

## Contents

**Preface to Second Edition** *XVII*

**Preface to the First Edition** *XIX*

<b>1</b>	<b>Introduction</b>	<i>1</i>
1.1	Brief Review of Useful Concepts	<i>2</i>
1.2	Laser with Modulated Losses	<i>4</i>
1.3	Objectives of a New Analysis Procedure	<i>11</i>
1.4	Preview of Results	<i>12</i>
1.5	Organization of This Work	<i>14</i>
<b>2</b>	<b>Discrete Dynamical Systems: Maps</b>	<i>19</i>
2.1	Introduction	<i>19</i>
2.2	Logistic Map	<i>20</i>
2.3	Bifurcation Diagrams	<i>22</i>
2.4	Elementary Bifurcations in the Logistic Map	<i>25</i>
2.4.1	Saddle–Node Bifurcation	<i>25</i>
2.4.2	Period-Doubling Bifurcation	<i>29</i>
2.5	Map Conjugacy	<i>32</i>
2.5.1	Changes of Coordinates	<i>32</i>
2.5.2	Invariants of Conjugacy	<i>33</i>
2.6	Fully Developed Chaos in the Logistic Map	<i>34</i>
2.6.1	Iterates of the Tent Map	<i>35</i>
2.6.2	Lyapunov Exponents	<i>36</i>
2.6.3	Sensitivity to Initial Conditions and Mixing	<i>37</i>
2.6.4	Chaos and Density of (Unstable) Periodic Orbits	<i>38</i>
2.6.4.1	Number of Periodic Orbits of the Tent Map	<i>38</i>
2.6.4.2	Expansiveness Implies Infinitely Many Periodic Orbits	<i>39</i>
2.6.5	Symbolic Coding of Trajectories: First Approach	<i>40</i>
2.7	One-Dimensional Symbolic Dynamics	<i>42</i>
2.7.1	Partitions	<i>42</i>
2.7.2	Symbolic Dynamics of Expansive Maps	<i>44</i>
2.7.3	Grammar of Chaos: First Approach	<i>48</i>
2.7.3.1	Interval Arithmetics and Invariant Interval	<i>48</i>

2.7.3.2	Existence of Forbidden Sequences	49
2.7.4	Kneading Theory	51
2.7.4.1	Ordering of Itineraries	52
2.7.4.2	Admissible Sequences	54
2.7.5	Bifurcation Diagram of the Logistic Map Revisited	55
2.7.5.1	Saddle–Node Bifurcations	55
2.7.5.2	Period-Doubling Bifurcations	56
2.7.5.3	Universal Sequence	57
2.7.5.4	Self-Similar Structure of the Bifurcation Diagram	58
2.8	Shift Dynamical Systems, Markov Partitions, and Entropy	59
2.8.1	Shifts of Finite Type and Topological Markov Chains	59
2.8.2	Periodic Orbits and Topological Entropy of a Markov Chain	61
2.8.3	Markov Partitions	63
2.8.4	Approximation by Markov Chains	65
2.8.5	Zeta Function	65
2.8.6	Dealing with Grammars	66
2.8.6.1	Simple Grammars	67
2.8.6.2	Complicated Grammars	69
2.9	Fingerprints of Periodic Orbits and Orbit Forcing	70
2.9.1	Permutation of Periodic Points as a Topological Invariant	70
2.9.2	Topological Entropy of a Periodic Orbit	72
2.9.3	Period 3 Implies Chaos and Sarkovskii’s Theorem	74
2.9.4	Period 3 Does Not Always Imply Chaos: Role of Phase-Space Topology	75
2.9.5	Permutations and Orbit Forcing	75
2.10	Two-Dimensional Dynamics: Smale’s Horseshoe	77
2.10.1	Horseshoe Map	77
2.10.2	Symbolic Dynamics of the Invariant Set	78
2.10.3	Dynamical Properties	81
2.10.4	Variations on the Horseshoe Map: Baker Maps	82
2.11	Hénon Map	85
2.11.1	A Once-Folding Map	85
2.11.2	Symbolic Dynamics of the Hénon Map: Coding	87
2.11.3	Symbolic Dynamics of the Hénon Map: Grammar	93
2.12	Circle Maps	96
2.12.1	A New Global Topology	96
2.12.2	Frequency Locking and Arnold Tongues	96
2.12.3	Chaotic Circle Maps as Limits of Annulus Maps	100
2.13	Annulus Maps	100
2.14	Summary	104
<b>3</b>	<b>Continuous Dynamical Systems: Flows</b>	<b>105</b>
3.1	Definition of Dynamical Systems	105
3.2	Existence and Uniqueness Theorem	106
3.3	Examples of Dynamical Systems	107

3.3.1	Duffing Equation	107
3.3.2	Van der Pol Equation	109
3.3.3	Lorenz Equations	111
3.3.4	Rössler Equations	113
3.3.5	Examples of Nondynamical Systems	114
3.3.5.1	Equation with Non-Lipschitz Forcing Terms	115
3.3.5.2	Delay Differential Equations	115
3.3.5.3	Stochastic Differential Equations	116
3.3.6	Additional Observations	117
3.4	Change of Variables	120
3.4.1	Diffeomorphisms	120
3.4.2	Examples	121
3.4.3	Structure Theory	124
3.5	Fixed Points	125
3.5.1	Dependence on Topology of Phase Space	125
3.5.2	How to Find Fixed Points in $R^n$	126
3.5.3	Bifurcations of Fixed Points	127
3.5.4	Stability of Fixed Points	130
3.6	Periodic Orbits	131
3.6.1	Locating Periodic Orbits in $R^{n-1} \times S^1$	131
3.6.2	Bifurcations of Fixed Points	132
3.6.3	Stability of Fixed Points	133
3.7	Flows Near Nonsingular Points	134
3.8	Volume Expansion and Contraction	136
3.9	Stretching and Squeezing	137
3.10	The Fundamental Idea	138
3.11	Summary	139
<b>4</b>	<b>Topological Invariants</b>	<b>141</b>
4.1	Stretching and Squeezing Mechanisms	141
4.2	Linking Numbers	145
4.2.1	Definitions	146
4.2.2	Reidemeister Moves	147
4.2.3	Braids	148
4.2.4	Examples	151
4.2.5	Linking Numbers for a Horseshoe	153
4.2.6	Linking Numbers for the Lorenz Attractor	154
4.2.7	Linking Numbers for the Period-Doubling Cascade	154
4.2.8	Local Torsion	155
4.2.9	Writhe and Twist	156
4.2.10	Additional Properties	158
4.3	Relative Rotation Rates	159
4.3.1	Definition	160
4.3.2	Computing Relative Rotation Rates	160
4.3.3	Horseshoe Mechanism	163

4.3.4	Additional Properties	168
4.4	Relation between Linking Numbers and Relative Rotation Rates	169
4.5	Additional Uses of Topological Invariants	170
4.5.1	Bifurcation Organization	170
4.5.2	Torus Orbits	171
4.5.3	Additional Remarks	171
4.6	Summary	174
<b>5</b>	<b>Branched Manifolds</b>	<b>175</b>
5.1	Closed Loops	175
5.2	What Does This Have to Do with Dynamical Systems?	178
5.3	General Properties of Branched Manifolds	178
5.4	Birman–Williams Theorem	181
5.4.1	Birman–Williams Projection	182
5.4.2	Statement of the Theorem	183
5.5	Relaxation of Restrictions	184
5.5.1	Strongly Contracting Restriction	184
5.5.2	Hyperbolic Restriction	185
5.6	Examples of Branched Manifolds	186
5.6.1	Smale–Rössler System	186
5.6.2	Lorenz System	188
5.6.3	Duffing System	189
5.6.4	Van der Pol System	192
5.7	Uniqueness and Nonuniqueness	194
5.7.1	Local Moves	195
5.7.2	Global Moves	197
5.8	Standard Form	200
5.9	Topological Invariants	201
5.9.1	Kneading Theory	202
5.9.2	Linking Numbers	205
5.9.3	Relative Rotation Rates	207
5.10	Additional Properties	207
5.10.1	Period as Linking Number	208
5.10.2	EBK-Like Expression for Periods	208
5.10.3	Poincaré Section	209
5.10.4	Blow-Up of Branched Manifolds	210
5.10.5	Branched-Manifold Singularities	211
5.10.6	Constructing a Branched Manifold from a Map	212
5.10.7	Topological Entropy	213
5.11	Subtemplates	216
5.11.1	Two Alternatives	216
5.11.2	A Choice	218
5.11.3	Topological Entropy	219
5.11.4	Subtemplates of the Smale Horseshoe	221

5.11.5	Subtemplates Involving Tongues	222
5.12	Summary	224
<b>6</b>	<b>Topological Analysis Program</b>	<b>227</b>
6.1	Brief Summary of the Topological Analysis Program	227
6.2	Overview of the Topological Analysis Program	228
6.2.1	Find Periodic Orbits	228
6.2.2	Embed in $R^3$	229
6.2.3	Compute Topological Invariants	230
6.2.4	Identify Template	230
6.2.5	Verify Template	231
6.2.6	Model Dynamics	232
6.2.7	Validate Model	233
6.3	Data	234
6.3.1	Data Requirements	235
6.3.2	Processing in the Time Domain	236
6.3.3	Processing in the Frequency Domain	238
6.3.3.1	High-Frequency Filter	238
6.3.3.2	Low-Frequency Filter	238
6.3.3.3	Derivatives and Integrals	239
6.3.3.4	Hilbert Transforms	240
6.3.3.5	Fourier Interpolation	241
6.3.3.6	Transform and Interpolation	242
6.4	Embeddings	243
6.4.1	Embeddings for Periodically Driven Systems	244
6.4.2	Differential Embeddings	244
6.4.3	Differential–Integral Embeddings	247
6.4.4	Embeddings with Symmetry	248
6.4.5	Time-Delay Embeddings	249
6.4.6	Coupled-Oscillator Embeddings	251
6.4.7	SVD Projections	252
6.4.8	SVD Embeddings	254
6.4.9	Embedding Theorems	254
6.5	Periodic Orbits	256
6.5.1	Close Returns Plots for Flows	256
6.5.1.1	Close Returns Histograms	258
6.5.1.2	Tests for Chaos	258
6.5.2	Close Returns in Maps	259
6.5.2.1	First Return Map	259
6.5.2.2	$p$ th Return Map	260
6.5.3	Metric Methods	261
6.6	Computation of Topological Invariants	262
6.6.1	Embed Orbits	262
6.6.2	Linking Numbers and Relative Rotation Rates	262
6.6.3	Label Orbits	263

6.7	Identify Template	263
6.7.1	Period-1 and Period-2 Orbits	263
6.7.2	Missing Orbits	264
6.7.3	More Complicated Branched Manifolds	264
6.8	Validate Template	264
6.8.1	Predict Additional Topological Invariants	265
6.8.2	Compare	265
6.8.3	Global Problem	265
6.9	Model Dynamics	265
6.10	Validate Model	268
6.10.1	Qualitative Validation	269
6.10.2	Quantitative Validation	269
6.11	Summary	270
<b>7</b>	<b>Folding Mechanisms: <math>A_2</math></b>	<b>271</b>
7.1	Belousov–Zhabotinskii Chemical Reaction	272
7.1.1	Location of Periodic Orbits	273
7.1.2	Embedding Attempts	274
7.1.3	Topological Invariants	278
7.1.4	Template	281
7.1.5	Dynamical Properties	281
7.1.6	Models	283
7.1.7	Model Verification	283
7.2	Laser with Saturable Absorber	285
7.3	Stringed Instrument	288
7.3.1	Experimental Arrangement	288
7.3.2	Flow Models	290
7.3.3	Dynamical Tests	291
7.3.4	Topological Analysis	291
7.4	Lasers with Low-Intensity Signals	294
7.4.1	SVD Embedding	295
7.4.2	Template Identification	296
7.4.3	Results of the Analysis	297
7.5	The Lasers in Lille	297
7.5.1	Class B Laser Model	298
7.5.2	CO <sub>2</sub> Laser with Modulated Losses	304
7.5.3	Nd-Doped YAG Laser	308
7.5.4	Nd-Doped Fiber Laser	311
7.5.5	Synthesis of Results	318
7.6	The Laser in Zaragoza	322
7.7	Neuron with Subthreshold Oscillations	328
7.8	Summary	334
<b>8</b>	<b>Tearing Mechanisms: <math>A_3</math></b>	<b>337</b>
8.1	Lorenz Equations	337
8.1.1	Fixed Points	338

8.1.2	Stability of Fixed Points	339
8.1.3	Bifurcation Diagram	339
8.1.4	Templates	341
8.1.5	Shimizu–Morioka Equations	343
8.2	Optically Pumped Molecular Laser	343
8.2.1	Models	344
8.2.2	Amplitudes	346
8.2.3	Template	346
8.2.4	Orbits	347
8.2.5	Intensities	350
8.3	Fluid Experiments	352
8.3.1	Data	352
8.3.2	Template	353
8.4	Why $A_3$ ?	354
8.5	Summary	354
<b>9</b>	<b>Unfoldings</b>	<b>357</b>
9.1	Catastrophe Theory as a Model	357
9.1.1	Overview	357
9.1.2	Example	358
9.1.3	Reduction to a Germ	359
9.1.4	Unfolding the Germ	361
9.1.5	Summary of Concepts	362
9.2	Unfolding of Branched Manifolds: Branched Manifolds as Germs	362
9.2.1	Unfolding of Folds	362
9.2.2	Unfolding of Tears	363
9.3	Unfolding within Branched Manifolds: Unfolding of the Horseshoe	365
9.3.1	Topology of Forcing: Maps	365
9.3.2	Topology of Forcing: Flows	366
9.3.3	Forcing Diagrams	369
9.3.3.1	Orbits with Zero Entropy	371
9.3.3.2	Orbits with Positive Entropy	372
9.3.3.3	Additional Comments	372
9.3.4	Basis Sets of Orbits	374
9.3.5	Coexisting Basins	375
9.4	Missing Orbits	375
9.5	Routes to Chaos	377
9.6	Orbit Forcing and Topological Entropy: Mathematical Aspects	378
9.6.1	General Outline	378
9.6.2	Basic Mathematical Concepts	379
9.6.2.1	Braids and Braid Types	379
9.6.2.2	Braids and Surface Homeomorphisms	380
9.6.2.3	Nielsen–Thurston Classification	381
9.6.2.4	Application to Periodic Orbits and Braid Types	382
9.7	Topological Measures of Chaos in Experiments	383

9.7.1	Mixing in Fluids	383
9.7.2	Chaos in an Optical Parametric Oscillator	385
9.8	Summary	389
<b>10</b>	<b>Symmetry</b>	<b>391</b>
10.1	Information Loss and Gain	391
10.1.1	Information Loss	391
10.1.2	Exchange of Symmetry	392
10.1.3	Information Gain	392
10.1.4	Symmetries of the Standard Systems	392
10.2	Cover and Image Relations	393
10.2.1	General Setup	393
10.3	Rotation Symmetry 1: Images	394
10.3.1	Image Equations and Flows	394
10.3.2	Image of Branched Manifolds	396
10.3.3	Image of Periodic Orbits	398
10.4	Rotation Symmetry 2: Covers	400
10.4.1	Topological Index	401
10.4.2	Covers of Branched Manifolds	402
10.4.3	Covers of Periodic Orbits	403
10.5	Peeling: a New Global Bifurcation	404
10.5.1	Orbit Perestroika	405
10.5.2	Covering Equations	405
10.6	Inversion Symmetry: Driven Oscillators	407
10.7	Duffing Oscillator	409
10.8	Van der Pol Oscillator	413
10.9	Summary	418
<b>11</b>	<b>Bounding Tori</b>	<b>419</b>
11.1	Stretching & Folding vs. Tearing & Squeezing	420
11.2	Inflation	421
11.3	Boundary of Inflation	422
11.4	Index	423
11.5	Projection	424
11.6	Nature of Singularities	426
11.7	Trinions	427
11.8	Poincaré Surface of Section	429
11.9	Construction of Canonical Forms	429
11.10	Perestroikas	432
11.10.1	Enlarging Branches	433
11.10.2	Starving Branches	433
11.11	Summary	435
<b>12</b>	<b>Representation Theory for Strange Attractors</b>	<b>437</b>
12.1	Embeddings, Representations, Equivalence	438
12.2	Simplest Class of Strange Attractors	439



12.3	Representation Labels	440
12.3.1	Parity	440
12.3.2	Global Torsion	441
12.3.3	Knot Type	445
12.4	Equivalence of Representations with Increasing Dimension	446
12.4.1	Parity	447
12.4.2	Knot Type	447
12.4.3	Global Torsion	448
12.5	Genus- $g$ Attractors	450
12.6	Representation Labels	451
12.6.1	Parity	451
12.6.2	Multitorsion Index	451
12.6.3	Knot Type	452
12.7	Equivalence in Increasing Dimension	453
12.7.1	Parity and Knot Type	453
12.7.2	Multitorsion Index	453
12.8	Summary	455
<b>13</b>	<b>Flows in Higher Dimensions</b>	<b>457</b>
13.1	Review of Classification Theory in $R^3$	457
13.2	General Setup	459
13.2.1	Spectrum of Lyapunov Exponents	459
13.2.2	Double Projection	461
13.3	Flows in $R^4$	462
13.3.1	Cyclic Phase Spaces	462
13.3.2	Floppiness and Rigidity	462
13.3.3	Singularities in Return Maps	463
13.4	Cusps in Weakly Coupled, Strongly Dissipative Chaotic Systems	466
13.4.1	Coupled Logistic Maps	466
13.4.2	Coupled Diode Resonators	469
13.5	Cusp Bifurcation Diagrams	470
13.5.1	Cusp Return Maps	472
13.5.2	Structure in the Control Plane	472
13.5.3	Comparison with the Fold	474
13.6	Nonlocal Singularities	475
13.6.1	Multiple Cusps	475
13.7	Global Boundary Conditions	477
13.8	From Braids to Triangulations: toward a Kinematics in Higher Dimensions	481
13.8.1	Knot Theory in Three Dimensions and Beyond	481
13.8.2	From Nonintersection to Orientation Preservation	482
13.8.3	Singularities in Higher Dimensions	490
13.9	Summary	490
<b>14</b>	<b>Program for Dynamical Systems Theory</b>	<b>493</b>
14.1	Reduction of Dimension	494

14.2	Equivalence	496
14.3	Structure Theory	497
14.3.1	Reducibility of Dynamical Systems	497
14.4	Germs	498
14.5	Unfolding	500
14.6	Paths	502
14.7	Rank	502
14.7.1	Stretching and Squeezing	503
14.8	Complex Extensions	504
14.9	Coxeter–Dynkin Diagrams	504
14.10	Real Forms	506
14.11	Local vs. Global Classification	507
14.12	Cover–Image Relations	508
14.13	Symmetry Breaking and Restoration	508
14.13.1	Entrainment and Synchronization	509
14.14	Summary	511

## **Appendix A Determining Templates from Topological Invariants** 513

A.1	The Fundamental Problem	513
A.2	From Template Matrices to Topological Invariants	515
A.2.1	Classification of Periodic Orbits by Symbolic Names	515
A.2.2	Algebraic Description of a Template	516
A.2.3	Local Torsion	517
A.2.4	Relative Rotation Rates: Examples	517
A.2.5	Relative Rotation Rates: General Case	519
A.3	Identifying Templates from Invariants	523
A.3.1	Using an Independent Symbolic Coding	524
A.3.2	Simultaneous Determination of Symbolic Names and Template	527
A.4	Constructing Generating Partitions	531
A.4.1	Symbolic Encoding as an Interpolation Process	531
A.4.2	Generating Partitions for Experimental Data	535
A.4.3	Comparison with Methods Based on Homoclinic Tangencies	536
A.4.4	Symbolic Dynamics on Three Symbols	538
A.5	Summary	539

## **Appendix B Embeddings** 541

B.1	Diffeomorphisms	541
B.2	Mappings of Data	543
B.2.1	Too Little Data	543
B.2.2	Too Much Data	545
B.2.3	Just the Right Amount of Data	547
B.3	Tests for Embeddings	547
B.4	Tests of Embedding Tests	549
B.4.1	Trial Data Set	549
B.5	Geometric Tests for Embeddings	550
B.5.1	Fractal Dimension Estimation	550

B.5.2	False Near Neighbor Estimates	553
B.6	Dynamical Tests for Embeddings	554
B.7	Topological Test for Embeddings	555
B.8	Postmortem on Embedding Tests	557
B.8.1	Generality	557
B.8.2	Computational Load	557
B.8.3	Variability	558
B.8.4	Statistics	559
B.8.5	Parameters	560
B.8.6	Noise	560
B.8.7	The Self-Intersection Problem	561
B.8.8	Reliability and Limitations	561
B.9	Stationarity	562
B.10	Beyond Embeddings	563
B.11	Summary	563
<b>Appendix C Frequently Asked Questions</b>		565
C.1	Is Template Analysis Valid for Non-Hyperbolic Systems?	565
C.2	Can Template Analysis Be Applied to Weakly Dissipative Systems?	566
C.3	What About Higher-Dimensional Systems?	567
<b>References</b>		569
<b>Index</b>		581

