models—Modifications of Delay Logistic Equations . . .</b>                     | <b>263</b> |
| 11.1      | Introduction . . . . .                                                                      | 263        |
| 11.2      | Generalized Logistic Equation with Several Delays . . . . .                                 | 265        |
| 11.2.1    | Preliminaries . . . . .                                                                     | 265        |
| 11.2.2    | Sublinear Case $\alpha_k < 1$ , $k = 1, \dots, m$ . . . . .                                 | 266        |
| 11.2.3    | Superlinear Case $\alpha_k > 1$ , $k = 1, \dots, m$ . . . . .                               | 269        |
| 11.2.4    | Mixed Cases . . . . .                                                                       | 272        |
| 11.2.5    | Generalized Logistic Equation—Main Results . . . . .                                        | 275        |
| 11.3      | Multiplicative Delay Logistic Equation . . . . .                                            | 277        |
| 11.3.1    | Preliminaries . . . . .                                                                     | 277        |
| 11.3.2    | Nonoscillation Criteria . . . . .                                                           | 278        |

|           |                                                                                                       |            |
|-----------|-------------------------------------------------------------------------------------------------------|------------|
| 11.3.3    | Multiplicative Logistic Equation—Main Results . . . . .                                               | 281        |
| 11.4      | Discussion and Open Problems . . . . .                                                                | 282        |
| <b>12</b> | <b>First-Order Linear Delay Impulsive Differential Equations . . . . .</b>                            | <b>285</b> |
| 12.1      | Introduction . . . . .                                                                                | 285        |
| 12.2      | Preliminaries . . . . .                                                                               | 286        |
| 12.3      | Nonoscillation Criteria for Impulsive Equations . . . . .                                             | 287        |
| 12.4      | Explicit Nonoscillation Tests and Comparison Theorems . . . . .                                       | 292        |
| 12.5      | Reduction to Equations Without Impulses . . . . .                                                     | 295        |
| 12.6      | Impulsive Equations with a Distributed Delay . . . . .                                                | 296        |
| 12.7      | Discussion and Open Problems . . . . .                                                                | 299        |
| <b>13</b> | <b>Second-Order Linear Delay Impulsive Differential Equations . . . . .</b>                           | <b>301</b> |
| 13.1      | Introduction . . . . .                                                                                | 301        |
| 13.2      | Preliminaries . . . . .                                                                               | 302        |
| 13.3      | Nonoscillation Criteria . . . . .                                                                     | 303        |
| 13.4      | Comparison Theorems . . . . .                                                                         | 308        |
| 13.5      | Explicit Nonoscillation and Oscillation Conditions . . . . .                                          | 309        |
| 13.6      | Impulsive Equations with Damping Terms . . . . .                                                      | 315        |
| 13.7      | Discussion and Open Problems . . . . .                                                                | 318        |
| <b>14</b> | <b>Linearized Oscillation Theory for Nonlinear Delay Impulsive<br/>Equations . . . . .</b>            | <b>319</b> |
| 14.1      | Introduction . . . . .                                                                                | 319        |
| 14.2      | Preliminaries . . . . .                                                                               | 319        |
| 14.3      | Oscillation and Nonoscillation . . . . .                                                              | 321        |
| 14.4      | Applications to Equations of Mathematical Biology . . . . .                                           | 329        |
| 14.4.1    | Logistic Equation: Theoretical Results . . . . .                                                      | 329        |
| 14.4.2    | Logistic Equation: Numerical Simulations . . . . .                                                    | 333        |
| 14.4.3    | Generalized Lasota-Ważewska Equation . . . . .                                                        | 334        |
| 14.5      | Discussion and Open Problems . . . . .                                                                | 336        |
| <b>15</b> | <b>Maximum Principles and Nonoscillation Intervals . . . . .</b>                                      | <b>339</b> |
| 15.1      | Introduction . . . . .                                                                                | 339        |
| 15.2      | Preliminaries . . . . .                                                                               | 340        |
| 15.3      | Maximum Principles in the Case of Positive Volterra<br>Operator $(-B)$ . . . . .                      | 346        |
| 15.4      | Nonoscillation and Positivity of Green's Functions for Positive<br>Volterra Operator $B$ . . . . .    | 350        |
| 15.5      | Nonoscillation on the Semiaxis . . . . .                                                              | 356        |
| 15.6      | Positivity Tests for Green's Functions Through Choice of $v(t)$ . . . . .                             | 357        |
| 15.7      | The Generalized Periodic Problem for Positive Volterra<br>Operator $B$ . . . . .                      | 360        |
| 15.8      | Regular Behavior of the Green's Function to a One-Point<br>Boundary Value Problem . . . . .           | 362        |
| 15.9      | Positivity of Green's Functions for Equations Including<br>Difference of Positive Operators . . . . . | 363        |

|                   |                                                                                  |            |
|-------------------|----------------------------------------------------------------------------------|------------|
| 15.10             | Positivity of the Cauchy and Green's Functions . . . . .                         | 367        |
| 15.11             | Equations with an Oscillating Coefficient . . . . .                              | 376        |
| 15.12             | Positivity of the Cauchy Function and Exponential Stability . . .                | 384        |
| 15.13             | General Boundary Value Problems . . . . .                                        | 388        |
| 15.14             | Discussion and Open Problems . . . . .                                           | 391        |
| <b>16</b>         | <b>Systems of Functional Differential Equations on Finite Intervals . .</b>      | <b>399</b> |
| 16.1              | Introduction . . . . .                                                           | 399        |
| 16.2              | Nonnegativity and Nonpositivity of Green's Matrices . . . . .                    | 401        |
| 16.3              | Positivity of the $n$ -th Row of the Cauchy Matrix . . . . .                     | 408        |
| 16.4              | Positivity of the Fixed $n$ -th Row of Green's Matrices . . . . .                | 417        |
| 16.5              | Nonpositivity Conditions for the $n$ -th Row of Green's Matrices .               | 420        |
| 16.6              | Discussion and Open Problems . . . . .                                           | 426        |
| <b>17</b>         | <b>Nonoscillation Intervals for <math>n</math>-th-Order Equations . . . . .</b>  | <b>429</b> |
| 17.1              | Introduction . . . . .                                                           | 429        |
| 17.2              | Homogeneous Functional Differential Equations of the $n$ -th<br>Order . . . . .  | 430        |
| 17.3              | Wronskian of the Fundamental System . . . . .                                    | 432        |
| 17.4              | Nonoscillation of Functional Differential Equations . . . . .                    | 434        |
| 17.5              | Nonoscillation and Regular Behavior of Green's Functions . . .                   | 437        |
| 17.6              | Tests for Differential Equations with Deviating Arguments . . .                  | 444        |
| 17.7              | Discussion and Open Problems . . . . .                                           | 451        |
| <b>Appendix A</b> | <b>Useful Theorems from Analysis . . . . .</b>                                   | <b>455</b> |
| A.1               | Vector Spaces . . . . .                                                          | 455        |
| A.2               | Functional Spaces . . . . .                                                      | 456        |
| A.3               | Sets in Functional Spaces . . . . .                                              | 458        |
| A.4               | Linear Operators in Functional Spaces . . . . .                                  | 458        |
| A.5               | Nonlinear Operators . . . . .                                                    | 462        |
| A.6               | Gronwall-Bellman and Coppel Inequalities . . . . .                               | 463        |
| <b>Appendix B</b> | <b>Existence and Uniqueness Theorems, Solution<br/>Representations . . . . .</b> | <b>465</b> |
| B.1               | Linear Functional Differential Equations . . . . .                               | 465        |
| B.1.1             | Differential Equations with Several Concentrated Delays .                        | 465        |
| B.1.2             | Mixed Equations with an Infinite Number of Delays . . .                          | 468        |
| B.1.3             | Equations with a Distributed Delay . . . . .                                     | 472        |
| B.1.4             | Equations of Neutral Type . . . . .                                              | 475        |
| B.1.5             | Higher-Order Scalar Delay Differential Equations . . . .                         | 477        |
| B.2               | Estimations of the Fundamental Matrix . . . . .                                  | 480        |
| B.3               | Nonlinear Delay Differential Equations . . . . .                                 | 482        |
| B.4               | Linear Delay Impulsive Differential Equations . . . . .                          | 490        |
| B.4.1             | First-Order Impulsive Equations . . . . .                                        | 490        |
| B.4.2             | Second-Order Impulsive Equations . . . . .                                       | 494        |
| B.5               | Bohl-Perron Theorems . . . . .                                                   | 499        |
| <b>References</b> | <b>. . . . .</b>                                                                 | <b>503</b> |
| <b>Index</b>      | <b>. . . . .</b>                                                                 | <b>517</b> |