

---

# Contents

<b>1</b>	<b>Introduction</b> .....	1
<b>2</b>	<b>Ranks and Cranks, Part I</b> .....	9
2.1	Introduction .....	9
2.2	Proof of Entry 2.1.1 .....	17
2.3	Background for Entries 2.1.2 and 2.1.4 .....	18
2.4	Proof of Entry 2.1.2 .....	24
2.5	Proof of Entry 2.1.4 .....	32
2.6	Proof of Entry 2.1.5 .....	33
<b>3</b>	<b>Ranks and Cranks, Part II</b> .....	45
3.1	Introduction .....	45
3.2	Preliminary Results .....	47
3.3	The 2-Dissection for $F(q)$ .....	48
3.4	The 3-Dissection for $F(q)$ .....	51
3.5	The 5-Dissection for $F(q)$ .....	56
3.6	The 7-Dissection for $F(q)$ .....	60
3.7	The 11-Dissection for $F(q)$ .....	63
3.8	Conclusion .....	69
<b>4</b>	<b>Ranks and Cranks, Part III</b> .....	71
4.1	Introduction .....	71
4.2	Key Formulas on Page 59 .....	71
4.3	Proofs of Entries 4.2.1 and 4.2.3 .....	72
4.4	Further Entries on Pages 58 and 59 .....	74
4.5	Congruences for the Coefficients $\lambda_n$ on Pages 179 and 180 .....	75
4.6	Page 181: Partitions and Factorizations of Crank Coefficients .....	82
4.7	Series on Pages 63 and 64 Related to Cranks .....	84
4.8	Ranks and Cranks: Ramanujan's Influence Continues .....	86
4.8.1	Congruences and Related Work .....	87

4.8.2 Asymptotics and Related Analysis . . . . . 87  
 4.8.3 Combinatorics . . . . . 88  
 4.8.4 Inequalities . . . . . 88  
 4.8.5 Generalizations . . . . . 88

**5 Ramanujan’s Unpublished Manuscript on the Partition and Tau Functions . . . . . 89**

5.0 Congruences for  $\tau(n)$  . . . . . 92  
 5.1 The Congruence  $p(5n + 4) \equiv 0 \pmod{5}$  . . . . . 94  
 5.2 Divisibility of  $\tau(n)$  by 5 . . . . . 95  
 5.3 The Congruence  $p(25n + 24) \equiv 0 \pmod{25}$  . . . . . 97  
 5.4 Congruences Modulo  $5^k$  . . . . . 98  
 5.5 Congruences Modulo 7 . . . . . 99  
 5.6 Congruences Modulo 7, Continued . . . . . 100  
 5.7 Congruences Modulo 49 . . . . . 102  
 5.8 Congruences Modulo 49, Continued . . . . . 104  
 5.9 The Congruence  $p(11n + 6) \equiv 0 \pmod{11}$  . . . . . 106  
 5.10 Congruences Modulo 11, Continued . . . . . 107  
 5.11 Divisibility by 2 or 3 . . . . . 110  
 5.12 Divisibility of  $\tau(n)$  . . . . . 113  
 5.13 Congruences Modulo 13 . . . . . 119  
 5.14 Congruences for  $p(n)$  Modulo 13 . . . . . 121  
 5.15 Congruences to Further Prime Moduli . . . . . 123  
 5.16 Congruences for  $p(n)$  Modulo 17, 19, 23, 29, or 31 . . . . . 125  
 5.17 Divisibility of  $\tau(n)$  by 23 . . . . . 127  
 5.18 The Congruence  $p(121n - 5) \equiv 0 \pmod{121}$  . . . . . 129  
 5.19 Divisibility of  $\tau(n)$  for Almost All Values of  $n$  . . . . . 130  
 5.20 The Congruence  $p(5n + 4) \equiv 0 \pmod{5}$ , Revisited . . . . . 132  
 5.21 The Congruence  $p(25n + 24) \equiv 0 \pmod{25}$ , Revisited . . . . . 134  
 5.22 Congruences for  $p(n)$  Modulo Higher Powers of 5 . . . . . 135  
 5.23 Congruences for  $p(n)$  Modulo Higher Powers of 5, Continued . . 136  
 5.24 The Congruence  $p(7n + 5) \equiv 0 \pmod{7}$  . . . . . 138  
 5.25 Commentary . . . . . 139  
 5.1 The Congruence  $p(5n + 4) \equiv 0 \pmod{5}$  . . . . . 140  
 5.2 Divisibility of  $\tau(n)$  by 5 . . . . . 141  
 5.4 Congruences Modulo  $5^k$  . . . . . 143  
 5.5 Congruences Modulo 7 . . . . . 144  
 5.6 Congruences Modulo 7, Continued . . . . . 144  
 5.7 Congruences Modulo 49 . . . . . 144  
 5.8 Congruences Modulo 49, Continued . . . . . 145  
 5.9 The Congruence  $p(11n + 6) \equiv 0 \pmod{11}$  . . . . . 145  
 5.10 Congruences Modulo 11, Continued . . . . . 146  
 5.11 Divisibility by 2 or 3 . . . . . 147  
 5.12 Divisibility of  $\tau(n)$  . . . . . 150  
 5.13 Congruences Modulo 13 . . . . . 151

5.14 Congruences for  $p(n)$  Modulo 13 ..... 153

5.15 Congruences to Further Prime Moduli ..... 153

5.16 Congruences for  $p(n)$  Modulo 17, 19, 23, 29, or 31 ..... 159

5.17 Divisibility of  $\tau(n)$  by 23 ..... 176

5.18 The Congruence  $p(121n - 5) \equiv 0 \pmod{121}$  ..... 177

5.19 Divisibility of  $\tau(n)$  for Almost All Values of  $n$  ..... 177

5.20 The Congruence  $p(5n + 4) \equiv 0 \pmod{5}$ , Revisited ..... 178

5.23 Congruences for  $p(n)$  Modulo Higher Powers of 5, Continued .. 179

5.24 The Congruence  $p(7n + 5) \equiv 0 \pmod{7}$  ..... 179

**6 Theorems about the Partition Function on Pages 189 and 182** ..... 181

6.1 Introduction ..... 181

6.2 The Identities for Modulus 5 ..... 183

6.3 The Identities for Modulus 7 ..... 186

6.4 Two Beautiful, False, but Correctable Claims of Ramanujan .. 193

6.5 Page 182 ..... 195

6.6 Further Remarks ..... 199

**7 Congruences for Generalized Tau Functions on Page 178** .. 205

7.1 Introduction ..... 205

7.2 Proofs ..... 207

**8 Ramanujan’s Forty Identities for the Rogers–Ramanujan Functions** ..... 217

8.1 Introduction ..... 217

8.2 Definitions and Preliminary Results ..... 219

8.3 The Forty Identities ..... 222

8.4 The Principal Ideas Behind the Proofs ..... 229

8.5 Proofs of the 40 Entries ..... 243

8.6 Other Identities for  $G(q)$  and  $H(q)$  and Final Remarks ..... 333

**9 Circular Summation** ..... 337

9.1 Introduction ..... 337

9.2 Proof of Entry 9.1.1 ..... 339

9.3 Reformulations ..... 342

9.4 Special Cases ..... 345

**10 Highly Composite Numbers** ..... 359

**Scratch Work** ..... 403

**Location Guide** ..... 407

**Provenance** ..... 411

**References** ..... 413